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COUNTING CONJUGACY CLASSES IN $\mathrm{Out}(F_N)$

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Abstract

We show that if a finitely generated group G has a nonelementary WPD action on a hyperbolic metric space X, then the number of G-conjugacy classes of X-loxodromic elements of G coming from a ball of radius R in the Cayley graph of G grows exponentially in R. As an application we prove that for $N \geq 3$ the number of distinct $\mathrm{Out}(F_N)$-conjugacy classes of fully irreducible elements φ from an R-ball in the Cayley graph of $\mathrm{Out}(F_N)$ with $\log \lambda(\varphi)$ of the order of R grows exponentially in R.

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1. Introduction

The study of the growth of the number of periodic orbits of a dynamical system is an important theme in dynamics and geometry. A classic and still incredibly influential result of Margulis [20] computes the precise asymptotics of the number of closed geodesics of length less than or equal to R (that is, of periodic orbits of geodesic flow of length less than or equal to R) for a compact hyperbolic manifold. A recent result of Eskin and Mirzakhani [12], which served as a motivation for this note, shows that for the moduli space $\mathcal{M}_g$ of a closed oriented surface S_g of genus $g \geq 2$, the number $N(R)$ of closed Teichmüller geodesics in $\mathcal{M}_g$ of length less than or equal to R grows as $N(R) \sim e^{hR}/hR$ as $R \to \infty$, where $h = 6g - 6$. Note that in the context of a group G acting by isometries on a geodesic metric space, counting closed geodesics in the quotient space amounts to counting conjugacy classes of elements of G rather than counting elements displacing a basepoint by distance less than or equal to R. Problems about counting conjugacy classes with various metric restrictions are harder than those about counting group elements and many group-theoretic tricks (for example, embeddings of free subgroups or of free subsemigroups) do not, *a priori*, give any useful information about the growth of the number of conjugacy classes.

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In the context of the Eskin–Mirzakhani result mentioned above, a closed Teichmüller geodesic of length less than or equal to R in $\mathcal{M}_g$ comes from an axis $L(\varphi)$ in the Teichmüller space $\mathcal{T}_g$ of a pseudo-Anosov element $\varphi \in \mathrm{Mod}(S_g)$ with translation length $\tau(\varphi) \leq R$. Note that $\tau(\varphi) = \log \lambda(\varphi)$, where $\lambda(\varphi)$ is the dilatation of φ. Thus $N(R)$ is the number of $\mathrm{Mod}(S_g)$-conjugacy classes of pseudo-Anosov elements $\varphi \in \mathrm{Mod}(S_g)$ with $\log \lambda(\varphi) \leq R$, where $\mathrm{Mod}(S_g)$ is the mapping class group.

In this note we study a version of this question for $\mathrm{Out}(F_N)$ where $N \geq 3$. The main analogue of being pseudo-Anosov in the $\mathrm{Out}(F_N)$ context is the notion of a fully irreducible element. Every fully irreducible element $\varphi \in \mathrm{Out}(F_N)$ has a well-defined *stretch factor* $\lambda(\varphi) > 1$ (see [6]) which is an invariant of the $\mathrm{Out}(F_N)$-conjugacy class of φ. For specific φ one can compute $\lambda(\varphi)$ via train track methods, but counting the number of distinct $\lambda(\varphi)$ with $\log(\lambda(\varphi)) \leq R$ (where $\varphi \in \mathrm{Out}(F_N)$ is fully irreducible) appears to be an unapproachable task. Other $\mathrm{Out}(F_N)$-conjugacy invariants such as the index, the index list and the ideal Whitehead graph of fully irreducibles [7, 8, 15, 22] also do not appear to be well suited for counting problems. Unlike the Teichmüller space, the Outer space CV_N does not have a nice local analytic/geometric structure similar to the Teichmüller geodesic flow. Moreover, as we explain in Remark 3.3 below, it is reasonable to expect that (for $N \geq 3$) the number of $\mathrm{Out}(F_N)$-conjugacy classes of fully irreducibles $\varphi \in \mathrm{Out}(F_N)$ with $\log \lambda(\varphi) \leq R$ probably grows as a double exponential in R (rather than as a single exponential in R, as in the case of $\mathcal{M}_g$). Therefore, to get an exponential growth of the number of conjugacy classes one needs to consider a more restricted context, namely where the elements come from an R-ball in the Cayley graph of $\mathrm{Out}(F_N)$.

Here we obtain the following result.

Theorem 1.1. *Let $N \geq 3$. Let S be a finite generating set for $\mathrm{Out}(F_N)$. For $R \geq 0$, let B_R be the ball of radius R in the Cayley graph of $\mathrm{Out}(F_N)$ with respect to S. There exist constants $\sigma > 1$ and $C_2 > C_1 > 0$, $R_0 \geq 1$ such that the following result holds. If c_R denotes the number of distinct $\mathrm{Out}(F_N)$-conjugacy classes of fully irreducible elements $\varphi \in B_R$ such that $C_1 R \leq \log \lambda(\varphi) \leq C_2 R$, then*

$$c_R \geq \sigma^R \quad \text{for all } R \geq R_0.$$

In the course of proving Theorem 1.1, we establish the following general result.

Theorem 1.2. *Suppose G has a cobounded WPD action on a hyperbolic metric space X and L is a nonelementary subgroup of G for this action. Let S be a generating set of G. For $R \geq 1$, let b_R be the number of distinct G-conjugacy classes of elements of $g \in L$ that act loxodromically on X and such that $|g|_S \leq R$. Then there exist $R_0 \geq 1$ and $c > 1$ such that, for all $R \geq R_0$,*

$$b_R \geq c^R.$$

This statement is most relevant in the case where G is finitely generated and S is finite, but the conclusion of Theorem 1.2 is not obvious even if S is infinite. Theorem 1.2 is a generalisation of [17, Theorem 1.1], which states (under a different but equivalent hypothesis (see [21])) that such G has exponential conjugacy growth. The proofs of Theorem 1.2 and [17, Theorem 1.1] are similar and are both based on the theory of hyperbolically embedded subgroups developed in [10], and specifically the construction of virtually free hyperbolically embedded subgroups in [10, Theorem 6.14].

Both Theorems 1.1 and 1.2 are derived using the following result.

Theorem 1.3. *Suppose that G has a cobounded WPD action on a hyperbolic metric space X and L is a nonelementary subgroup of G for this action. Then, for every $n \geq 2$, there exist a subgroup $H \leq L$ and a finite group $K \leq G$ such that:*

(1) $H \cong F_n$;
(2) $H \times K \hookrightarrow_h G$;
(3) *the orbit map $H \to X$ is a quasi-isometric embedding.*

In particular, every element of H is loxodromic with respect to the action on X and two elements of H are conjugate in G if and only if they are conjugate in H.

Here, for a subgroup A of a group G, writing $A \hookrightarrow_h G$ means that A is *hyperbolically embedded in G*.

The proof of Theorem 1.1 also uses, as an essential ingredient, a result of Dowdall and Taylor [11, Theorem 4.1] about quasigeodesics in the Outer spaces and the free factor complex. Note that it is still not known whether the action of $\mathrm{Out}(F_N)$ on the free factor complex is acylindrical and, in a sense, the use of Theorem 1.2 provides a partial workaround here. Note that in the proof of Theorem 1.1, instead of using Theorem 1.3, we could have used an argument about stable subgroups having finite *width*. It is proved in [3] that convex cocompact (that is, finitely generated and with the orbit map to the free factor complex being a quasi-isometric embedding) subgroups of $\mathrm{Out}(F_N)$ are *stable* in $\mathrm{Out}(F_N)$. In turn, it is proved in [2] that if H is a stable subgroup of a group G, then H has finite width in G (see [2] for the relevant definitions). Having finite width is a property sufficiently close to being malnormal to allow counting arguments with conjugacy classes to go through. The proof given here, based on using Theorem 1.3 above, is different in flavour and proceeds from rather general assumptions. Note that in the conclusion of Theorem 1.3 the fact that $H \times K \hookrightarrow_h G$ implies that H and $H \times K$ are stable in G.

Another possible approach to counting conjugacy classes involves the notion of 'statistically convex cocompact actions' recently introduced and studied by Yang; see [23, 24] (particularly [24, Theorem B] about genericity of conjugacy classes of strongly contracting elements). However, Yang's results only apply to actions on proper geodesic metric spaces with finite critical exponent for the action, such as the action of the mapping class group on the Teichmüller space. For essentially the same reasons as explained in Remark 3.3 below, the action of $\mathrm{Out}(F_N)$ (where $N \geq 3$) on CV_N, with either the asymmetric or symmetrised Lipschitz metric, has infinite critical exponent. Still, it is possible that the statistical convex cocompactness methods may be applicable to the actions on CV_N of some interesting subgroups of $\mathrm{Out}(F_N)$.

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2. Conjugacy classes of loxodromics for WPD actions

We assume throughout that all metric spaces under consideration are geodesic and all group actions on metric spaces are by isometries. Given a generating set $\mathcal{A}$ of a group G, we let $\mathrm{Cay}(G, \mathcal{A})$ denote the Cayley graph of G with respect to $\mathcal{A}$. In order to apply results from [10] directly, it is more convenient to consider actions on Cayley graphs. By the well-known Milnor–Svarc lemma, this is equivalent to considering cobounded actions.

Lemma 2.1 (Milnor–Svarc). *If G acts coboundedly on a geodesic metric space X, then there exists $\mathcal{A} \subseteq G$ such that* $\mathrm{Cay}(G, \mathcal{A})$ *is G-equivalently quasi-isometric to X.*

Definition 2.2. Let H be a subgroup of G and S a subset of G such that $\langle H \cup S \rangle = G$. We identify $\mathrm{Cay}(H, H)$ with the corresponding complete subgraph of $\mathrm{Cay}(G, H \cup S)$. We say that H is *hyperbolically embedded in G with respect to S* if the following two conditions are satisfied.

(1) $\mathrm{Cay}(G, H \cup S)$ is a hyperbolic metric space.
(2) For each $n \in \mathbb{N}$, there are at most finitely many $h \in H$ such that there exists a path in $\mathrm{Cay}(G, H \cup S)$ from 1 to h of length at most n which contains no edges of $\mathrm{Cay}(H, H)$.

We use $H \hookrightarrow_h (G, S)$ to denote that H is hyperbolically embedded in G with respect to S, or simply $H \hookrightarrow_h G$ if $H \hookrightarrow_h (G, S)$ for some $S \subseteq G$.

This definition can be naturally extended to a collection of subgroups; we refer to [10] for more details. The only property of a hyperbolically embedded subgroup H that we use is that H is *almost malnormal*, that is, for $g \in G \setminus H$, the intersection of H and H^g is finite [10, Proposition 4.33]. Note that this implies that any two infinite-order elements of H are conjugate in G if and only if they are conjugate in H.

For a metric space X and an isometry g of X, the asymptotic translation length $\|g\|_X$ is defined as $\|g\|_X = \lim_{i\to\infty} d(g^i x, x)/i$, where $x \in X$. It is well known that this limit always exists and is independent of $x \in X$. If $\|g\|_X > 0$ then g is called *loxodromic*. For a group G acting on X, a loxodromic element is called a *WPD element* if, for all $\varepsilon > 0$ and all $x \in X$, there exists $m \in \mathbb{N}$ such that

$$|\{h \in G \mid d(x, hx) < \varepsilon, d(g^m x, hg^m x) < \varepsilon\}| < \infty.$$

We say that the action of G on X is *WPD* if every loxodromic element is a WPD element.

We now fix a subset $\mathcal{A} \subseteq G$ such that $\mathrm{Cay}(G, \mathcal{A})$ is hyperbolic and the action of G on $\mathrm{Cay}(G, \mathcal{A})$ is WPD. We say that $g \in G$ is loxodromic if it is loxodromic with respect to the action of G on $\mathrm{Cay}(G, \mathcal{A})$. Each such element is contained in a unique, maximal, virtually cyclic subgroup $E(g)$ [10, Lemma 6.5].

Lemma 2.3 [16, Corollary 3.17]. *If $g_1, \ldots, g_n$ are noncommensurable loxodromic elements, then $\{E(g_1), \ldots, E(g_n)\} \hookrightarrow_h (G, \mathcal{A})$.*

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A subgroup $L \leq G$ is called *nonelementary* if L contains two noncommensurable loxodromic elements. Let $K_G(L)$ denote the maximal finite subgroup of G normalised by L. When L is nonelementary, this subgroup is well defined by [16, Lemma 5.5].

The following lemma was proved in [16] under the assumption that the action is acylindrical, but the proof only requires that the action is WPD.

LEMMA 2.4 [16, Lemma 5.6]. *Let L be a nonelementary subgroup of G. Then there exist noncommensurable, loxodromic elements $g_1, \ldots, g_n$ contained in L such that $E(g_i) = \langle g_i \rangle \times K_G(L)$.*

PROOF OF THEOREM 1.3. First we note that statement (1.3) in the theorem implies that every element of H is loxodromic with respect to the action on X. Also, the fact that two elements of H are conjugate in G if and only if they are conjugate in H follows from the fact that $H \times K$ is almost malnormal in G and K acts trivially on H by conjugation.

We use the construction from [10, Theorem 6.14]. As in [10, Theorem 6.14], we let $n = 2$ since the construction from this case can be easily modified for any n.

By Lemma 2.1, we can assume $X = \mathrm{Cay}(G, \mathcal{A})$ for some $\mathcal{A} \subseteq G$. Let $g_1, \ldots, g_6$ be elements of L given by Lemma 2.4. Then each $E(g_i) = \langle g_i \rangle \times K_G(L)$ and, furthermore, $\{E(g_1), \ldots, E(g_6)\} \hookrightarrow_h (G, \mathcal{A})$ by Lemma 2.3. Let

$$\mathcal{E} = \bigcup_{i=1}^{6} E(g_i) \setminus \{1\}.$$

Let $H = \langle x, y \rangle$, where $x = g_1^n g_2^n g_3^n$ and $y = g_4^n g_5^n g_6^n$ for sufficiently large n. It is shown in [10] that x and y generate a free subgroup of G and this subgroup is quasiconvex in $\mathrm{Cay}(G, \mathcal{A} \cup \mathcal{E})$. Hence H (with the natural word metric) is quasi-isometrically embedded in $\mathrm{Cay}(G, \mathcal{A} \cup \mathcal{E})$, and since the map $\mathrm{Cay}(G, \mathcal{A}) \to \mathrm{Cay}(G, \mathcal{A} \cup \mathcal{E})$ is 1-Lipschitz H is also quasi-isometrically embedded in $\mathrm{Cay}(G, \mathcal{A})$. Let $K = K_G(L)$. Since x and y both commute with K, it follows that $\langle H, K \rangle \cong H \times K$. Finally, we can apply [10, Theorem 4.42] to see that $H \times K \hookrightarrow_h G$. Verifying the assumptions of [10, Theorem 4.42] is identical to the proof of [10, Theorem 6.14].

Note that Theorem 1.2 is an immediate consequence of Theorem 1.3.

3. The case of $\mathrm{Out}(F_N)$

We assume familiarity on the part of the reader with the basics related to $\mathrm{Out}(F_N)$ and Outer space. For background information on these topics we refer the reader to [5, 6, 9, 13].

In what follows we assume that $N \geq 2$ is an integer, CV_N is the (volume-one normalised) Culler–Vogtmann Outer space, $\mathcal{F}_N$ is the free factor complex for F_N, d_C is the asymmetric Lipschitz metric on CV_N and d_C^{sym} is the symmetrised Lipschitz metric on CV_N. When we talk about the Hausdorff distance d_{Haus} in CV_N, we always mean the Hausdorff distance with respect to d_C^{sym}. For $K \geq 1$, by a K-quasigeodesic in CV_N

we mean that a function $\gamma : I \to CV_N$ (where $I \subseteq \mathbb{R}$ is an interval) such that, for all $s, t \in I$ with $s \le t$,

$$\frac{1}{K}(t-s) - K \le d_C(\gamma(s), \gamma(t)) \le K(t-s) + K.$$

For $\epsilon > 0$ we denote by $CV_{N,\epsilon}$ the ϵ-thick part of CV_N.

Remark 3.1 (Left and right actions on CV_N). There is a natural right action of $\mathrm{Out}(F_N)$ on CV_N. If $T \in CV_N$ is an $\mathbb{R}$-tree with a minimal free discrete isometric action of F_N and if $\Phi \in \mathrm{Aut}(F_N)$ is an automorphism, then the point $T\Phi \in CV_N$ is defined as the tree T with the action of F_N twisted via Φ: for $x \in T$ and $u \in F_N$, we have $u \underset{T\Phi}{\cdot} x := \Phi(u) \underset{T}{\cdot} x$. This action of $\mathrm{Aut}(F_N)$ descends to the action of $\mathrm{Out}(F_N)$ on CV_N. At the level of translation length functions, for $T \in CV_N$, $\varphi \in \mathrm{Out}(F_N)$ and $u \in F_N$, we have $\|u\|_{T\varphi} = \|\varphi(u)\|_T$. This right action of $\mathrm{Out}(F_N)$ on CV_N can also be converted to a left action by setting $\varphi T := T\varphi^{-1}$ for $T \in CV_N$, $\varphi \in \mathrm{Out}(F_N)$. We will need to work with both the right and left actions of $\mathrm{Out}(F_N)$ on CV_N. Note, however, that periodic folding lines in CV_N coming from train track maps naturally correspond to the right action of $\mathrm{Out}(F_N)$ on CV_N. More precisely, if $f : G \to G$ is a train track map representing some $\varphi \in \mathrm{Out}(F_N)$, then f naturally defines a 'folding' path from $\tilde{G}$ to $\tilde{G}\varphi$ in CV_N, which can be extended to a bi-infinite φ-periodic folding path. See [4, 13, 15] for more details.

We recall a portion of one of the main technical results of Dowdall and Taylor, [11, Theorem 4.1].

Proposition 3.2. *Let $K \ge 1$ and let $\gamma : \mathbb{R} \to CV_N$ be a K-quasigeodesic such that its projection $\pi \circ \gamma : \mathbb{R} \to \mathcal{F}_N$ is also a K-quasigeodesic. There are constants $D > 0, \epsilon > 0$, depending only on K and N, such that the following result holds. If $\rho : \mathbb{R} \to CV_N$ is any geodesic with the same endpoints as γ, then*

(1) $\gamma(\mathbb{R}), \rho(\mathbb{R}) \subset CV_{N,\epsilon}$;
(2) $d_{\mathrm{Haus}}(\gamma(\mathbb{R}), \rho(\mathbb{R})) \le D$.

Here saying that γ and ρ have the same endpoints means that $\sup_{t\in\mathbb{R}} d_C^{\mathrm{sym}}(\gamma(t), \rho(t)) < \infty$.

Proof of Theorem 1.1. By [5], the action of $\mathrm{Out}(F_N)$ on $\mathcal{F}_N$ satisfies the hypothesis of Theorem 1.3. Let H be the subgroup provided by Theorem 1.3 with $L = \mathrm{Out}(F_n)$. We fix a free basis $\mathcal{A} = \{a, b\}$ for the free group H, and let $d_{\mathcal{A}}$ be the corresponding word metric on H.

Note that the assumptions on H imply that every nontrivial element of H is fully irreducible. Moreover, if we pick a basepoint p in $\mathcal{F}_N$, then there is $K \ge 1$ such that the image of every geodesic in the Cayley graph $\mathrm{Cay}(H, \mathcal{A})$ in $\mathcal{F}_N$ under the orbit map is a (parameterised) K-quasigeodesic.

Pick a basepoint $G_0 \in CV_N$. Since the projection $\pi : (CV_N, d_C) \to \mathcal{F}_N$ is coarsely Lipschitz [5], and since the orbit map $H \to \mathcal{F}_N$ is a quasi-isometric embedding, it follows that the orbit map $(H, d_{\mathcal{A}}) \to (CV_N, d_C), u \mapsto uG_0$ is a K_1-quasi-isometric embedding for some $K_1 \geq 1$. Moreover, the image of this orbit map lives in an ϵ_0-thick part of CV_N (where ϵ_0 is the injectivity radius of G_0). Since on CV_{N,ϵ_0} the metrics d_C and d_C^{sym} are bi-Lipschitz equivalent (see [1, Theorem 24]), it follows that the orbit map $(H, d_{\mathcal{A}}) \to (CV_N, d_C^{\mathrm{sym}}), u \mapsto uG_0$ is a K_2-quasi-isometric embedding for some $K_2 \geq 1$. For every $c \in \mathcal{A}^{\pm 1}$, fix a d_C-geodesic τ_c from G_0 to cG_0.

Now let $\gamma : I \to \mathrm{Cay}(H, \mathcal{A})$ be a geodesic such that $\gamma^{-1}(H) = I \cap \mathbb{Z}$ and such that the endpoints of I (if any) are integers. We then define a path $\underline{\gamma} : I \to CV_N$ as follows. Whenever $n \in \mathbb{Z}$ is such that $[n, n+1] \subseteq I$, then $\gamma(n) = g$ and $\gamma(n+1) = gc$ for some $c \in \mathcal{A}^{\pm 1}$. In this case we define $\underline{\gamma}|_{[n,n+1]}$ to be $g\tau_c$. Then, for every geodesic $\gamma : I \to \mathrm{Cay}(H, \mathcal{A})$ as above, the path $\underline{\gamma} : I \to CV_N$ is K_3-quasigeodesic with respect to both d_C and d_C^{sym}, for some $K_3 \geq 1$ independent of γ. Moreover, $\underline{\gamma}(I) \subset CV_{N,\epsilon_1}$ for some $\epsilon_1 > 0$ independent of γ.

Let w be a cyclically reduced word of length $n \geq 1$ in H. Consider the bi-infinite w^{-1}-periodic geodesic $\gamma : \mathbb{R} \to \mathrm{Cay}(H, \mathcal{A})$ with $\gamma(0) = 1$ and $\gamma(n) = w^{-1}$. Thus the path $\underline{\gamma} : \mathbb{R} \to CV_N$ is K_3-quasigeodesic, with respect to both d_C and d_C^{sym}, and $\underline{\gamma}(\mathbb{R}) \subset CV_{N,\epsilon_1}$. Since $1 \neq w \in H$, it follows that w is fully irreducible as an element of $\mathrm{Out}(F_N)$. Hence w can be represented by an expanding irreducible train track map $f : G \to G$ with the Perron–Frobenius eigenvalue $\lambda(f)$ equal to the stretch factor $\lambda(w)$ of the outer automorphism $w \in \mathrm{Out}(F_N)$. There exists a volume-one 'eigenmetric' d_f on G with respect to which f is a local $\lambda(f)$-homothety. If we view (G, d_f) as a point of CV_N, then $d_C(G, Gw) = \log \lambda(w)$. Moreover, in this case we can construct a w-periodic d_C-geodesic folding line $\rho : \mathbb{R} \to CV_N$ with the property that $\rho(i) = Gw^i$ for any integer i. Hence, $d_C(Gw^i, Gw^j) = (j - i) \log \lambda(w)$ for integers $i < j$. Thus, for any $i > 0$, we have $d_C(G, w^{-i}G) = i \log \lambda(w)$.

The bi-infinite lines ρ and $\underline{\gamma}$ are both w^{-1}-periodic (in the sense of the left action of w^{-1}) and therefore $\sup_{t \in \mathbb{R}} d_C^{\mathrm{sym}}(\underline{\gamma}(t), \rho(t)) < \infty$. Hence, by Proposition 3.2, there exist constants $D > 0$ and $\epsilon > 0$ (independent of w) such that $\rho \subset CV_{N,\epsilon}$ and $d_{\mathrm{Haus}}(\rho, \underline{\gamma}) \leq D$. The fact that $\rho \subset CV_{N,\epsilon}$ implies that ρ is a K_4-quasigeodesic with respect to d_C^{sym} for some constant $K_4 \geq 1$ independent of w.

Consider the asymptotic translation length $||w||_{CV}$, where w is viewed as an isometry of $(CV_N, d_C^{\mathrm{sym}})$. On the one hand (using the line ρ),

$$\frac{1}{K_4} \log \lambda(w^{-1}) \leq ||w||_{CV} \leq K_4 \log \lambda(w^{-1}),$$

and on the other hand (using the line $\underline{\gamma}$),

$$\frac{1}{K_3} n \leq ||w||_{CV} \leq K_3 n.$$

Therefore

$$\frac{1}{K_3 K_4} n \leq \log \lambda(w^{-1}) \leq K_3 K_4 n.$$

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Recall also that, by a result of Handel and Mosher [14], there exists a constant $M = M(N) \geq 1$ such that, for every fully irreducible $\varphi \in \mathrm{Out}(F_N)$,

$$\frac{1}{M} \leq \frac{\log \lambda(\varphi)}{\log \lambda(\varphi^{-1})} \leq M.$$

Therefore, for w as above,

$$\frac{1}{K_3 K_4 M} n \leq \log \lambda(w) \leq K_3 K_4 M n.$$

Now recall that $\mathcal{A} = \{a, b\}$ and that S is a finite generating set for $\mathrm{Out}(F_N)$. Put $M' = \max\{|a|_S, |b|_S\}$ so that, for every freely reduced word w in H, we have $|w|_S \leq M' |w|_{\mathcal{A}}$.

For $R \gg 1$, put $n = \lfloor R/M' \rfloor$. The number of distinct H-conjugacy classes represented by cyclically reduced words w of length n is equal to or greater than 2^n, for n big enough. Recall that two elements of H are conjugate in H if and only if they are conjugate in $\mathrm{Out}(F_N)$. Therefore, we get at least $2^n \geq 2^{R/M'-1}$ distinct $\mathrm{Out}(F_N)$-conjugacy classes from such words w. As we have seen above, each such w gives us a fully irreducible element of $\mathrm{Out}(F_N)$ with

$$\frac{1}{K_3 K_4 M} \frac{R}{2M'} \leq \log \lambda(w) \leq K_3 K_4 M n \leq K_3 K_4 M \frac{R}{M'},$$

and the statement of Theorem 1.1 is verified.

Remark 3.3. As noted in the introduction, unlike the mapping class group case, we expect that, for $N \geq 3$, the number of $\mathrm{Out}(F_N)$-conjugacy classes of fully irreducibles $\varphi \in \mathrm{Out}(F_N)$ with $\log \lambda(\varphi) \leq R$ grows as a double exponential in R. A double exponential upper bound follows from general Perron–Frobenius considerations. Every fully irreducible $\varphi \in \mathrm{Out}(F_N)$ can be represented by an expanding irreducible train track map $f : G \to G$, where G is a finite connected graph with $b_1(G) = N$ and with all vertices in G of degree 3 or less. The number of possibilities for such G is finite in terms of N. By [19, Proposition A.4], if f is as above and $\lambda := \lambda(f)$, then $\max m_{ij} \leq k\lambda^{k+1}$ (where k is the number of edges in G and $M = (m_{ij})_{ij}$ is the transition matrix of f). If $\log \lambda \leq R$, we get $\max \log m_{ij} \leq \log k + (k+1)R$ and $\max m_{ij} \leq ke^{(k+1)R}$. Thus we get exponentially many (in terms of R) possibilities for the transition matrix M of f. Since, for a given length L, there are exponentially many paths of length L in G, this yields an (*a priori*) double exponential upper bound for the number of train track maps representing fully irreducible elements of $\mathrm{Out}(F_N)$ with $\log \lambda \leq R$.

For the prospective double exponential lower bound we give the following explicit construction for the case of F_3. Let $w \in F(b, c)$ be a nontrivial positive word containing the subwords b^2, c^2, bc and cb. Consider the automorphism φ_w of $F_3 = F(a, b, c)$ defined by $\varphi_w(a) = b$, $\varphi_w(b) = c$, $\varphi_w(c) = aw(b, c)$. We can also view φ_w as a graph map $f_w : R_3 \to R_3$, where R_3 is the 3-rose with the petals marked by a, b, c. Then f_w is an expanding irreducible train track map representing φ_w. Moreover, a direct check

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shows that, under the assumptions made on w, the Whitehead graph of f_w is connected. Additionally, for a given $n \geq 1$, 'most' positive words of length n in $F(b, c)$ satisfy the above conditions and define fully irreducible automorphisms φ_w. To see this, observe that the free-by-cyclic group $G_w = F_3 \rtimes_{\varphi_w} \mathbb{Z}$ can be rewritten as a one-relator group:

$$\begin{aligned} G_w &= \langle a, b, c, t \mid t^{-1}at = b, t^{-1}bt = c, t^{-1}ct = aw(b, c)\rangle \\ &= \langle a, t \mid t^{-3}at^3 = aw(t^{-1}at, t^{-2}at^2)\rangle. \end{aligned}$$

Moreover, one can check that if w was a $C'(1/20)$ word, then the above one-relator presentation of G_w satisfies the $C'(1/6)$ small cancellation condition, and therefore G_w is word-hyperbolic and the automorphism $\varphi_w \in \mathrm{Out}(F_3)$ is atoroidal. Since, as noted above, φ_w admits an expanding irreducible train track map on the rose with connected Whitehead graph, a result of Kapovich [18] implies that φ is fully irreducible. Moreover, if $|w| = L$, then it is not hard to check that $\log \lambda(f_w) = \log \lambda(\varphi_w)$ grows like $\log L$.

Since 'random' positive words $w \in F(b, c)$ are $C'(1/20)$ and contain b^2, c^2, cb, bc as subwords, for sufficiently large $R \geq 1$, the above construction produces doubly exponentially many atoroidal fully irreducible automorphisms φ_w with $|w| = e^R$ and $\log \lambda(\varphi_w)$ of the order of R. We conjecture that in fact most of these elements are pairwise nonconjugate in $\mathrm{Out}(F_3)$ and that this method yields doubly exponentially many fully irreducible elements φ of $\mathrm{Out}(F_3)$ with $\log \lambda(\varphi) \leq R$. However, verifying this conjecture appears to require some new techniques and ideas beyond the reach of this paper.

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