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NUCLEARITY

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ABSTRACT. The notion of nuclear object in an autonomous category is studied. It is shown that the full subcategory determined by the nuclear objects is also autonomous.

The concept of a nuclear object arose out of an attempt on the part of the author to characterize finite-dimensional objects categorically. It soon appeared that the appropriate ideas could best be expressed in the context of symmetric monoidal closed categories, or "autonomous" categories in the sense of Linton. We originally called such objects "finite-dimensional", but have been persuaded that the present term is to be preferred. If one interprets the idea of "nuclear" as incorporating the concept of "finite-dimensional", then our definition is equivalent to, but simpler than, one which is apparently part of the folklore (see 1.5, below).

In this paper we define nuclear objects in such categories, study their elementary properties, and show that the full subcategory of an autonomous category determined by the nuclear objects is also autonomous. In addition, we briefly consider the form this concept takes in some of the categories considered in [1] by Banaschewski and Nelson.

1. **Nuclear Objects.** Our starting point is the problem of characterizing finite-dimensional objects in the category of vector spaces over a field K (that is, the finite-dimensional vector spaces). Let V denote a finite-dimensional vector space over K, and let $f: V \to V$ denote any linear operator on V. For any ordered basis $\mathbf{a} = (a_1, \ldots, a_n)$ of V let

$$f[\mathbf{a}] = \sum A_{ij} a_i \otimes a_j^* \in V \otimes_K V^*,$$

where V^* is the dual space of V, $A = (A_{ij})$ is the matrix representing f with respect to the basis $\mathbf{a}$, and $(a_1^*, \ldots, a_n^*)$ is the dual basis of $\mathbf{a}$. It is easily verified that the element of $V \otimes_K V^*$ determined by f and $\mathbf{a}$ above is independent of the choice of $\mathbf{a}$ and that the map $f \mapsto f[\mathbf{a}]$ is inverse to the "natural" map $\phi_V: V \otimes_K V^* \mapsto \mathrm{Hom}_K(V, V)$ given by $\phi_V(a \otimes \alpha)(x) = \alpha(x)a$, and so is an

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isomorphism. On the other hand, if V is not finite-dimensional, then ϕ_V cannot be an isomorphism. For, if f is in $\mathrm{im}(\phi_V)$ then the rank of f is finite. But then the identity is not in $\mathrm{im}(\phi_V)$. Hence, V is finite-dimensional if ϕ_V is an isomorphism.

Since the analogue of ϕ_V can be constructed in any autonomous category, it would be natural to say that an object is finite-dimensional if the corresponding morphism is an isomorphism. However, other examples, some of which will be considered below, show that this is not quite the appropriate term, and we have adopted the term "nuclear" to describe such objects.

Throughout, $\mathbf{E}$ will denote a fixed autonomous category. This is a category equipped with:

a "tensor product" $\otimes:\mathbf{E} \times \mathbf{E} \to \mathbf{E}$;
an "internal hom" $[\_, \_]:\mathbf{E}^{\mathrm{op}} \times \mathbf{E} \to \mathbf{E}$;
a "base object" K;
a "unit" $\mu_B^A:B \to [A, B \otimes A]$;
an "evaluation" $ev_B^A:[A, B] \otimes A \to B$;
an "identity" $\sigma_A:K \otimes A \approx A$;
"symmetry isomorphisms" $\tau_{A,B}:A \otimes B \approx B \otimes A$.

We suppress explicit mention of the associativity isomorphisms $(A \otimes B) \otimes C \approx A \otimes (B \otimes C)$. This will cause no difficulty. $1_A:A \to A$ will denote the identity morphism. It is of course also assumed that we have natural isomorphisms $[A, [B, C]] \approx [A \otimes B, C]$ for all objects A, B, C of $\mathbf{E}$.

The remaining data are defined by:

$$j_A = [1_A, \sigma_A] \cdot \mu_K^A:K \to [A, A];$$

$$i_A = [1_K, \sigma_A, \tau_K^A] \cdot \mu_A^k:A \approx [K, A];$$

$$\Gamma_{A,B,C} = [1_A, ev_C^B] \cdot [1_A, 1_{[B,C]} \otimes ev_B] \cdot \mu_X^A:[B, C] \otimes [A, B] \to [A, C],$$

where $X = [B, C] \otimes [A, B]$.

If A is any object of $\mathbf{E}$, we set $A^* = [A, K]$. As usual, there is a natural transformation $\theta_A:A \to A^{**}$ and a morphism $\phi_A:A \otimes A^* \to [A, A]$ whose internal adjoint is given by the morphism $A \otimes A^* \otimes A \to A \otimes K \approx A$.

1.1. DEFINITION. A is *nuclear* if ϕ_A is an isomorphism.

It is immediate that the base object K is nuclear. As usual, we say A is *reflexive* if θ_A is an isomorphism.

1.2. PROPOSITION. *If A is nuclear then it is reflexive.*

PROOF. The inverse of the morphism θ_A is given by the composition:

$$\begin{aligned} A^{**} &\approx K \otimes A^{**} \approx A^{**} \otimes K \to A^{**} \otimes [A, A] \to A^{**} \otimes A \otimes A^* \\ &\approx A^{**} \otimes A^* \otimes A \to K \otimes A \approx A \end{aligned}$$

as is easily seen using (internal) adjointness. □

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1.3. Corollary. *If* **E** *is cartesian closed (with* $\otimes$ = *product) and* A *is nuclear, then* $A \approx K$.

Proof. In this case, K is a terminal object, so $A^* \approx K$ for any object A. Hence, if A is nuclear, we have $A \approx A^{**} \approx K$. □

1.4. Proposition. *A retract of a nuclear object is nuclear.*

Proof. Let $f:A \to B$ and $g:B \to A$ be morphisms such that $gf = 1_A$, and let B be nuclear. Then the morphism

$$[A, A] \xrightarrow{[g, f]} [B, B] \to B \otimes B^* \xrightarrow{g \otimes f^*} A \otimes A^*$$

is easily seen to give the required inverse to the natural morphism $A \otimes A^* \to [A, A]$.

1.5. Remark. There is a definition of nuclearity (under the name of "finite-dimensionality") in the "folklore" as follows: A is nuclear if

(i) A is reflexive; and

(ii) given any object B, the canonical morphism $A \otimes B^* \to [B, A]$ is an isomorphism.

Clearly, the "folklore" definition implies ours. On the other hand, if A is nuclear in our sense then, as we have seen, it is reflexive and the morphism given as the composition:

$$[B, A] \approx K \otimes [B, A] \to [A, A] \otimes [B, A] \to A \otimes A^* \otimes [B, A] \to A \otimes B^*$$

is readily seen to be the inverse of the canonical morphism used in the folklore definition. Hence, these two definitions coincide.

2. **The Autonomous Category of Nuclear Objects.**

In this section we give the method of proof of the following theorem:

2.1. Theorem. *Let* **E** *be an autonomous category. Then the full subcategory of* **E** *determined by the nuclear objects of E is again an autonomous category.*

Proof. We will not actually go through the laborious diagram-chasing required to prove this. We will instead simply describe the required morphisms, leaving the verification that they actually work to the reader. For this purpose, recourse may be had to Szabo's Theorem (see [2], page 103).

As may be expected, the proof boils down to showing that if A and B are nuclear, then so are $A \otimes B$ and $[A, B]$, its being known that K is always nuclear.

Suppose that A and B are nuclear.

(i) *$A \otimes B$ is nuclear.* Set $X = A \otimes B$. Then the morphism given as the composite:

$$[X, X] \approx [X, X] \otimes K \otimes K \to [X, X] \otimes [A, A] \otimes [B, B]$$
$$\approx [X, X] \otimes A \otimes A^* \otimes B \otimes B^* \to [X, X] \otimes (A \otimes B) \otimes (A^* \otimes B^*)$$
$$\to [X, X] \otimes X \otimes X^* \to X \otimes X^*,$$

where the penultimate morphism is given by the adjoint of the following composite:

$$A^* \otimes B^* \otimes A \otimes B \approx A^* \otimes A \otimes B^* \otimes B \to K \otimes K \approx K,$$

may be shown to be the inverse of $\phi_X: X \otimes X^* \to [X, X]$.

(ii) $[A, B]$ *is nuclear*. Set $Y = [A, B]$. Then the morphism given by the composite:

$$[Y, Y] \approx [Y, Y] \otimes K \otimes K \to [Y, Y] \otimes [A, A] \otimes [B, B]$$
$$\approx [Y, Y] \otimes A \otimes A^* \otimes B \otimes B^*$$
$$\approx [Y, Y] \otimes B \otimes A^* \otimes A \otimes B^* \approx [Y, Y] \otimes Y \otimes Y^* \to Y \otimes Y^*,$$

where the penultimate morphism is derived from the morphism described above in (i) and the morphism given as the composite:

$$A \otimes B^* \to [\,[A, B], A \otimes B^* \otimes [A, B]\,] \to [\,[A, B], A \otimes A^*]\,] \to [A, B]^*$$

may be shown to be the inverse of $\phi_Y: Y \otimes Y^* \to [Y, Y]$.

These observations suffice to prove the theorem. □

2.2. Corollary. *If A is nuclear then so is A^*.* □

2.3. Proposition. *If A and B are nuclear then*

$$A \otimes B \approx [A, B^*]^*.$$

Proof. We always have $[A, B^*] = [A, [B, K]\,] \approx [A \otimes B, K] = (A \otimes B)^*$, so $[A, B^*]^* \approx (A \otimes B)^{**}$. Hence, if A and B are nuclear then

$$[A, B^*]^* \approx (A \otimes B)^{**} \approx A \otimes B. \qquad □$$

Suppose that A is a nuclear object in $\mathbf{E}$. Then we have a morphism

$$t_A: [A, A] \approx A \otimes A^* \approx A^* \otimes A \to K$$

which, in the category of vector spaces, is easily seen to give the trace. In general, the trace function is "natural". This means that if A and B are nuclear and $f: A \to B$ is any morphism, then the diagram

$$\begin{tikzcd}
{[B, A]} \arrow[r, "{[1, f]}"] \arrow[d, "{[f, 1]}"'] & {[B, B]} \arrow[d, "t_B"] \\
{[A, A]} \arrow[r, "t_A"] & K
\end{tikzcd}$$

commutes. (In the category of vector spaces this is merely the statement that trace(ST) = trace(TS).)

If $\mathbf{E}$ denotes an autonomous category in which *every* object is nuclear (such categories are called "compact" in [2]), then every object has a trace which is natural in the above sense. We refer to an autonomous category in which there is such a "trace" morphism for each object and which is natural as one which has a natural trace.

2.4. THEOREM. *Let* $\mathbf{E}$ *denote an autonomous category. Then the following are equivalent*:

(a) *Every object of* $\mathbf{E}$ *is nuclear*;

(b) *Every object of* $\mathbf{E}$ *is reflexive and* $\mathbf{E}$ *has a natural trace.*

PROOF. It remains only to see that (b) implies (a). The composite

$$[A, A] \to [[A, A], [A, A] \otimes [A, A]] \to [[A, A], [A, A]] \to [A, A]^*,$$

in which the penultimate morphism is derived from the internal composition and the last morphism is given by the trace, is easily seen to be the inverse of the canonical morphism $A \otimes A^* \to [A, A]$, where we have used reflexivity to replace $A \otimes A^*$ by $[A, A]^*$.

3. **B-N Categories.** In [1], Banaschewski and Nelson consider concrete categories $\mathbf{E}$ which are equipped with a "functional internal hom" $[\_, \_]$ and have a "dualizer" D. By "functional" they mean that any function $f:|A| \to E(B, C)$ for which there exists for each $b \in B$ an actual morphism $h_b:A \to C$ satisfying the obvious consistency condition is the underlying function of an actual morphism $A \to [B, C]$, where $|X|(|f|)$ denotes the underlying set (function) of the object X (morphism f), and, for any objects A, B there is a natural morphism $A \to [[A, B], B]$. A "dualizer" D is an object satisfying the condition that all the morphisms $A \to [[A, D], D]$ are isomorphisms.

We call a category equipped with this structure and a dualizer D a "B-N Category" and denote the object $[A, D]$ by $A^{\#}$.

It is implicit in [1] that if $\mathbf{E}$ is a B-N category with dualizer D then $\mathbf{E}$ is an autonomous category with tensor product defined by $A \otimes B = [A, B^{\#}]^{\#}$ and base object $K = D^{\#}$. Thus, we have $A^{\#} = [A, K] \approx (A \otimes D)^{\#}$ and the concept of "reflexive" becomes the statement that $(A \otimes D)^{\#} \otimes D^{\#} \approx A$. In the following, by "$D$ is self-dual" we mean that $D \approx [D, D]$.

At present, we do not know how to characterize the nuclear objects of a B-N category. We can, however, show:

3.1. PROPOSITION. *The dualizer D of a* B-N *category is nuclear if and only if it is self-dual.*

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PROOF.

$$D^{**} = [D^*, D^\#] = [\,[D, D^\#], D^\#].$$

Suppose D is nuclear. Then

$$D^\# = [D, D] \approx D \otimes D^* \approx D^* \otimes D = [D^*, D^\#]^\#.$$

Hence,

$$D \approx D^\# \approx [D^*, D^\#]^{\#\#} \approx [D^*, D^\#] = D^{**}.$$

The reverse implication is obvious. □

4. **Some Examples.** Most, but not all, of our examples are also examples of B-N categories and have been considered in [1].

Our first example, the category of vector spaces over a field, has already been discussed. Generalizing this basic example, we consider:

R-MODULES. Here, R is a commutative ring with unit. We show:

4.1. PROPOSITION. *Let R be a commutative ring and A an R-module. Then A is nuclear if and only if A is finitely-generated and projective.*

PROOF. Let A be nuclear. Then the canonical homomorphism $\phi_A : A \otimes_R \mathrm{Hom}_R(A, R) \to \mathrm{Hom}_R(A, A)$ has an inverse. Set $\phi_A^{-1}(1_A) = \sum a_k \otimes f_k$. Since for any $a \in A$ we have

$$a = (\phi(\phi^{-1}(1_A)))(a) = \sum f_k(a)a_k$$

we deduce that A is projective by Proposition 3.1, p. 132, of [3]. This same formula shows that A is finitely-generated.

Now suppose A is finitely generated, say by $\{a_1, \ldots, a_n\}$, and projective. Then we have split exact sequence

$$R^n \xrightarrow{f} A \to 0.$$

Let $g : A \to R^n$ denote the splitting map and let, for $k = 1, \ldots, n$, $\pi_k : R^n \to R$ denote that "kth projection". Then it is easily shown that $\phi_A^{-1}(h) = \sum h(a_k) \otimes (\pi_k g)$. □

POINTED SETS. This is the category S_* in which an object $\mathbf{X}$ is a pair consisting of a set $|\mathbf{X}|$ and a point $p(\mathbf{X}) \in |\mathbf{X}|$, while a morphism $f\colon \mathbf{X} \to \mathbf{Y}$ is a function $f : |\mathbf{X}| \to |\mathbf{Y}|$ such that $f(p(\mathbf{X})) = p(\mathbf{Y})$. The internal hom in this category is the set of all morphisms with "base point" the morphism which takes every point in the domain to the specified point in the codomain. The tensor product is the so-called "smash product" obtained from the cartesian product by identifying the "wedge" to a point, where, if $\mathbf{X}$ and $\mathbf{Y}$ are pointed

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sets, their wedge is the set $(|\mathbf{X}| \times \{p(\mathbf{Y})\}) \cup (\{p(\mathbf{X})\} \times |\mathbf{Y}|)$. The base object in this category is given by $|\mathbf{K}| = 2 = \{0, 1\}$, $p(\mathbf{K}) = 0$.

For any two objects $\mathbf{A}$ and $\mathbf{B}$ in this category, let $\pi:\mathbf{A} \times \mathbf{B} \to \mathbf{A} \otimes \mathbf{B}$ denote the canonical map. The isomorphism $\mathbf{K} \otimes \mathbf{A} \to \mathbf{A}$ is given by $\sigma(\pi(0, a)) = p(\mathbf{A})$, $\sigma(\pi(1, a)) = a$. The canonical morphism $\phi_{\mathbf{A}}:\mathbf{A} \otimes \mathbf{A}^* \to [\mathbf{A}, \mathbf{A}]$ is given by

$$(\phi_A(\pi(a, f)))(x) = \begin{cases} a & \text{if } f(x) = 1 \\ p(\mathbf{A}) & \text{if } f(x) = 0 \end{cases}$$

as is easily seen. If $\phi_{\mathbf{A}}$ has an inverse $\phi_{\mathbf{A}}^{-1}$ set $\phi_{\mathbf{A}}^{-1}(1_{\mathbf{A}}) = \pi(a_0, f)$. Then, for all $x \in |\mathbf{A}|$, we have

$$x = (\phi_{\mathbf{A}}(\pi(a_0, f)))(x) = \begin{cases} a_0 & \text{if } f(x) = 1 \\ p(\mathbf{A}) & \text{if } f(x) = 0. \end{cases}$$

We claim that the only objects in this category which are nuclear are those whose underlying sets have cardinality 1 or 2. For, if $\mathbf{A}$ is nuclear, $\phi_{\mathbf{A}}^{-1}(1_{\mathbf{A}})$ is given as above, and if $a \in |\mathbf{A}|$, then $a = a_0$ if $f(a) = 1$ while if $f(a) = 0$ then $a = p(\mathbf{A})$. Hence, $\text{card}(|\mathbf{A}|) \leqq 2$. On the other hand, if $\text{card}(|\mathbf{A}|) \leqq 2$ it is easy to see that $\mathbf{A}$ is nuclear.

BANACH SPACES. Let **Ban** denote the category of (real or complex) Banach spaces in which the morphisms are just bounded (not necessarily bounded by 1). This is an autonomous category as may be found in, for example, [5]. An object in this category is nuclear if and only if it is a nuclear as a vector space, as may easily be checked. Indeed, the only thing to show is that the inverse to the (vector space) isomorphism ϕ_V, for any nuclear Banach space V, is itself bounded. On the other hand, if we consider the category $\mathbf{Ban}_1$ of Banach spaces with morphisms bounded by 1, we find that if $\phi:A \otimes A^* \to [A, A]$, which is easily seen to be in $\mathbf{Ban}_1$, has an inverse, such inverse is a vector space inverse and is bounded but, if $\dim(A) > 1$, then the inverse cannot be in $\mathbf{Ban}_1$. Indeed, we claim that if ϕ has a (vector space) inverse then $\|\phi^{-1}\| \geqq \dim(A)$. For it is easily seen that if $(a_1, \ldots, a_n)$ is a basis of the (necessarily finite-dimensional) Banach space A then ϕ^{-1} must be given by

$$\phi^{-1}(f) = \sum f(a_i) \otimes a_i^*,$$

where $(a_1^*, \ldots, a_n^*)$ is the "dual" basis of A^*. In particular, $\phi^{-1}(I) = \sum a_i \otimes a_i^*$. Now,

$$\|a_i^*\| = \sup_{x \neq 0} \frac{|a_i^*(x)|}{\|x\|} \frac{|a_i^*(a_i)|}{\|a_i\|} = \frac{1}{\|a_i\|},$$

so that $\|a_i^*\| \, \|a_i\| \geqq 1$ for each i. Summing, we have $\sum \|a_i^*\| \, \|a_i\| \geqq n$. But we also have $\|\sum a_i \otimes a_i^*\| = \sum \|a_i^*\| \, \|a_i\|$. Since

$$\|\phi^{-1}\| \leqq \frac{\|\phi^{-1}(I)\|}{\|I\|},$$

the claimed inequality follows.

COMPLETE SEMILATTICES. This is the category **CSL** whose objects are pairs $(X, \sup_X)$ consisting of a set X and a function $\sup_X:P(X) \to X$, where $P(X)$ denotes the "power set" of X, satisfying:

(1) $\sup_X(\{x\}) = x$ for all $x \in X$; and

(2) $\sup_X(\cup_{i \in I} A_i) = \sup_X(\{\sup_X(A_i) \mid i \in I\})$

and whose morphisms $F:(X, \sup_X) \to (Y, \sup_Y)$ are functions $F:X \to Y$ satisfying $\sup_Y(\{F(a) \mid a \in A\}) = F(\sup_X(A))$.

The present author and D. A. Higgs have shown [4] that the nuclear complete semilattices are precisely the completely distributive semilattices.

5. **Appendix.** One of the properties of B-N categories which seems to have been overlooked in [1] and which might be useful in this connection is contained in the following:

5.1. THEOREM. *Let* **E** *be a* B-N *category in which the dualizer D is self-dual (that is, nuclear). Then, given any morphism $f:A \to [B, C]$ and any $b \in |B|$ there exists a morphism $h_b:A \to C$ such that $|h_b|(a) = |\,|f|(a)\,|(b)$ for all $a \in |A|$.*

PROOF. Since $D^{\#} \approx D$, we may identify these objects. Hence, we may identify $A^{\#} = [A, D]$ and $A^* = [A, D^{\#}]$ for any object A. In the following, we let $\sigma_A:A \to A^{**}$ denote the natural isomorphism.

By Proposition 5 of [1], there is an isomorphism $\theta:[A, [B, C]] \approx [[A, B^*]^*, C]$ and hence a bijection $|\theta|:\mathbf{E}(A, [B, C]) \approx \mathbf{E}([A, B^*]^*, C)$. Since, in general in a B-N category, we have $[X, Y] \approx [Y^*, X^*]$, we may regard θ as an isomorphism $\theta:[A, [B, C]] \approx [C^*, [B, A^*]]$, which induces a bijection $|\theta|:\mathbf{E}(A, [B, C]) \approx \mathbf{E}(C^*, [B, A^*])$. Thus, to be given a morphism $A \to [B, C]$ is equivalent to being given a morphism $g:C^* \to [B, A^*]$. Given such a g and a $b \in |B|$, we may define $m_b:|C^*| \to (A, D)$ by $m_b(\gamma) = |\,|g|(\gamma)\,|(b)$. By the functionality condition, there is a unique morphism $k_b:C^* \to A^*$ such that $|k_b| = m_b$. Finally, we define h_b by requiring that $\sigma_c \cdot h_b = (k_b)^* \cdot \sigma_B$. □

5.2. COROLLARY. *Every morphism $f:A \to B$ in a* B-N *category with a self-dual dualizer is given by a family of morphisms $h_\beta:A \to D$ (where $\beta \in |B^*|$) such that $|h_\beta|(a) = |\beta|(\,|f|(a)\,)$ for all $a \in |A|$.*

PROOF. Simply apply the above theorem, regarding f as a morphism $A \to B^{**}$. □

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