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Corrigenda

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'On projective planes of type (6, m)'

BY P. LORIMER

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Alan Rahilly has pointed out to me that the proof of the Theorem in my paper (2) is incomplete. This correction will now complete it. I would also like to acknowledge here that the results which are actually established in (2) can also be found in the paper (5) by Praeger and Rahilly.

The trouble was in the proof of Proposition 2. Although the group G has a subgroup H_1 which intersects each of its conjugates trivially, the same is not necessarily true of the image H_1N/N of H_1 in the 2-transitive representation G/N of G referred to in the paper. A theorem of M. E. O'Nan from (4) was used along with Proposition 2 to establish my theorem. What should have been done was to look at O'Nan's results more deeply and combine them with other results. Here is the way that it is done.

There is nothing actually wrong with the argument in my paper; it is just that it is not complete. The proof here will continue the argument of (2) using the notations established there.

First, a straightforward lemma.

LEMMA 4. *If* $i \neq j$ *then*

(1) $H_i \cap K_j$ *has order* m^2-m *and index* m^2+m *in* H_i;

(2) $K_i \cap K_j$ *is the direct product of* $H_i \cap K_j$ *and* $H_j \cap K_i$;

(3) $N \neq K_i \cap K_j$.

Proof. The argument at the end of section 1 in (2) establishes that $H_i \cap K_j$ has index at least m^2+m in H_i; as this is the maximum possible it must actually be the index. As the order of H_i is m^4-m^2, $H_i \cap K_j$ must have order m^2-m. Then $H_i \cap K_j$ and $H_j \cap K_i$ are two normal subgroups of order m^2-m in the group $K_i \cap K_j$ of order $(m^2-m)^2$. As $(H_i \cap K_j) \cap (H_j \cap K_i) \subseteq H_i \cap H_j = 1$, result (ii) follows.

To prove (iii) suppose that $N = K_j \cap K_j$. Then N contains the m^2-m members of $H_i \cap K_j$. As H_i has m^2+m+1 conjugates intersecting one another trivially and N is a normal subgroup of G, N has order at least

$$1+(m^2+m+1)(m^2-m-1) = m^4-m^2-2m.$$

But, as N has order $(m^2-m)^2$,

$$m^4-m^2-2m \leqslant m^4-2m^3+m^2,$$

therefore

$$m^3 \leqslant m^2+m$$

which is impossible.

This proves Lemma 4.

In that proposition 3 of (2) relies on proposition 2, its proof is also incomplete. However, what is written there can be interpreted as a proof of the following proposition which can also be found in (5).

PROPOSITION 5. *If* $PSL(n,q) \subseteq G/N \subseteq P\Gamma L(n,q)$ *then* $N = 1$, $n = 3$ *and* $q = 2$ *or* 3.

The clue to completing the proof is O'Nan's concept of an (H, K, L) configuration: this is a triple of groups with L a proper subgroup of K, H a subgroup of the automorphism group of K and with the centralizer of each nonidentity member of H equal to L ((4), p. 2); it is called constrained if H is isomorphic to L.

PROPOSITION 6. *$(H_2 \cap K_1, H_1, H_1 \cap K_2)$ is a constrained $(H_2 \cap K_1, H_1, H_1 \cap K_2)$ configuration and either*

(i) *$K_1 \cap K_2$ is abelian or*

(ii) *H_1 and $H_1 \cap K_2$ are Frobenius groups and $H_1 \cap K_2$ intersects the centre of the Frobenius kernel of H_1 nontrivially.*

Proof. The fact that this triple is a constrained $(H_2 \cap K_1, H_1, H_1 \cap K_2)$ configuration is clear.

From Proposition 4·9 of O'Nan (4), it can now be concluded that either

(i) $H_2 \cap K_1$ is a Frobenius complement or

(ii) $H_2 \cap K_1$ is abelian or

(iii) $H_2 \cap K_1$ and H_1 are Frobenius groups and $H_1 \cap K_2$ intersects the centre of the Frobenius kernel of H_1 nontrivially.

Conclusion (ii) of the Proposition comes from the third case.

If $H_2 \cap K_1$ is a non-abelian Frobenius complement then O'Nan also shows (Proposition 4·11) that $H_1 \cap K_2$ is a Hall subgroup of H_1. That is not the case here as $H_1 \cap K_2$ has order $m^2 - m$ and H_1 has order $m^4 - m^2$. Thus the case that $H_2 \cap K_1$ is a Frobenius complement is subsumed under the case that it is abelian.

If $H_2 \cap K_1$ is abelian, so is $H_1 \cap K_2$ and even $K_1 \cap K_2$ because, by Lemma 4, it is the direct product of these two groups.

This proves Proposition 6.

The two possible outcomes of Proposition 6 are now considered and the following is proved.

PROPOSITION 7. *For some $n \geq 3$ and some prime power q*

$$PSL(n,q) \subseteq G/N \subseteq P\Gamma L(n,q).$$

Proof. Suppose first that $K_1 \cap K_2$ is abelian. As $N \subseteq K_1 \cap K_2$, G/N is a 2-transitive permutation group in which the stabilizer of two points is abelian and the result of M. Aschbacher (1) can be applied. The degree of G is $m^2 + m + 1$ and as $m^2 + m$ cannot be a prime power it follows from his theorem that either $G/N = PSL(3,2)$, fitting in with the conclusion of this Proposition, or G/N has a normal regular subgroup. Suppose that the latter is the case.

As $m^2 + m + 1$ is odd, it follows from the remarks at the bottom of p. 114 of (1) that if there is an involution in G/N fixing $s > 1$ points, then $m^2 + m + 1 = s^2$. Then $m + 1 = (s - m)(s + m)$ which is impossible. Thus $(K_1 \cap K_2)/N$ is odd. As $H_1 \cap K_2$ has even order $m^2 - m$, $H_1 \cap K_2$ intersects N nontrivially, and, because of the 2-transitivity of G/N, so does each subgroup $H_i \cap K_j$, $i \neq j$. Suppose $H_i \neq H_1, H_2$ and $x \in H_i \cap N$, $x \neq 1$. Then $K_1 \cap K_2 \subseteq C(x) \subseteq K_i$. As N is the intersection of all the subgroups K_i, this means that $N = K_1 \cap K_2$, contradicting Lemma 4.

Thus $K_1 \cap K_2$ is abelian only when $G = PSL(3,2)$.

Following Proposition 6, the alternative is that H_1 and $H_1 \cap K_2$ are Frobenius groups and $H_1 \cap K_2$ intersects the centre of the Frobenius kernel of H_1 non-trivially.

Let L_1 be the Frobenius kernel of H_1. As H_1 has order $m^2(m^2-1)$, L_1 has order at least m^2. As $H_1 \cap K_2$ has order m^2-m, L_1 is not a subset of K_2; i.e. $L_1 \cap K_2$ is a proper subgroup of L_1.

By a result of Thompson (6), L_1 is nilpotent and hence so is its image L_1N/N in G/N. As L_1 is not a subgroup of K_2 but N is, L_1N/N is not the trivial group. Hence H_1 has a characteristic abelian subgroup A_1 such that A_1N/N is non-trivial. In these circumstances theorem A of another paper (3) of O'Nan can be applied and it can be deduced that either $PSL(n,q) \subseteq G/N \subseteq P\Gamma L(n,q)$, fitting in with the result of this proposition, or A_1N/N is semiregular.

Suppose the latter. Then $A_1 \cap K_2 \subseteq N$.

Let p be a prime number which divides the order of L_1 but not the order of $L_1 \cap K_2$ and let P be a Sylow p-subgroup of L_1: then $P \cap K_2 = 1$. As L_1 is nilpotent, P is a characteristic subgroup of L_1 and hence a normal subgroup of K_1. Thus $H_2 \cap K_1$ acts on P. Suppose $h \in H_2 \cap K_1$, $x \in P$ and $hx = xh$. Then $h \in H_2 \cap x^{-1}H_2x$ so that either $H_2 = x^{-1}H_2x$ in which case $x \in N(H_2) \cap P = K_2 \cap P = 1$ or $h = 1$: hence $H_2 \cap K_1$ acts fixed point free on P. Thus $H_2 \cap K_1$ is a Frobenius complement, contradicting the fact that it is a Frobenius group. This establishes the fact that if p is a prime divisor of the order of L_1 it is also a divisor of the order of $L_1 \cap K_2$.

Now, $H_1 \cap K_2$ has order $m(m-1)$ and H_1 has order $m^2(m-1)(m+1)$. Thus, if m is even, $m+1$ does not divide the order of L_1 and so it does divide the order of the Frobenius complement of L_1 in H_1: if m is odd the same applies to $\frac{1}{2}(m+1)$. Suppose the order of the complement is $\alpha(m+1)$ or $\frac{1}{2}\alpha(m+1)$ respectively.

Both A_1 and $A_1 \cap N$ are normal subgroups of K_1 and thus the Frobenius complements act fixed point free on the factor group $A_1/A_1 \cap N$. Hence the order of $A_1/A_1 \cap N$ is $\alpha\beta(m+1)+1$ when m is even or $\frac{1}{2}\alpha\beta(m+1)+1$ when m is odd, for some integer β.

As A_1 is a normal subgroup of K_1, the group $A_1(H_1 \cap K_2)$ has order

$$\begin{aligned}\frac{|A_1|\,|H_1 \cap K_2|}{|A_1 \cap K_2|} &= |A_1N/N|\,|H_1 \cap K_2| \\ &= (m^2-m)\,|A_1N/N|.\end{aligned}$$

This is a subgroup of H_1 which has order $(m^2-m)(m^2+m)$. Hence m^2+m is divisible by $\alpha\beta(m+1)+1$ when m is even and $\frac{1}{2}\alpha\beta(m+1)+1$ when m is odd. In any case

$$m^2+m = \gamma(\tfrac{1}{2}\delta(m+1)+1) \quad \text{for} \quad \delta = 2\alpha\beta \text{ or } \alpha\beta \text{ and some integer } \gamma.$$

Thus $m+1$ divides γ, say $\gamma = \epsilon(m+1)$ and

$$\begin{aligned} m &= \tfrac{1}{2}\delta\epsilon(m+1)+\epsilon \\ &> \tfrac{1}{2}\delta\epsilon(m+1). \end{aligned}$$

Thus

$$\delta < \frac{2m}{\epsilon(m+1)} < 2,$$

which implies that $\delta = 1$. Then

$$\begin{aligned} m^2+m &= \gamma(\tfrac{1}{2}(m+1)+1) \\ &= \gamma(\tfrac{1}{2}m+\tfrac{3}{2}). \end{aligned}$$

But

$$2(m^2+m) = (2m-4)(m+3)+12.$$

Hence $m+3$ divides 12. As $m > 3$ the only solution is $m = 9$. Suppose that this is the case. Then $A_1/A_1 \cap N$ has order $\frac{1}{2}m+\frac{3}{2} = 6$ and the Frobenius complement acting on it fixed point free has order at least $\frac{1}{2}(m+1) = 5$ which is not possible.

This establishes Proposition 7.

Propositions 5 and 7 combine to finish the proof of the Theorem in (2).

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