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Generalizations of Frobenius' Theorem on Manifolds and Subcartesian Spaces

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Abstract. Let $\mathcal{F}$ be a family of vector fields on a manifold or a subcartesian space spanning a distribution D. We prove that an orbit O of $\mathcal{F}$ is an integral manifold of D if D is involutive on O and it has constant rank on O. This result implies Frobenius' theorem, and its various generalizations, on manifolds as well as on subcartesian spaces.

1 Introduction

1.1 Distributions

A *generalized distribution* on a manifold M is a subset D of TM such that, for each $p \in M$, $D_p = D \cap T_pM$ is a subspace of T_pM. The dimension of D_p is called the *rank* of D at p. A generalized distribution is *smooth* if it is locally spanned by smooth vector fields. For the sake of brevity we use here the term *distribution* to mean a smooth generalized distribution. In differential geometry the term distribution is usually restricted to distributions of constant rank. In the definition adopted here, rank of a distribution need not be constant.

An *integral manifold* of a distribution D on M is an immersed submanifold N of M such that, for every $q \in N$, $T\iota_{NM}(T_qN) = D_{\iota(q)}$, where $\iota_{NM} \colon N \hookrightarrow M$ is the inclusion map. A distribution D on M is said to be *integrable* if, for every $p \in M$, there exists an integral manifold of D containing p. If D is integrable, then every integral manifold of D can be extended to a maximal integral manifold of D.

Let $\mathcal{D}$ be the family of all vector fields on M with values in D. We say that D is *involutive* if the family $\mathcal{D}$ is closed under the Lie bracket of vector fields. An integrable distribution is involutive. The classical theorem of Frobenius is usually reformulated in differential geometry as a criterion for integrability of distributions of constant rank, see [12].

Theorem 1 (Frobenius' Theorem) *A constant rank distribution D on a manifold M is integrable if it is involutive.*

A necessary and sufficient condition for integrability of a distribution with variable rank has been studied by Sussmann, who proved the following result [10].

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Theorem 2 (Sussmann's Theorem) *A distribution D on a manifold M is integrable if and only if it is preserved by local one-parameter groups of local diffeomorphisms of M generated by vector fields on M with values in D.*

Integrability of a distribution D on manifold M implies that D is involutive and is preserved by local one-parameter groups of local diffeomorphisms of M generated by vector fields on M with values in D. A more general sufficient condition for integrability of a distribution that contains sufficient conditions of both Frobenius' theorem and Sussmann's theorem can be found in [2, p. 28, Theorem 3.25, part 2].

Theorem 3 (Kolář, Michor and Slovak) *A distribution D on a manifold M is integrable if it is involutive and its rank is constant on integral curves of vector fields on M with values in D.*

Frobenius' theorem is clearly a special case of the theorem of Kolář, Michor and Slovak since the assumption that rank D is constant implies that it is constant on integral curves of vector fields on M with values in D. Similarly, if D is preserved by local one-parameter groups of local diffeomorphisms of M generated by vector fields on M with values in D, then D is involutive and rank D is constant on integral curves of vector fields on M with values in D. Thus, Sussmann's theorem is also a special case of this theorem.

A distribution D on M gives rise to an equivalence relation $\sim$ on M defined by $p \sim q$ if p can be joined to q by a piece-wise smooth curve γ whose tangent vectors lie in D. Equivalence classes of this relation are *orbits* of the family $\mathcal{D}$ of vector fields on M with values in D. For $p \in M$, the orbit through p of $\mathcal{D}$ can be expressed as follows:

(1)
$$O = \{\exp(t_n X_n) \circ \cdots \circ \exp(t_1 X_1)(p) \mid n \in \mathbb{N}, (t_1, \dots, t_n) \in \mathbb{R}^n, X_1, \dots, X_n \in \mathcal{D}\},$$

where $\exp tX$ is the local one-parameter group of local diffeomorphisms of M generated by the vector field X. If $q \in O$, then in equation (1) we can replace p by q. If the distribution D is integrable, then integral manifolds of D are orbits of $\mathcal{D}$.

In equation (1), we get the same orbits if we replace the family $\mathcal{D}$ of all vector fields on M with values in D by any family $\mathcal{F}$ of vector fields on M that span the distribution D, [10]. In other words

(2)
$$O = \{\exp(t_n X_n) \circ \cdots \circ \exp(t_1 X_1)(p) \mid n \in \mathbb{N}, (t_1, \dots, t_n) \in \mathbb{R}^n, X_1, \dots, X_n \in \mathcal{F}\}.$$

Thus, we can rewrite the theorem of Kolář, Michor and Slovak in an alternative, equivalent form.

Theorem 4 (Kolář, Michor and Slovak, alternative version) *Let D be a distribution on M and $\mathcal{F}$ a family of vector fields on M that spans D. The distribution D is integrable if and only if it is involutive and has constant rank on every orbit of $\mathcal{F}$.*

All the generalizations of Frobenius' theorem discussed above deal with integrability of a distribution as a whole, and not with individual integral manifolds. In

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particular, the theorem of Kolář, Michor and Slovak implies that if a family $\mathcal{F}$ of vector fields on M spans an involutive distribution D such that the rank of D is constant on every orbit of $\mathcal{F}$, then all orbits of $\mathcal{F}$ are maximal integral manifolds of D. In particular, all orbits of $\mathcal{F}$ are immersed submanifolds of M.

Here, we weaken assumptions of the theorem of Kolář, Michor and Slovak and investigate necessary and sufficient conditions for a single orbit O of a family $\mathcal{F}$ of vector fields on M to be an integral manifold of the distribution D spanned by $\mathcal{F}$. In this way, we obtain criteria for deciding when an orbit of a family of vector fields has a structure of an immersed submanifold of M.

Definition A distribution D spanned by a family $\mathcal{F}$ of vector fields on M is said to be *involutive on an orbit* O of $\mathcal{F}$ if, for every $X, Y \in \mathcal{F}$, the Lie bracket $[X, Y](p) \in D_p$ for every $p \in O$.

We obtain the following result.

Theorem 5 *Let $\mathcal{F}$ a family of vector fields on M, and D the distribution on M spanned by $\mathcal{F}$. An orbit O of $\mathcal{F}$ is an integral manifold of D if and only if the rank of D is constant on O and D is involutive on O.*

Clearly, Theorem 5 implies the theorem of Kolář, Michor and Slovak Hence, it implies Frobenius' theorem and Sussmann's theorem.

The statement of Theorem 5 has not been given in the literature, even though its proof could have been distilled from the proof of the theorem of Kolář, Michor and Slovak [2]. Here, we give a proof Theorem 5 that is an adaptation of the proof of Frobenius' theorem given in [12]. It has an advantage that it can be easily generalized to subcartesian spaces.

1.2 Differential Spaces

A differential space is a topological space S endowed with a family $C^\infty(S)$ of functions on S satisfying the following conditions:

(i) The family $\{f^{-1}((a, b)) \mid f \in C^\infty(S), a, b \in \mathbb{R}\}$ is a sub-basis for the topology of S.

(ii) If $f_1, \dots, f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \dots, f_n) \in C^\infty(S)$.

(iii) If $f : S \to \mathbb{R}$ is such that for every $x \in S$ there exist an open neighbourhood U_x of x and a function $f_x \in C^\infty(S)$ satisfying $f_{x|U_x} = f_{|U_x}$, then $f \in C^\infty(S)$.

Differential spaces were introduced by Sikorski [4], see also [5, 6].

Let R and S be differential spaces with differential structures $C^\infty(R)$ and $C^\infty(S)$, respectively. A map $\rho : R \to S$ is said to be smooth if $\rho^* f \in C^\infty(R)$ for all $f \in C^\infty(S)$. A smooth map between differential spaces is a diffeomorphism if it is invertible and its inverse is smooth.

Clearly, smooth manifolds are differential spaces. However, the category of differential spaces is much larger than the category of manifolds.

If R is a differential space with differential structure $C^\infty(R)$ and S is a subset of R, then we can define a differential structure $C^\infty(S)$ on S as follows. A function

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$f\colon S \to \mathbb{R}$ is in $C^\infty(S)$ if and only if for every $x \in S$, there is an open neighborhood U of x in R and a function $f_x \in C^\infty(R)$ such that $f_{|S\cap U} = f_{x|(S\cap U)}$. The differential structure $C^\infty(S)$ described above is the smallest differential structure on S such that the inclusion map $\iota\colon S \to R$ is smooth. We shall refer to S with the differential structure $C^\infty(S)$ described above as a differential subspace of R. It should be noted that the topology of a differential subspace S of R is the same as the topological subspace induced by the inclusion map $\iota\colon S \to R$. If S is a closed subset of R, then the differential structure $C^\infty(S)$ described above consists of restrictions to S of functions in $C^\infty(R)$.

A Hausdorff differential space S such that every point of S has a neighbourhood diffeomorphic to a differential subspace of $\mathbb{R}^n$ is called a subcartesian space. The original definition of subcartesian space was given by Aronszajn in terms of a singular atlas [1]. The characterization of subcartesian spaces used here can be found in [9, 11]. A review of properties of subcartesian spaces needed here can be found in [8].

Let S be a differential space. A *derivation* of $C^\infty(S)$ at $p \in S$ is a linear map $u\colon C^\infty(S) \to \mathbb{R}\colon f \mapsto u\cdot f$ satisfying Leibniz' rule $u\cdot(fh) = (u\cdot f)h(p) + (u\cdot h)f(p)$ for all $f, h \in C^\infty(S)$. The space T^Z_pS of derivations of $C^\infty(S)$ is called the *(Zariski) tangent space* of S at p. The space $T^ZS = \bigcup_{p\in S} T^Z_pS$ has the structure of a differential space and the projection map $\tau\colon T^ZS \to S$ is smooth. If S is subcartesian, so is T^ZS.

A global derivation on S is a linear map $X\colon C^\infty(S) \to C^\infty(S)\colon f \mapsto X\cdot f$ that satisfies Leibniz' rule $X\cdot(fh) = (X\cdot f)h + (X\cdot h)f$ for all $f, h \in C^\infty(S)$. For each $p \in S$, the value of a global derivation X at p is $X(p) \in T^Z_pS$ defined by $X(p)\cdot f = (X\cdot f)(p)$ for every $f \in C^\infty(S)$. Thus, evaluation at p gives a map from the space $\mathrm{Der}(C^\infty(S))$ to T^Z_pS. It should be noted that this map need not be an epimorphism. Hence, the different notions of tangent vectors, which are equivalent for manifolds, need not be equivalent for differential spaces. For this reason, we refer to T^ZS as the *Zariski tangent bundle space* of S.

In [7], we proved existence and uniqueness of maximal integral curves of derivations on subcartesian spaces. We say that a global derivation X of $C^\infty(S)$ is a vector field on S if translations along integral curves of X are local diffeomorphisms $\exp tX$ of S.

As in the case of a smooth manifold, we define a *distribution on a differential space* S to be a linear subset D of T^ZS that is locally spanned by smooth vector fields. For each $p \in S$, $D_p = D \cap T^Z_pS$ is a subspace of T^Z_pS. The dimension of D_p is the *rank* of D at p.

In [8], we generalized Sussmann's theorem to subcartesian spaces. The main result of this paper is a generalization of Theorem 5 to subcartesian spaces.

Theorem 6 *Let S be a subcartesian space, $\mathcal{F}$ a family of vector fields on S, and D the distribution on S spanned by $\mathcal{F}$. An orbit O of $\mathcal{F}$ is an integral manifold of D if and only if the rank of D is constant on O and D is involutive on O.*

Since, by definition, a distribution on a subcartesian space is spanned by vector fields, and vector fields on a subcartesian space M generate local one-parameter groups of local diffeomorphisms of M, it is easy to see that Theorem 6 implies Theorem 5, the theorem of Kolář, Michor and Slovak, Sussmann's theorem, and Frobenius' theorem for subcartesian spaces. In other words, Frobenius' theorem, Sussmann's theorem, the theorem of Kolář, Michor and Slovak, and Theorem 5 remain valid if we replace *manifold M* by *subcartesian space S*.

2 Proof of Theorem 5

Let O be an orbit of a family $\mathcal{F}$ of vector fields on a manifold M which spans a distribution D such that rank $D_{|O} = m$, where $D_{|O}$ denotes the restriction of D to points of O. In addition we assume that distribution D is involutive on O.

2.1 Covering of the Orbit by Manifolds

For every $q \in O$, there exists $\xi = (X_1, \dots, X_m) \in \mathcal{F}^m$ such that the vector fields $X_1, \dots, X_m$ are independent in an open neighbourhood V of q in M and span D restricted to $V \cap O$.

Let $\rho_{\xi,q}$ be a map from a neighbourhood of $0 \in \mathbb{R}^m$ to M given by

$$\rho_{\xi,q}(t_1, \dots, t_m) = \exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q). \tag{3}$$

For each $i = 1, \dots, m$, and $u \in \mathbb{R}$,

$$u \frac{d}{dt} \exp(tX_i)(q) = uX_i(\exp(tX_i)(q).$$

Hence, for each $(u_1, \dots, u_m) \in \mathbb{R}^m$,

$$\begin{aligned} T\rho_{\xi,q|(t_1,\dots,t_m)}&(u_1, \dots, u_m) \\ &= u_1 \frac{d}{dt_1}(\exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1))(q) + \cdots \\ &\quad + u_m \frac{d}{dt_m}(\exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1))(q) \\ &= u_1 T\exp(t_m X_m) \circ \cdots \circ T\exp(t_2 X_2)(X_1(\exp(t_1 X_1)(q))) + \cdots \\ &\quad + u_m X_m(\exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q)) \end{aligned} \tag{4}$$

In particular,

$$T\rho_{\xi,q|(0)}(u_1, \dots, u_m) = u_1 X_1(q) + \cdots + u_m X_m(q). \tag{5}$$

Since the vectors $X_1(q), \dots, X_m(q)$ are linearly independent, it follows that

$$T\rho_{\xi,q|(0)} \colon \mathbb{R}^m \to T_q M$$

is one-to-one. Hence, there exists an open connected neighbourhood $W_{\xi,q}$ of 0 in $\mathbb{R}^m$ such that the restriction of $\rho_{\xi,q}$ to $W_{\xi,q}$ is a one-to one immersion of $W_{\xi,q}$ into M. We redefine $\rho_{\xi,q}$ so that it has domain $W_{\xi,q} \subseteq \mathbb{R}^m$. Thus,

$$\rho_{\xi,q} \colon W_{\xi,q} \to M \colon (t_1, \dots, t_m) \mapsto \exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q) \tag{6}$$

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is a one-to-one immersion. Therefore, the image

$$U_{\xi,q} = \rho_{\xi,q}(W_{\xi,q}) \tag{7}$$

is an immersed submanifold of M, contained in O. Equation (1) implies that

$$O = \bigcup_{\xi \in \mathcal{F}^m} \bigcup_{q \in O} \rho_{\xi,q}(W_{\xi,q}). \tag{8}$$

2.2 Topology of the orbit

Let $\iota\colon O \to M$ denote the inclusion map, and let the map $\sigma_{\xi,q}\colon W_{\xi,q} \to O$ be defined by $\rho_{\xi,q} = \iota \circ \sigma_{\xi,q}$. We topologize O with the strongest topology $\mathcal{T}$ which makes all maps $\sigma_{\xi,q}$ continuous. In this topology each $\sigma_{\xi,q}$ is an open map. In particular, each $\rho_{\xi,q}(W_{\xi,q})$ is open in O. The topology $\mathcal{T}$ has a basis of open sets consisting of sets $\rho_{\xi,q}(W_{\xi,q})$ and all their open subsets.

Since each $\rho_{\xi,q}\colon\ W_{\xi,q} \to M$ is continuous, it follows that, for every V open in M, $\rho^{-1}_{\xi,q}(V)$ is open in $W_{\xi,q}$. Therefore, $\sigma^{-1}_{\xi,q}(\iota^{-1}(V)) = \rho^{-1}_{\xi,q}(V)$ is open in $W_{\xi,q}$ which implies that $\iota^{-1}(V)$ is open in O. Hence, the inclusion map $\iota\colon O \to M$ is continuous. Note that the topology $\mathcal{T}$ on O may be finer than the subspace topology induced by $\iota\colon O \to M$.

2.3 Local Integral Manifolds

We show here that every point $q \in O$ has a connected open neighbourhood in O which is an integral manifold of D. The argument follows the proof of Frobenius' theorem given in [12]. It is based on induction on $m = \operatorname{rank} D_{|O}$, where $D_{|O}$ is the restriction of D to O.

If $\operatorname{rank} D_{|O} = 1$, then every point $q \in O$ has a neighbourhood U in O that is the range of an integral curve of a vector field X in $\mathcal{D}$ which spans $D_{|U}$, and $T_pU = D_p$ for every $p \in U$. Hence, U is an integral manifold of D through q. This holds for every manifold M, every family $\mathcal{F}$ of vector fields on M, and every orbit O of $\mathcal{F}$ such that the rank of the distribution D spanned by $\mathcal{F}$ is equal to 1 on the orbit O. Since $\operatorname{rank} D_{|O} = 1$, it follows that D is involutive on O.

Suppose now that for every manifold M, every family $\mathcal{F}$ of vector fields on M, and every orbit O of $\mathcal{F}$ such that the rank of the distribution D spanned by $\mathcal{F}$ is $m-1$ on the orbit O and D is involutive on O, each point $q \in O$ has an open neighbourhood in O that is an integral manifold of D. We prove that the same property holds in the case when $\operatorname{rank} D_{|O} = m$ and D is involutive on O.

We now use the notation established in preceding sections. As before, O is an orbit of a family $\mathcal{F}$ of vector fields on a given manifold M which spans a distribution D such that $\operatorname{rank} D_{|O} = m$, and D is involutive on O. Each point $q \in O$ has a neighbourhood $U_{\xi,q}$ which is open in O and has the structure of an immersed submanifold of M. We show here that there is a connected open neighbourhood of q in O contained in $U_{\xi,q}$ that is an integral manifold of D.

Without loss of generality, we may assume there are local coordinates $(y_1, \ldots, y_n)$ on M, with domain V containing q, such that $y_i(q) = 0$ for $i = 1, \ldots, n = \dim M$,

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the connected component W of q in $U_{\xi,q} \cap V$ is given by the slice $y_j = 0$, for $j = m+1, \dots, n$, and

$$X_m = \frac{\partial}{\partial y_m} \text{ on } V. \tag{9}$$

On V let

$$Y_i = X_i - (X_i \cdot y_m)X_m, \text{ for } i = 1, 2, \dots, m-1, \qquad Y_m = X_m. \tag{10}$$

Then, $Y_1, \dots, Y_m$ are smooth vector fields on V and $(Y_1(p), \dots, Y_m(p))$ span D_p for all $p \in U_{\xi,q} \cap V$. Let S be the slice $y_m = 0$, and let

$$Z_i = Y_{i|S} \text{ for } i = 1, 2, \dots, m-1. \tag{11}$$

Since equations (9) and (10) imply that

$$Y_i \cdot y_m = 0 \text{ for } i = 1, 2, \dots, m-1, \tag{12}$$

it follows that Z_i are vector fields on S. In other words, $Z_i(p) \in T_pS$ for all $p \in S$, and $i = 1, \dots, m-1$.

Consider a distribution D' on S spanned by vector fields Z_i. Let $\mathcal{D}'$ be the family of all vector fields on S with values in D. Let $\zeta = (Z_1, \dots, Z_{m-1})$ and $U_{\zeta,q}$ be a neighbourhood of q in the orbit O' of $\mathcal{D}'$ constructed as in Section 2.3. Taking into account equations (7) and (3), we can make the identification

$$U_{\zeta,q} = U_{\xi,q} \cap S. \tag{13}$$

On $U_{\zeta,q}$, the rank of D' is constant and equal to $m-1$. Moreover, for every $i, j = 1, \dots, m-1$ and $p \in U_{O'}$, we have $[Z_i, Z_j](p) \in T_qS \cap D_p \subseteq D'_p$. Hence, D' is involutive on O', and we may invoke the inductive hypothesis. By shrinking $U_{\zeta,q}$ if necessary, we may assume that $U_{\zeta,q}$ is an integral manifold of D' passing through q.

In terms of the coordinates $(y_1, \dots, y_n)$ on M with domain V, introduced above, equation (13) implies that $U_{\zeta,q} \cap V$ is given by the slice $y_i = 0$, for $i = m, \dots, n = \dim M$. Moreover, the induction hypothesis implies that

$$(Z_i \cdot y_k)_{|(y_m, y_{m+1}, \dots, y_n) = 0} = 0 \text{ for } i = 1, \dots, m-1 \text{ and } k = m+1, \dots, n. \tag{14}$$

Equations (11) and (14) imply that

$$(Y_i \cdot y_k)_{|(y_m, y_{m+1}, \dots, y_n) = 0} = 0 \text{ for } i = 1, \dots, m-1 \text{ and } k = m+1, \dots n. \tag{15}$$

We are going to prove that

$$(Y_i \cdot y_k)_{|(y_{m+1}, \dots, y_n) = 0} = 0 \text{ for } i = 1, \dots, m \text{ and } k = m+1, \dots n. \tag{16}$$

Equation (16) implies that the vector fields $Y_1, \dots, Y_m$ are tangent to the slice

$$S_{\xi,q} = \{p \in V \mid y_k = 0 \text{ for } k = m+1, \dots, n\}, \tag{17}$$

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which is equivalent to the statement that $S_{\xi,q}$ is an integral manifold of D.

Equation (9) implies that

$$(Y_m \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} = 0 \text{ for } k = m+1,\dots n. \tag{18}$$

For $i = 1,\dots,m-1$ and $j,k = m+1,\dots,n$, we differentiate the left-hand side of equation (16) with respect to y_m and obtain

$$\begin{aligned}\frac{\partial}{\partial y_m}(Y_i \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} &= Y_m \cdot (Y_i \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} \\ &= Y_i \cdot (Y_m \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} + ([Y_m, Y_i] \cdot y_k)_{|(y_{m+1},\dots,y_n)=0}.\end{aligned}$$

The first term on the right-hand side vanishes on account of equation (18). Moreover, the assumption that D is involutive on O implies that there exist smooth functions $c_{mil}(y_1,\dots,y_m)$ such that

$$([Y_m, Y_i] \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} = \sum_{l=1}^{m} c_{mil}(y_1,\dots,y_m)(Y_l \cdot y_k)_{|(y_{m+1},\dots,y_n)=0}. \tag{19}$$

Hence, equation (19), for $i = 1,\dots,m-1$ and $k = m+1,\dots,n$, reads

$$\frac{\partial}{\partial y_m}(Y_i \cdot y_k)_{|(y_{m+1},\dots,y_n)=0} = \sum_{l=1}^{m-1} c_{mil}(y_1,\dots,y_m)(Y_l \cdot y_k)_{|(y_{m+1},\dots,y_n)=0}. \tag{20}$$

Fix a slice of $W_{\xi,q}$ of the form $y_1 = \text{const},\dots,y_{m-1} = \text{const}$, $y_{m+1} = 0,\dots,y_n = 0$. On such a slice, for every $k = m+1,\dots,n$, $(Y_i \cdot y_k)$ are functions of y_m alone, and equation (20) becomes a system of $m-1$ homogeneous linear differential equations for $m-1$ functions $(Y_i \cdot y_k)(y_m)$, where $i = 1,\dots,m-1$. Such a system has a unique solution with given initial data at $y_m = 0$, which vanish identically by equation (15). Hence, $(Y_i \cdot y_k)(y_m) = 0$ for each i. This implies that $(Y_i \cdot y_k)\ |_{(y_{m+1},\dots,y_n)=0} = 0$ for $i = 1,\dots,m-1$ and all $y_1,\dots,y_m$. This result, together with equation (20) implies that equation (16) holds. Hence, the slice $S_{\xi,q} \subseteq U_{\xi,q}$ given by equation (17) is an integral manifold of D. Moreover, $S_{\xi,q}$ is an open connected neighbourhood of q in the orbit O.

2.4 Transition Functions

For each $q \in O$ and $\xi = (X_1,\dots,X_m) \in \mathfrak{F}^m$, we have an open neighbourhood $S_{\xi,q}$ of q in O that is an integral manifold of D. Moreover, $S_{\xi,q} = \rho_{\xi,q}(W'_{\xi,q})$, where

$$\rho_{\xi,q}(t_1,\dots,t_m) = \exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q),$$

see equation (6). Moreover, $W'_{\xi,q} = \rho_{\xi,q}^{-1}(S_{\xi,q})$ is an open neighbourhood of $0 \in \mathbb{R}^m$.

Suppose that $S_{\xi,q}$ and $S_{\zeta,p}$ are two such integral manifolds of D such that $S_{\xi,q} \cap S_{\zeta,p} \neq 0$. Since $S_{\xi,q}$ and $S_{\zeta,p}$ are open in O, the intersection $S_{\xi,q} \cap S_{\zeta,p}$ is open. Let

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$W_1 = \rho_{\xi,q}^{-1}(S_{\xi,q} \cap S_{\zeta,p})$ and $W_2 = \rho_{\zeta,p}^{-1}(S_{\xi,q} \cap S_{\zeta,p})$. Since $\rho_{\zeta,p}$ is invertible, there exists a function $\tau_{12} \colon W_1 \to W_2$ such that

$$\tau_{12}(r) = \rho_{\zeta,p}^{-1} \circ \rho_{\xi,q}(r), \tag{21}$$

for all $r \in W_1$. We want to show that τ_{12} is smooth. Let $r = \rho_{\xi,q}(t_1, \dots, t_m)$ and $u = (u_1, \dots, u_m)$ in $\mathbb{R}^m$. Since $S_{\xi,q}$ is an integral manifold of D, the vector $T\rho_{\xi,q|(t_1,\dots,t_m)}(u_1, \dots, u_m)$ is in D and there exist real numbers $d_{ij}(p)$ such that

$$T\rho_{\xi,q|(t_1,\dots,t_m)}(u_1, \dots, u_m) = \sum_{i,j} u_i d_{ij}(r) X_j(r), \tag{22}$$

where $(X_1, \dots, X_n) = \xi$. The matrix $(d_{ij}(r))$ is invertible and it depends smoothly on r. Similarly, $r = \rho_{\zeta,p}(s_1, \dots, s_m)$ for a unique $(s_1, \dots, s_m) \in W_2$, and there is a unique invertible matrix $(e_{ij}(r))$, which depends smoothly on r, such that

$$T\rho_{\zeta,p|(s_1,\dots,s_m)}(v_1, \dots, v_m) = \sum_{i,j} v_i e_{ij}(r) Z_j(r), \tag{23}$$

where $(Z_1, \dots, Z_m) = \zeta$. Expressing ζ in terms of ξ, we get

$$Z_j(r) = \sum_k f_{jk}(r) X_k(r), \tag{24}$$

where the matrix (f_{jk}) is invertible and depends smoothly on r. If

$$T\rho_{\zeta,p|(s_1,\dots,s_m)}(v_1, \dots, v_m) = T\rho_{\xi,q|(t_1,\dots,t_m)}(u_1, \dots, u_m),$$

then linear independence of vectors $X_1, \dots, X_m$ together with equations (22), (23) and (24) give

$$\sum_{i,j} v_i e_{ik}(r) f_{kj}(r) = \sum_i u_i d_{ij}(r), \tag{25}$$

which is a system of m linear equations in m unknowns $(v_1, \dots, v_m)$. Since the matrices $(e_{ik}(r))$ and $(f_{kj}(r))$ are invertible, their product $\sum_k (e_{ik}(r) f_{kj}(r))$ is invertible, with inverse $(g_{jk}(r))$ which depends smoothly on r. Hence there exists a unique solution

$$v_k = \sum_{i,j} u_i d_{ij}(r) g_{jk}(r) \tag{26}$$

of equation (25). Equation (26) gives an explicit expression for the derivative of $\tau_{12} \colon W_1 \to W_2$, (21), at $(t_1, \dots, t_m)$. Since $(d_{ij}(r) g_{jk}(r))$ depends smoothly on r, and $r = \rho_{\xi,q}(t_1, \dots, t_m)$ depends smoothly on $(t_1, \dots, t_m)$, it follows that τ_{12} is a smooth map.

Summarizing, we have shown that as q varies over O, and ξ varies over elements of $\mathcal{F}^k$ which locally span D, the maps $\rho_{\xi,q|S_{\xi,q}}^{-1} \colon S_{\xi,q} \to \mathbb{R}^m$ are compatible charts of a smooth structure on O. The orbit O with the maximal atlas generated by these charts is a maximal (connected) integral manifold of D. This completes a proof of Theorem 5.

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3 Proof of Theorem 6

In this section S is a subcartesian space, $\mathcal{F}$ a family of vector fields on S which spans a distribution D such that $\operatorname{rank} D_{|O} = m$, where $D_{|O}$ denotes the restriction of D to points of O. In addition, we assume that distribution D is involutive on O.

In the first section of the proof of Theorem 5 for manifolds, the manifold structure of M was used only to show that a smooth map from a neighbourhood of zero in $\mathbb{R}^m$ to M with one-to-one derivative at zero is locally a smooth immersion. An extension of this result to subcartesian spaces was proved in [3]. However, for the sake of completeness, we give here an independent proof of the covering of O by smooth manifolds.

Lemma 7 *Let S be a subcartesian space, $\xi = (X_1, \dots, X_m)$ vector fields on S, and q a point in S such that $X_1(q), \dots, X_m(q)$ are linearly independent. Let $\rho_{\xi,q}$ be a smooth map from a neighbourhood of $0 \in \mathbb{R}^m$ to S given by*

$$\rho_{\xi,q}(t_1, \dots, t_m) = \exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q). \tag{27}$$

Then there exists an open neighbourhood W of $0 \in \mathbb{R}^m$ such that $\rho_{\xi,q}(W)$ is a smooth manifold, the restriction $\rho_{\xi,q|W}$ of $\rho_{\xi,q}$ to W is a diffeomorphism onto the image $\rho_{\xi,q}(W)$, and the inclusion map $\rho_{\xi,q}(W) \hookrightarrow S$ is smooth.

Proof There exists a neighbourhood V_0 of $q = \rho_{\xi,q}(0)$ in S and a smooth map $\phi_0\colon V_0 \to \mathbb{R}^n$ that induces a diffeomorphism of V onto its image $\phi_0(V_0)$. By [7, Proposition 2], for each $i = 1, \dots, m$, there exists a neighbourhood V_i of q in S contained in V_0, a neighbourhood U_i of $\phi_0(q)$ in $\mathbb{R}^n$ and a vector field $\tilde{X}_i$ on U_i such that $\phi_0(V_i) = U_i \cap \phi_0(V_0)$ and

$$T\phi_0(X_i(p)) = \tilde{X}_i(\phi_0(p)). \tag{28}$$

for all $p \in V_i$.

Take $V = \bigcap_{i=1}^m V_i$ and $U = \bigcap_{i=1}^m U_i$. For $i = 1, \dots, m$, let Y_i be the restriction of $\tilde{X}_i$ to U, and let $\phi = \phi_{0|V}$ be the restriction of ϕ_0 to V. It is a diffeomorphism of V onto its image $\phi(V)$ contained in U. Equation (28) implies that for every $i = 1, \dots, m$,

$$T\phi \circ X_i = Y_i \circ \phi. \tag{29}$$

For every $i = 1, \dots, m$ and $(t_1, \dots, t_m)$ in the neighbourhood of $0 \in \mathbb{R}^m$, we have

$$\phi \circ \exp t_i X_i = (\exp t_i Y_i) \circ \phi. \tag{30}$$

Consider a map $\sigma = \phi \circ \rho_{\xi,q}$ from a neighbourhood of $0 \in \mathbb{R}^m$ to $\mathbb{R}^n$. Equation (30) implies

$$\begin{aligned} \sigma &= \phi \circ \rho_{\xi,q} \\ &= \phi \circ \exp(t_m X_m) \circ \cdots \circ \exp(t_1 X_1)(q) \\ &= \exp(t_m Y_m) \circ \cdots \circ \exp(t_1 X_1)(\phi(q)). \end{aligned} \tag{31}$$

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Hence, taking into account equation (4), we get

$$\begin{aligned} T\sigma_{|(t_1,\dots,t_m)}(u_1,\dots,u_m) &= u_1 T\exp(t_mY_m)\circ\cdots\circ T\exp(t_2Y_2)(Y_1(\exp(t_1Y_1)(q))) + \cdots \\ &\quad + u_mY_m(\exp(t_mY_m)\circ\cdots\circ\exp(t_1Y_1)(q)) \end{aligned}$$

and

$$\begin{aligned} T\sigma_{|(0)}(u_1,\dots,u_m) &= u_1Y_1(\phi(q)) + \cdots + u_mY_m(\phi(q)) \\ &= u_1T\phi(X_1(q)) + \cdots + u_mT\phi(X_m(q)). \end{aligned}$$

Since the vectors $X_1(q),\dots,X_m(q)$ are linearly independent, the vectors

$$T\phi(X_1(q)),\dots,T\phi(X_m(q))$$

are also linearly independent. Hence, by the inverse function theorem, there exists a neighbourhood W of 0 in $\mathbb{R}^m$ such that $\sigma(W)$ is an immersed submanifold of $\mathbb{R}^n$ and the restriction $\sigma_{|W}$ of σ to W is a diffeomorphism of W onto $\sigma(W)$.

Equation (31) implies that

$$\rho_{\xi,q}(W) = \phi^{-1}(\sigma(W)). \tag{32}$$

Since $\sigma(W)$ is a submanifold of $\mathbb{R}^n$ and ϕ is a diffeomorphism, it follows that $\rho_{\xi,q}(W)$ is a manifold. Moreover, $\rho_{\xi,q|W} = \phi^{-1}\circ\sigma$ is a diffeomorphism of W onto $\rho_{\xi,q}(W)$.

It remains to prove that the inclusion map $\rho_{\xi,q}(W)\hookrightarrow S$ is smooth. This will be accomplished if we show that for every $f\in C^\infty(S)$, the restriction $f_{|\rho_{\xi,q}(W)}$ of f to $\rho_{\xi,q}(W)$ is a smooth function on $\rho_{\xi,q}(W)$. Since $f\in C^\infty(S)$, it follows that for every $q\in\rho_{\xi,q}(W)$ there are a neighbourhood V_q of q in S and a function $F\in C^\infty(\mathbb{R}^n)$ such that $f_{|V_q} = F\circ\phi_{|V_q}$. Restricting the above equality to $\rho_{\xi,q}(W)$, we get

$$f_{|V_q\cap\rho_{\xi,q}(W)} = (F\circ\phi)_{|\rho_{\xi,q}(W)\cap V_q} = (F_{|\sigma(W)}\circ\phi_{|\rho_{\xi,q}(W)})_{|\rho_{\xi,q}(W)\cap V_q}.$$

But, $\phi_{|\rho_{\xi,q}(W)}\colon \rho_{\xi,q}(W)\to\sigma(W)\subseteq\mathbb{R}^n$ is a diffeomorphism, and $F\in C^\infty(\mathbb{R}^n)$ implies that $F_{|\sigma(W)}$ is smooth. Hence, $F_{|\sigma(W)}\circ\phi_{|\rho_{\xi,q}(W)}\in C^\infty(\rho_{\xi,q}(W))$, and $f_{|\rho_{\xi,q}(W)}$ coincides on $\rho_{\xi,q}(W)\cap V_q$ with a smooth function $F_{|\sigma(W)}\circ\phi_{|\rho_{\xi,q}(W)}$. This holds for every $q\in\rho_{\xi,q}(W)$. Hence, $f_{|\rho_{\xi,q}(W)}$ is smooth. ■

It follows from Lemma 7 that we have a covering of the orbit O by manifolds $\rho_{\xi,q}(W_{\xi,q})$. Hence, we may continue with topological arguments in Section 2.2, and topologize the orbit O with the topology $\mathcal{T}$ generated by a basis of open sets consisting of sets $\rho_{\xi,q}(W_{\xi,q})$ and all their open subsets.

In the arguments of Section 2.3 the manifold structure of M is used in an essential way. In order to extend results of Section 2.3 to subcartesian spaces, we use Theorem 5 to show that for each point p of O there is a neighbourhood V of p in S such that connected components of $V\cap O$ are integral manifolds of $D_{|V}$.

In the proof of Lemma 7, we have shown that for every point $p \in O$, there exists a neighbourhood V of $p \in S$, a diffeomorphism ϕ of V onto the image $\phi(V)$, contained in an open subset U of $\mathbb{R}^n$, and m vector fields $Y_1, \dots, Y_m$ on U that are ϕ-related to vector fields $X_1, \dots, X_n$ that span the restriction of D to V, equation (29).

Consider now the family

$$\mathcal{F}_\phi = (Y_1, \dots, Y_m) \tag{33}$$

of vector fields on the open set U of $\mathbb{R}^n$. Let D_ϕ be the distribution on U spanned by family $\mathcal{F}_\phi$ and O_ϕ be the orbit of $\mathcal{F}_\phi$ through $\phi(p)$. In other words,

$$O_\phi = \Big\{ \exp(t_k Y_{i_k}) \circ \cdots \circ \exp(t_1 Y_{i_1})(\phi(p)) \;\Big|\; k \in \mathbb{N}, (t_1, \dots, t_k) \in \mathbb{R}^k, \\ Y_{i_1}, \dots, Y_{i_k} \in \mathcal{F}_\phi \Big\}.$$

Equation (30) implies that

$$\exp(t_k Y_{i_k}) \circ \cdots \circ \exp(t_1 Y_{i_1})(\phi(q)) = \phi \circ \exp(t_k X_{i_k}) \circ \cdots \circ \exp(t_1 X_{i_1})(p), \tag{34}$$

whenever the left-hand side is defined. Hence,

$$O_\phi \subseteq \phi(O \cap V). \tag{35}$$

By assumption, the distribution D spanned by $\mathcal{F}$ is involutive on the orbit O. Hence, for every $q \in O$ and $i, j = 1, \dots, m$,

$$[X_i, X_j](q) = \sum_{k=1}^{m} c_{ij}^k(q) X_k(q), \tag{36}$$

for some coefficients $c_{ij}^k(q)$. Since the vector fields X_i and Y_i are ϕ-related, equation (29), taking into account equations (35) and (36) we obtain

$$[Y_i, Y_j](x) = \sum_{k=1}^{m} \bar{c}_{ij}^k(x) Y_k(x), \tag{37}$$

for every $x \in O_\phi$ and some coefficients $\bar{c}_{ij}^k(x)$. This means that the distribution D_ϕ on U is involutive on the orbit O_ϕ. By hypothesis, $\operatorname{rank} D_{|O} = m$, and the vector fields $X_1, \dots, X_m$ are linearly independent on $O \cap V$. Hence, the vector fields $Y_1, \dots, Y_m$ are linearly independent on $\phi(V \cap O) \supseteq O_\phi$. Thus, the distribution D_ϕ is involutive on the orbit O_ϕ and has constant rank on O_ϕ. By Theorem 5, the orbit O_ϕ is an integral manifold of D_ϕ.

Since $O_\phi \subseteq \phi(V)$, and ϕ is a diffeomorphism of V onto $\phi(V)$, equation (29) implies that $\phi^{-1}(O_\phi)$ is an integral manifold through p of the family $(X_{1|V}, \dots, X_{m|V})$ of vector fields on V obtained by restricting to V vector fields $X_1, \dots, X_m$ on S. By assumption, $X_1, \dots, X_m$ span D restricted to V. Hence, $\phi^{-1}(O_\phi)$ is an integral manifold of D through p. This implies that $\phi^{-1}(O_\phi)$ is a neighbourhood of p in O. In

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particular, we can choose a sufficiently small neighbourhood W of $0 \in \mathbb{R}^m$ so that $\phi^{-1}(O_\phi) = \rho_{\xi,p}(W)$, where $\xi = (X_1, \dots, X_m)$ and $\rho_{\xi,p}$ is defined by equation (27).

We have shown that for every $p \in O$, there exists a neighbourhood W of 0 in $\mathbb{R}^m$ and $\xi = (X_1, \dots, X_m) \in \mathcal{F}^m$ such that $\rho_{\xi,p}(W)$ is an integral manifold of D. This completes the differential space analogue of the argument given in Section 2. The arguments of Section 2.4, leading to a smooth atlas on the orbit, do not use the assumption that M is a manifold. Hence, they apply also to a differential space S. This completes the proof of Theorem 6.

References

[1] N. Aronszajn, *Subcartesian and subriemannian spaces.* Notices Amer. Math. Soc. **14**(1967), 111.

[2] I. Kolář, P. W. Michor and J. Slovák, *Natural Operators in Differential Geometry.* Springer-Verlag, Berlin, 1993.

[3] A. Kowalczyk, *The open immersion invariance of differential spaces of class D_0.* Demonstratio Math. **13**(1980), no. 2, 539–550.

[4] R. Sikorski, *Abstract covariant derivative.* Colloq. Math. **18**(1967), 251–272.

[5] ———, *Differential modules.* Colloq. Math. **24**(1971/72), 45–79.

[6] ———, *Wstęp do Geometrii Różniczkowej*, PWN, Warszawa, 1972.

[7] J. Śniatycki, *Integral curves of derivations on locally semi-algebraic differential spaces.* In: Dynamical Systems and Differential Equations, American Institute of Mathematical Sciences Press, Springfield, MO, 2003, pp. 825–831.

[8] ———, *Orbits of families of vector fields on subcartesian spaces.* Ann. Inst. Fourier (Grenoble), **53**(2003), no. 7, 2257–2296.

[9] K. Spallek, *Differenzierbare Räume.* Math. Ann **180**(1969), 269–296.

[10] H. J. Sussmann, *Orbits of families of vector fields and integrability of distributions.* Trans. Amer. Math. Soc. **180**(1973), 171–188.

[11] P. G. Walczak, *A theorem on diffeomorphisms in the category of differential spaces.* Bull. Acad. Polon. Sci., Sér. Sci. Math Astronom. Phys. **21**(1973), 325–329.

[12] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups.* Graduate Texts in Mathematics 94, Springer-Verlag, New York, 1983.

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