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NOTES ON LYAPUNOV GRAPHS AND NON-SINGULAR SMALE FLOWS ON THREE MANIFOLDS

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§1. Introduction

In the 1980s, Franks, Pugh and Shub raised the question "Given any subshift of finite type σ_A: $\Sigma_A \to \Sigma_A$ is there a non-singular Smale flow (or an NS flow for short) on S^3 with the suspension of σ_A as a basic set?" (See [5] and [12]).

In 1985, Franks introduced the concept of a Lyapunov graph, and using this graph, he obtained an affirmative answer about the question (see [6]). Simultaneously, he characterized the Lyapunov graphs appearing as that of NS flows on S^3. (See Theorem 1 [6]). On the other hand, J. Birman, R.F. Williams studied what kinds of knots appear as sets of closed orbits of knot holders of an NS flow on S^3. (See [2]). For non-singular Morse Smale flows (or NMS flows for short), which are special cases of NS flows, Sasano and Wada characterized the knot types or link types which appear as sets of closed orbits. Kobayashi also studied the types of primitive links in the case of special Seifert manifolds. (See [10], [14] and [16]).

Our purpose of this notes is to decide what kinds of Lyapunov graphs appear on certain three manifolds associated with NS flows. By using this graph, the global conditions of flows on manifolds are more visible than other methods.

Our first result in Theorem A is a characterization of the Lyapunov graphs of NS flows on $L(2p-1, q)$ by using a standard technique of 3-dimensional topology, which extends Theorem 1 of Franks' paper [6]. For NMS flows, Franks characterized the type of Lyapunov graphs assciated with NMS flows on S^3. (See Theorem 2 [6]). In our notes, we define a notion of a singular vertex for Lyapunov graphs associated with NMS flows on the irreducible, orientable, closed 3-manifolds which can

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not admit incompressible tori. By observing that the Lyapunov graphs on the case of the above manifolds which are not lens space must have one singular vertex in Theorem B, we extend Theorem 2 [6] to the case of the irreducible, orientable, closed 3-manifolds which can not admit incompressible tori. Moreover, we define a special Lyapunov graph which distinguishes a singular vertex from an ordinary vertex. We characterize the types of the special graphs on the case of lens space. (See Proposition 5 and Proposition 6).

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§ 2. Definitions and preliminaries

For the definition of an irreducible 3-manifold, a Seifert fibered space, an incompressible surface, etc., we refer to Hempel [7] and Jaco [8].

DEFINITION 1. Suppose that ϕ_t: $M \to M$ is a continuous flow and $\varepsilon > 0$, we say there is an ε-chain from x to y provided that there exist points $x_1 = x, x_2, \cdots, x_{n+1} = y$ and real numbers $t(i) > 1$ such that $d(\phi_{t(i)}(x_i), x_{i+1}) < \varepsilon$ for all $1 \leqq i \leqq n$. A point x is called chain recurrent if for any $\varepsilon > 0$ there is an ε-chain from x to x. The set R of all chain recurrent points is called the chain recurrent set.

DEFINITION 2. If ϕ_t: $M \to M$ is a smooth flow, then a smooth function g: $M \to \mathbb{R}$ will be called a Lyapunov function provided

(1) $d(g(\phi_t(x))/dt < 0$ if $x \in R$ and

(2) when $x, y \in R$, $g(x) = g(y)$ if and only if for each $\varepsilon > 0$ there are ε-chains from x to y and y to x.

DEFINITION 3. A smooth flow ϕ_t: $M \to M$ on a compact manifold is called a Smale flow provided that

(1) its chain recurrent set R has a hyperbolic structure, $\dim R \leqq 1$ and that

(2) it satisfies the transversality condition.

Remark 1. When a Smale flow does not have any singular points, it is called a non-singular Smale flow (or an NS flow for short).

Remark 2. A flow ϕ_t: $M \to M$ is especially called an NMS flow when its chain recurrent set R consists of finitely many hyperbolic closed orbits and satisfies the transversality condition.

DEFINITION 4. An (abstract) Lyapunov graph is a finite connected oriented graph which possesses no oriented cycles, and each vertex of which is labelled with a chain recurrent flow on a compact space.

From now on we assume that M is an irreducible, orientable, closed 3-manifold. Since we study an NS flow on M, each vertex of a Lyapunov graph will be labelled with a basic set, which in this case is topologically equivalent to either a suspension of a subshift of finite type σ_A: $\Sigma \to \Sigma$ corresponding to a matrix A (see Bowen [1]), an attracting closed orbit, a repelling closed orbit, a closed orbit of twisted saddle type, or a closed orbit of untwisted saddle type.

We note that all attractor and repeller of an NS flow are closed orbit; because a hyperbolic chain recurrent set satisfies Axiom A (see Franke and Selgrade [3], and Smale [15]), each basic set has a closed orbit. If a basic set Λ is attracting or repelling and $\dim \Lambda = 1$, Λ must be an isolated orbit.

DEFINITION 5. Suppose ϕ_t is a Smale flow on M, g: $M \to \mathbb{R}$ is a Lyapunov function and Λ is a basic set with $g(\Lambda) = C$. We will say that $X = g^{-1}([C - \varepsilon, C + \varepsilon])$ is a basic block for Λ, if the following three conditions are satisfied.

(1) X contains only one basic set.

(2) There exist (not necessarily connected) codimension one submanifolds U and V with boundary in X such that $U \subset V$ and that they are transverse to the flow.

(3) Let H be a finite set of one handles h_i ($\cong D^1 \times D^1$) in U; $H = \bigcup_i h_i$ the first return map r: $U \to \operatorname{int} V$ is well defined, smooth and there is a hyperbolic handle set $H \subset \operatorname{int} U$, with every orbit of Λ intersecting H and $h_i \subset H$ intersecting Λ. Here we call a handle set H hyperbolic handle, if it satisfies the followings.

(1) If $x \in h_i$ and $r(x) \in h_j$, then $\operatorname{int}(r(W_i^u(x)) \supset W_j^u(r(x))$ and $r(W_i^s(x)) \subset \operatorname{int}(W_j^s(r(x))$.

(2) There is a $\lambda \in (0, 1)$ such that for each $x \in h_i$, with $r(x) \in H$ and each $v \in T_x(W_i^s(x))$, $w \in T_x(W_i^u(x))$ we have $\|dr(v)\| \leqq \lambda \|v\|$, $\|dr(w)\| \geqq \lambda^{-1} \|w\|$. Here $W_i^u(x)$ denotes the interval $D^1 \times \{p\} \subset h_i$ which contains a point x. Similarly $W_i^s(x)$ denotes the interval $\{q\} \times D^1 \subset h_i$ containing x.

DEFINITION 6. Suppose Λ_j is a basic set of a Smale flow contained in a basic block X_j with hyperbolic handles H, then we call $K(\Lambda_j) =$

$\bigcup_{t \in \mathbb{R}} \phi_t(H)$ the saturated handles. $K(\Lambda_j)$ has stable foliations $W^s(x)$ with leaves containing $x \in K$, and $y \in W^s(x)$ if and only if there exists $t_1, t_2 \in R$ and $z \in h_i \subset H$ such that $\phi_{t1}(x)$, $\phi_{t2}(y)$ are both in $W_i^s(z) \subset h_i$. The unstable leaf $W^u(x)$ has a similar property.

DEFINITION 7. A Smale flow ϕ_t on M will be called fitted provided

(1) there exists a Lyapunov function g: $M \to \mathbb{R}$ with respect to which a basic set Λ_j has a basic block X_j and $\{X_j\}$ are pairwise disjoint, and

(2) if $x \in K(\Lambda_i)$, $y \in K(\Lambda_j)$ and $g(x) > g(y)$, then $W^u(x)$ either contains $W^u(y)$ or is disjoint from it and $W^s(y)$ either contains $W^s(x)$ or is disjoint from it.

Remark 3. A fitted Smale flow satisfies the transversality conditions.

LEMMA 1. *Suppose that ϕ_t: $M \to M$ is a non-singular smooth flow on an irreducible, orientable, closed 3-manifold M, and suppose that g: $M \to \mathbb{R}$ is a Lyapunov function associated with a flow ϕ_t. Then each component of a level surface of a regular value of the Lyapunov function g is homeomorphic to a torus.*

Proof. Suppose that we cut manifold M at the level $g^{-1}(C)$, where C is a regular value of the Lyapunov function g. Let $g^{-1}([C, +\infty))$ be denoted by M^+. Suppose that a flow exits transversely on the boundary ∂M^+. Thus considering a double of M^+, we obtain $\chi(\partial M^+) = 0$, where $\chi(\partial M^+)$ denotes the Euler characteristic of ∂M^+. If there is a surface of genus greater than or equal to two in $g^{-1}(C) = \partial M^+$, the boundary of $g^{-1}(C)$ must contain S^2. Since M is irreducible, the S^2 bounds a ball in M. But there exists a singular point in the ball. This is a contradiction. Then each component of ∂M^+ is a torus.

LEMMA 2. *Let T^2 be a compressible torus in an irreducible, orientable, closed 3-manifold M. Then T^2 bounds a solid torus or there exists a non-trivial knot h in S^3 such that T^2 bounds a compact 3-manifold M which is homeomorphic to $S^3 - N(h)$, where $N(h)$ denotes a regular neighbourhood of h. Moreover, in the last case we can take a boundary of a compressing disk of T^2 for a meridian of h.*

Proof. Let T^2 be a compressible torus in M. Then there exists an essential simple closed curve on T^2 which bounds a disk D^2 in M. By Dehn's Lemma, D^2 is a non-singular disk. Thus the $(T^2 \cap \overline{(M - (D^2 \times I))}) \cup D^2 \times \partial I$ is homeomorphic to S^2. The S^2 bounds a ball in M. Here K

denotes a closure of an outside of the torus as in Figure (1). Let $\overline{M - (K \cup D^2 \times I)}$ be denoted by A. Then if the ball B^3 corresponds to A, we see that $A \cup D^2 \times I$ is homeomorphic to a solid torus. It means that the T^2 bounds a solid torus in M. If the ball B^3 corresponds to $K \cup D^2 \times I$, we see that T^2 bounds $K = B^3 - D^2 \times I$. Thus when K is embedded in S^3, it may be a non-trivial knot complement $S^3 - N(h)$. Moreover, in this case, the boundary of a compressible disk of the T^2 can be taken as the boundary of $D^2 \times \{p\}$ of the $D^2 \times I$ in M. Then we see that the disk $D^2 \times \{p\}$ is a meridian disk of h in S^3.

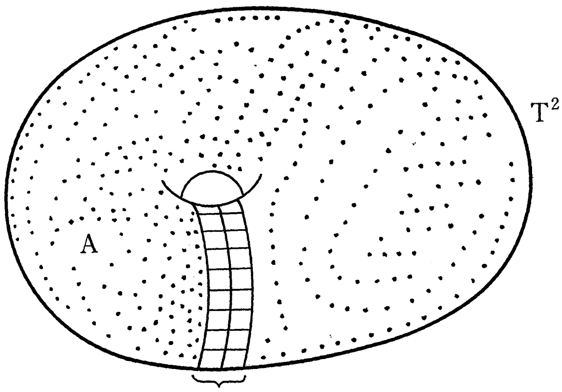

Figure (1)

Proposition 1. *Suppose that M is an orientable, irreducible, closed 3-manifold, and that M does not admit incompressible tori. Let ϕ_t: $M \to M$ be a non-singular smooth flow with a Lyapunov function g: $M \to \mathbb{R}$. Let Γ be a Lyapunov graph associated with the flow ϕ_t and Lyapunov function g. Then Γ is a tree.*

Proof. By Lemma 1, each component of $g^{-1}(C)$ is a torus, where C denotes a regular value of g. Since M is an irreducible manifold and since M does not have any incompressible torus, each component of $g^{-1}(C)$ bounds a solid torus or a non-trivial knot complement by Lemma 2. Hence Γ is a tree.

Remark 4. If $H_1(M, \mathbb{Q}) = \{0\}$ for a 3-manifold M, a Lyapunov graph Γ associated with a flow ϕ_t: $M \to M$ and a Lyapunov function g: $M \to \mathbb{R}$

is a tree. In this case, the flow ϕ_t is not necessarily non-singular (see [6]).

Remark 5. There exists an irreducible, orientable, closed 3-manifold M which does not admit incompressible tori and $H_1(M, \mathbf{Q}) \neq \{0\}$ in the category of Seifert manifolds whose orbit spaces are S^2 with three exceptional fibers (See [8]).

PROPOSITION 2 ([6]). *Let X_i $(i = 1, 2, \cdots, n-1)$ be a 3-dimensional basic block in a 3-manifold M. Suppose that h_i: $\partial X_i^- \to \partial X_{i+1}^+$ is a diffeomorphism $(1 \leqq i \leqq n-1)$, where ∂X_i^- is the part of the boundary ∂X_i on which the flow exits, and ∂X_{i+1}^+ is the part of the boundary ∂X_{i+1} on which the flow is entering $(1 \leqq i \leqq n-1)$. For each i, h_i is isotopic to g_i: $\partial X_i^- \to \partial X_{i+1}^+$ such that the flow on $X_1 \cup_{g_1} X_2 \cup_{g_2} \cdots \cup_{g_{n-1}} X_n$ is fitted Smale flow.*

§ 3. Main results

The main results of this paper is as follows.

(1) *Case of Smale flow*

THEOREM A. *Let ϕ_t be an NS flow on a lens space $L(2p-1, q)$ and let g be a Lyapunov function. Then a Lyapunov graph Γ associated with the flow ϕ_t and the function g satisfies the following conditions* (0), (1) *and* (2).

(0) *It is a tree with one edge attached to each source and sink vertex.*

(1) *Each source (resp. each sink) vertex is labelled with an attracting closed orbit (resp. a repelling closed orbit).*

(2) *If V is any other vertex, it is labelled with a suspension of the subshift of finite type $\sigma(A)$ with a transition matrix A which is an irreducible $n \times n$ matrix. Let e_V^+ be the number of entering edges and let e_V^- be the number of exiting edges. If $k_V = \dim\ker((I - \bar{A}_V): Z_2^n \to Z_2^n)$, where $\bar{A}_V$ is the* mod 2 *reduction of A_V, then $e_V^+ \leqq k_V + 1$, $e_V^- \leqq k_V + 1$ and $k_V + 1 \leqq e_V^+ + e_V^-$.*

Conversely, if an abstract Lyapunov graph Γ satisfies the above conditions (0), (1) *and* (2), *then there exists an NS flow on $L(2p-1, q)$ whose Lyapunov graph is Γ.*

(2) *Case of Morse-Smale flow*

DEFINITION 8. Suppose that Γ is a Lyapunov graph associated with an NS flow ϕ_t and a Lyapunov function g on an irreducible, orientable, closed 3-manifold M which does not admit incompressible tori. If there is a vertex of Γ which satisfies following conditions, we call the vertex a *singular vertex*.

(1) If we cut the graph Γ at any point C^+ of any entering edge of a vertex V, a submanifold of M which corresponds to a component of Γ containing the vertex V is neither a soild torus nor a non-trivial knot complement $S^3 - N(h)$ such that the boundary of a compressing disk of $\partial(S^3 - N(h))$ is a meridian of this knot h.

(2) If we cut the graph Γ at any point C^- of any exiting edge of the above vertex V, a submanifold of M which corresponds to a component of Γ containing the vertex V is also neither a solid torus nor a non-trivial knot complement which satisfies the condition of the case (1).

DEFINITION 9. If a vertex V corresponds the closed orbit of twisted saddle type, then we call the vertex V a *twisted vertex*, *An untwisted vertex* is defined similarly. Here a saddle type closed orbit is called *untwisted* or *twisted* if the associated unstable bandle E^U is oreintable or not.

THEOREM B. *Suppose that ϕ_t is an NMS flow on an irreducible, orientable, close 3-manifold M which does not admit incompressible tori and suppose that g is a Lyapunov function. Then a Lyapunov graph Γ associated with the flow ϕ_t and the function g satisfies the followings.*

(a) *Case of M being a lens space*:

  (1) *It is a tree with one edge attached to each source or sink vertex.*

  (2) *If V is a vertex labelled with the closed orbit of saddle type and the vertex V has e_V^+ enterning edges, and e_V^- exiting edges, then $e_V^+ = 1$ or 2, $e_V^- = 1$ or 2 and if V is twisted, then $e_V^+ = e_V^- = 1$.*

(b) *Case of M not being a lens space*:

(b1) *Suppose M admits a Seifert fibration whose orbit manifold is S^2 with three exceptional fibers. And suppose that it does not admit a Seifert fibration whose orbit space is a projective plane P^2.*

  (1) *The graph is a tree with one edge attached to each source or sink vertex.*

  (2) *There exists one (singular) vertex V^* labelled with a closed orbit of saddle type, and suppose V^* has $e_{V^*}^+$ entering edges and $e_{V^*}^-$ exiting edges. If V^* is untwisted then either $e_{V^*}^+ = 2$ and $e_{V^*}^- = 1$, or $e_{V^*}^+ = 1$ and $e_{V^*}^- = 2$. If V^* is twisted, then $e_{V^*}^+ = e_{V^*}^- = 1$.*

Suppose V is any other vertex labelled with a saddle orbit, and has e_V^+ entering edges and e_V^- exiting edges. If V is untwisted, then $e_V^+ = 2$ or 1, $e_V^- = 2$ or 1. If V is twisted, then $e_V^+ = e_V^- = 1$.

(b2) *Suppose that M admits a Seifert fibration whose orbit manifold is S^2 with three exceptional fibers. And suppose that it also admits a Seifert fibration whose orbit manifold is a projective plane P^2 with at most one exceptional fiber.*

(1) *The graph is a tree with one edge attached to each source or sink vertex.*

(2) *There exists one (singular) vertex V^* labelled with an orbit of saddle type. Suppose V^* has $e_{V^*}^+$ entering edges and $e_{V^*}^-$ exiting edges. If V^* is untwisted, then $e_{V^*}^+ = 2$ and $e_{V^*}^- = 1$, $e_{V^*}^+ = 1$ and $e_{V^*}^- = 2$, or $e_{V^*}^+ = e_{V^*}^- = 1$. If V^* is twisted, then $e_{V^*}^+ = e_{V^*}^- = 1$, Suppose V is any other vertex labelled with an orbit of saddle type and has e_V^+ entering edges and e_V^- existing edges. If V is untwisted, then $e_V^+ = 1$ or 2, $e_V^- = 1$ or 2 and if V is twisted, $e_V^+ = e_V^- = 1$*

Conversely, if an abstract Lyapunov graph Γ satisfies conditions (1) *and* (2) *of case* (*a*), *case* (*b*1), *and case* (*b*2) *respectively, then there exists an NMS flow ϕ_t on M whose Lyapunov graph is Γ.*

Remark 6. Kobayashi detects the following. If an irreducible, orientable, closed 3-manifold M which does not admit incompressible tori admits an NMS flow, then M is one of the following manifolds.

(1) lens space (2) Seifert manifold whose orbit space is S^2 with three exceptional fibers. (See Kobayashi [10] and also see Morgan [11]).

§ 4. Proof of Theorem A and Theorem B

We will use the next Lemma and Proposition for the necessary condition in Theorem A and Theorem B.

LEMMA 3. *Suppose V is a singular vertex of a Lyapunov graph Γ. If we cut the graph Γ at any cut point on its edges, the submanifold of M which corresponds to the component of Γ containing the vertex V is neither a solid torus nor a non-trivial knot-complement $S^3 - N(h)$ such that the boundary of compressing disk of $\partial(S^3 - (N(h))$ is a meridian of this knot h.*

Proof. Suppose that we cut the graph Γ at a cut point on an edge of a vertex V. Let Γ' denote a component of Γ which contains a singular vertex V, and let N denote a submanifold of M which corresponds to Γ'. Suppose N is a solid torus or a knot-complement. Attaching one solid torus which contains an attracting closed orbit or a repelling closed orbit along the boundary of N, we can construct a non-singular smooth flow on S^3. Then we can construct a new Lyapunov graph Γ''. By the Solid Torus Theorem (see [13]), any torus in S^3 bounds a solid torus on one side and bounds a knot-complement which satisfies the condition of Lemma 3 on the other side. Then if we cut the graph Γ'' at any point C^* of any entering edge of V_0 or any exiting edge of V_0, each component of Γ'' is a solid torus or a knot-complement. This is a contradiction. Thus N is neither a solid torus nor a knot-complement which satisfies the condition of Lemma 3.

PROPOSITION 3. *Suppose that ϕ_t is a non-singular Smale flow on an irreducible, orientable, closed 3-manifold M which does not admit incompressible tori, and suppose that Γ is a graph associated with a flow ϕ_t and a Lyapunov function g, then a vertex of Γ which is neither an attracting nor a repelling satisfies the following conditions.*

(1) *There exists at most one singular vertex V_0 labelled with the suspension of a subshift $\sigma(A_{V_0})$. Suppose V_0 has $e^+_{V_0}$ entering edges and $e^-_{V_0}$ exiting edges. If $k_V = \dim\ker((I - \overline{A}_{V_0})\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n)$, where $\overline{A}_{V_0}$ is the* mod 2 *reduction of A_{V_0}. $e^+_{V_0} \leqq k_{V_0} + 1$, $e^-_{V_0} \leqq k_{V_0} + 1$, $k_{V_0} + 1 - \dim H_1(M, \mathbb{Z}_2) \leqq e^+_{V_0} + e^-_{V_0}$.*

(2) *Any other vertex V satisfies condition* (2) *in Theorem* A. *Moreover, if M is not a lens space, Γ must have exactly one singular vertex.*

Proof. By Proposition 1, the Lyapunov graph Γ is a tree. Suppose that there are two singular vertices V_1 and V_2 in Γ. Then V_1 is contained in a component of $\Gamma - \{C\}$ corresponding to a solid torus or a knot-complement in M, where C is a cut point of an entering edge or an exiting edge of the vertex V_2. Then by Lemma 3, V_1 is not a singular vertex. Thus the number of singular vertices in Γ is at most one. Now we suppose that Γ has a singular vertex V^*, and we choose cut points $\{C_1, \cdots, C_n\}$ on all entering edges and all exiting edges of singular vertex V^*. By Lemma 1 and Lemma 2, each component which does not contain the singular vertex V^* corresponds a solid torus or a nontrivial knot-complement $S^3 - N(h)$. Now, we will cut Γ at a cut point

$C_i \in \{C_1, \cdots, C_n\}$ and let Γ' be the component of $\Gamma - \{C_i\}$ which does not contain V^*. Then Γ' corresponds a solid torus or a non-trivial knot-complement $S^3 - N(h)$ in M. By attaching a new solid torus which contains an attractive closed orbit or a repelling closed orbit along the boundary of the solid torus or the knot-complement corresponding to Γ', we construct a new graph Γ'' associated with a new flow on S^3. Then each vertex of Γ'' satisfies condition (2) in Theorem A by Franks' Theorem 1 in [6]. Thus the vertex which is not a singular vertex satisfies the condition (2) in Theorem A. Let C be a level of the singular vertex V^* of Γ for the Lyapunov function g. By Lemma 2, each component of Γ corresponding with $g^{-1}([C+\varepsilon, +\infty)) = Y$ is a solid torus or a non-trivial knot-complement, and also $g^{-1}((-\infty, C-\varepsilon]) = Z$ is a solid torus, or a knot-complement in M. Let X be $g^{-1}((-\infty, C+\varepsilon])$, then $X \cup Y = M$ and $X \cap Y = \partial X$ consists of disjoint $e^+_{V_0}$ tori.

Now, we consider the following Mayer-Vietoris exact sequence with in $\mathbb{Z}_2$.

$$H_3(X) \oplus H_3(Y) \longrightarrow H_3(X \cup Y) \longrightarrow H_2(X \cap Y) \xrightarrow{\alpha_*} H_2(X) \oplus H_2(Y) \xrightarrow{C_*} .$$

Since each component of Y is a solid torus or a knot-complement, $H_3(Y) \cong H_2(Y) \cong \{0\}$. And we see that $H_3(X \cup Y) \cong H_3(M) \cong \mathbb{Z}_2$, $H_3(X) \cong \{0\}$, and $H_2(X \cap Y) = \oplus_{i=1}^{e^+_{V_0}} \mathbb{Z}_2$ Hence,

$$\text{(1)} \qquad \dim \ker C_* = \dim \operatorname{Im} \alpha_* \leqq \dim H_2(X)$$

and

$$\text{(2)} \qquad \dim \operatorname{Im} \alpha_* = e^+_{V_0} - 1 .$$

Next, we consider the exact sequence of the pair (X, Z).

$$H_2(Z) \longrightarrow H_2(X) \xrightarrow{b_{1*}} H_2(X, Z) \longrightarrow H_1(Z) \longrightarrow .$$

Since $H_2(Z) = \{0\}$, b_{1*} is injective and $\dim H_2(X, Z) = \dim \ker (I - \bar{A}) = k_{V_0}$ (see Franks [6]). Then

$$\text{(3)} \qquad \dim H_2(X) \leqq k_{V_0} .$$

We see $e^+_{V_0} \leqq k_{V_0} + 1$ by (1), (2) and (3) and $e^-_{V_0} \leqq k_{V_0} + 1$ follows from considering the reverse flow.

We will show that $e^+_{V_0} + e^-_{V_0} \geqq k_{V_0} + 1 - \dim H_1(M)$. We see that

$$\text{(4)} \qquad \dim H_2(X, \mathbb{Z}_2) = \dim H^2(X, \mathbb{Z}_2)$$

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and that

(5) $$H^2(X) = H_1(M, M - X).$$

Now, we consider the following exact sequence.

$$\tilde{H}_1(M - X) \longrightarrow \tilde{H}_2(M) \longrightarrow H_1(M, M - X) \longrightarrow \tilde{H}_0(M - X) \longrightarrow .$$

Then

(6) $$\dim H_1(M, M - X) \leqq e^+_{V_0} - 1 + \dim H_1(M).$$

We consider the next exact sequence of the pair (X, Z) again.

$$H_2(Z) \longrightarrow H_2(X) \xrightarrow{b_{2*}} H_2(X, Z) \longrightarrow H_1(Z) \longrightarrow .$$

By (4), (5), (6) and the facts that b_{2*} is injective and that $H_1(Z) = \oplus_{i=1}^{e^-_{V_0}} \mathbb{Z}_2$, we obtain that $k_{V_0} + 1 - \dim H_1(M, Z_2) \leqq e^+_{V_0} + e^-_{V_0}$. If M is not a lens space, an NS flow on M has at least one basic set which is neither an attractor nor a repeller. Then an associated Lyapunov graph Γ has at least one vertex V_0 which is neither an attractor nor a repeller. Suppose that an associated graph Γ has no singular vertex. Then for any vertex V_0 which is neither an attractor nor a repeller, there exists a cut point C_0 on an entering edge V_0V_1 or an exiting edge V_0V_1. The component Γ_0 of $\Gamma - \{C_0\}$ which contains V_0 corresponds to a submanifold which is homeomorphic to a solid torus or a non-trivial knot-complement. Similarly, there is a cut point C_1 on an entering edge V_1V_2 or an exiting edge V_1V_2 such that the component of Γ_1 of $\Gamma - \{C_1\}$ which contains V_1 corresponds to a submanifold which is homeomorphic to a solid torus or a non-trivial knot-complement. Here, if $V_0 = V_2$, the manifold M is homeomorphic to one of (solid torus) $\bigcup_T$ (solid torus), (solid torus) $\bigcup_T$ (non-trivial knot-complement) and (non-trivial knot-complement) $\bigcup_T$ (non-trivial knot-complement), where $A \bigcup_T B$ means a manifold obtained from A and B by identifying ∂A and ∂B which are homeomorphic to a torus T. In the first case, M is homeomorphic to a lens space, in the second case, M is homeomorphic to S^3 by Lemma 2 and in the last case, M has an incompressible torus. Then this is a contradiction. Thus $V_0 \neq V_2$. Then, there is a cut point C_2 on an entering edge V_2V_3 or an exiting edge V_2V_3 such that the component Γ_2 of $\Gamma - \{C_2\}$ which contains V_3 corresponds to a submanifold which is homeomorphic to a solid torus or a knot complement. We see $V_3 \neq V_1$ by the above argument. Since Γ

is a tree, $V_3 \neq V_0$. This procedure can be continued indefinitely. But the Lyapunov graph has only finite vertices. This is a contradiction. Thus the Lyapunov graph has exactly one singular vertex.

Proof of Theorem A.

Necessity. By Proposition 3, all vertices (which are neither an attractor nor a repeller) satisfy $e_V^+ \leqq k_V + 1$, $e_V^- \leqq k_V + 1$ and $k_V + 1 - \dim H_1(M, \mathbb{Z}_2) \leqq e_V^+ + e_V^-$. We note that $H_1(L(2p-1, q), \mathbb{Z}_2) = \{0\}$. Then, it follows that $k_V + 1 \leqq e_V^+ + e_V^-$. Thus the condition (2) holds. And the condition (0) and (1) are clearly satisfied.

Sufficiency. Let $X_1, X_2, \cdots, X_n$ be basic blocks corresponding with vertices of a given abstract Lyapunov graph satisfying the conditions as in Theorem 1 [6], where X_n is especially a basic block corresponding with an attractive closed orbit. Since all boundaries of basic blocks $X_1, X_2, \cdots, X_{n-1}$ are standard tori. By the Solid Torus Theorem [13], we can construct unknotted solid torus $X_1 \bigcup_{g_1} X_2 \bigcup_{g_2} \cdots \bigcup_{g_{n-2}} X_{n-1}$ in S^3, where $g_1, g_2, \cdots, g_{n-2}$ are attaching diffeomorphisms. Because X_n is a basic block for an attractive closed orbit, it is a solid torus. Then, we can choose an attaching diffeomorphism g_{n-1} such that $(X_1 \cup_{g_1} X_2 \cup_{g_2} \cdots \bigcup_{g_{n-2}} X_{n-1}) \bigcup_{g_{n-1}} X_n$ is a lens space $L(2p-1, q)$. By Proposition 2, we can construct a fitted Smale flow on $L(2p-1, q)$.

Remark 7. Since the diffeomorphism h_i in Proposition 2 is isotopic to g_i, the type of lens space $L(2p-1, q)$ is unchanged.

QUESTION. For a singular vertex V_0, is there a basic block such that $k_{V_0} = e_{V_0}^+ + e_{V_0}^-$, $e_{V_0}^+ \leqq k_{V_0} + 1$ and $e_{V_0}^- \leqq k_{V_0} + 1$ in lens space $L(2p, q)$?

Concerning the above question, we can construct a basic block such that $k_{V_0} = e_{V_0}^+ + e_{V_0}^-$, $e_{V_0}^+ \leqq k_{V_0} + 1$, and $e_{V_0}^- \leqq k_{V_0} + 1$ in 3-torus. But a 3-torus is not a lens space and it has an incompressible torus.

Construction of a basic block in a 3-*torus.*

We consider a cross section of ϕ_t, whose first return map has two saddle points and one attractive fixed point as Figure (2). We next add $k_V - (e_V^+ + 1)$ pairs of a source and a saddle point on this cross section and also add $k_V - (e_V^- + 1)$ pairs of a sink and a saddle. Finally we add $(n - k_V)$ nilpotent handles but we do no further isotopy and which contain no chain recurrent points at this moment. (See Theorem 1 in [6]).

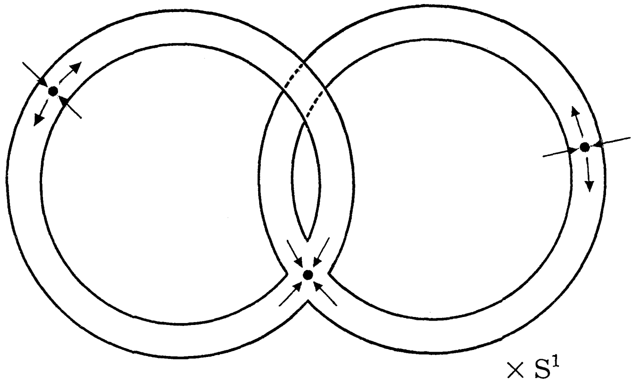

Figure (2)

Then, we can construct basic block such that $k_V = e_V^+ + e_V^-$, $e_V^+ \leqq k_V + 1$ and $e_V^- \leqq k_V + 1$. By attaching a standard solid torus containing one repelling closed orbit along the boundary of this basic block we obtain a 3-torus and an NS flow. (Here, $k_V \geqq 2$).

Remark 8. A closed orbit of saddle type has two dimensional unstable manifold and also two dimensional stable manifold and satisfies $k_V = 1$. Therefore, there is no saddle type vertex which satisfies $k_V = e_V^+ + e_V^-$ in an NMS flow on any 3-manifold.

To prove Theorem B, we need the following proposition.

PROPOSITION 4. *Suppose that M is an irreducible, orientable, closed 3-manifold which does not admit incompressible tori, and suppose that M is not a lens space. Let ϕ_t be an NMS flow on M with a Lyapunov function g. Then a Lyapunov graph associated with the flow ϕ_t and the function g satisfies one of* (1), (2) *and* (3).

(1) (a) *M admits a structure of a Seifert fibered manifold whose orbit space is a projective plane P^2 with at most one exceptional fiber, and* (b) *a graph Γ has one singular vertex V_0 such that $e_{V_0}^+ = e_{V_0}^- = 1$ and V_0 is untwisted. Any other saddle type vertex is of S^3-type, where a vertex of S^3 type is a vertex which is not a singular vertex, and which satisfies condition.* (2) *of the case of a lens space in Theorem* B.

(2) (a) *M admits a Seifert fibration whose orbit manifold is S^2 with three exceptional fibers and* (b) *the graph Γ has one singular vertex V_0 such that $e_{V_0}^+ = 2$ and $e_{V_0}^- = 1$ or $e_{V_0}^+ = 1$ and $e_{V_0}^- = 2$. Remaining saddle type vertices are of S^3-type.*

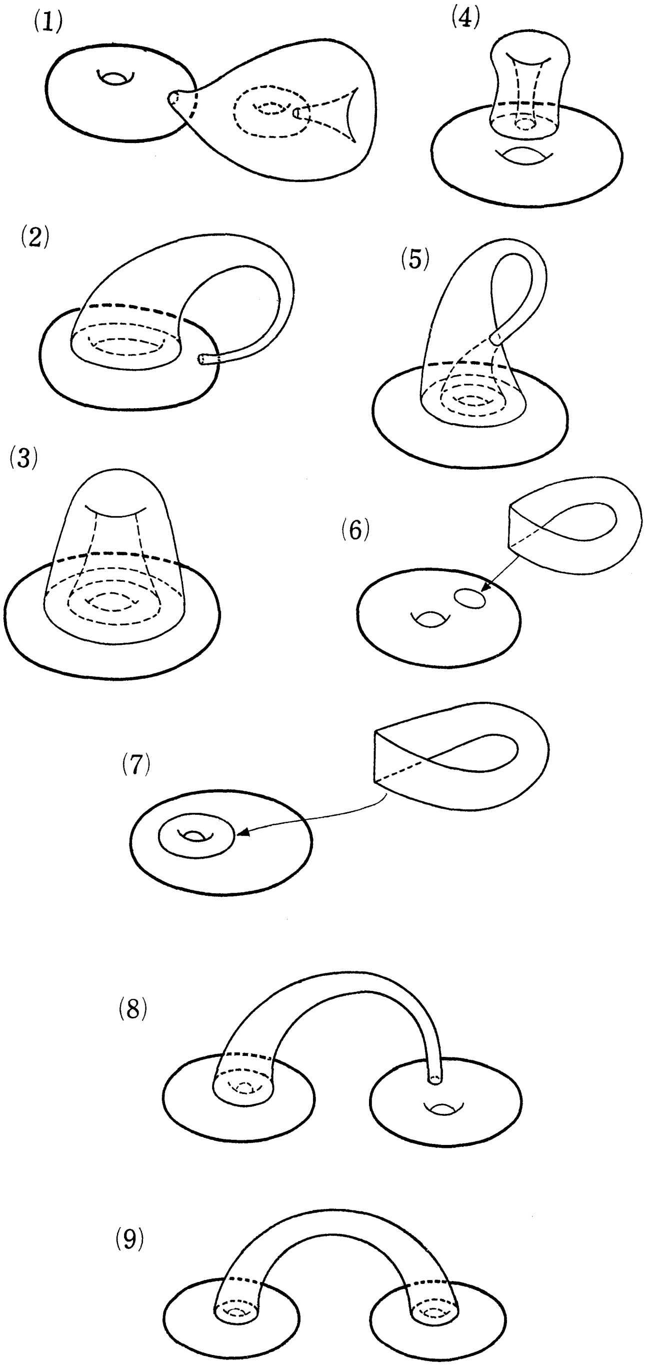

Figure (3)

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(3) (a) *M admits a Seifert fibration whose orbit manifold is S^2 with three exceptional fibers and* (b) *the associated graph Γ has one singular vertex V_0 such that $e^+_{V_0} = e^-_{V_0} = 1$ and V_0 is twisted. And any other saddle type vertex is of S^3-type.*

Proof. Because M is not a lens space, the Lyapunov graph must have one singular vertex (see Proposition 3). If V_0 is a vertex corresponding to a twisted closed orbit, then $e^+_{V_0} = e^-_{V_0} = 1$ (see proof of Theorem 2 in [5]). We also see a topological type of basic block containing a saddle type closed orbit corresponding to the singular vertex V_0 as in Figure (3). (See [10].) It is homeomorphic to one of the followings: $S^1 \times D^2 \# S^1 \times D^2$, $S^1 \times D^2 \# T^2 \times I$, $T^2 \times I \# T^2 \times I$, a Seifert fibered space whose orbit space is an annulus with one exceptional fiber of index 2, (two hole disk) $\times S^1$, an orientable S^1-bundle over a punctured Möbius band, and $P^3 \# T^2 \times I$. Suppose that a basic block B is homeomorphic to $T^2 \times I \# T^2 \times I$. Since M is irreducible, we can regard $T^2 \times I \# T^2 \times I$ as $T^2 \times I \# S^3$. And one component of a boundary torus of $T^2 \times I$ bounds a solid torus, because M has no incompressible torus. We also see that the Seifert structure of $T^2 \times I$ is extended to this solid torus. Otherwise, M is homeomorphic to S^3 or a lens space. Since a Seifert space with compressible boundary is only a solid torus (see Jaco [8]), $S^1 \times D^2 \cup S^1 \times D^2$ is homeomorphic to a lens space and $S^1 \times D^2 \bigcup_T$ (non-trivial knot-complement) is homeomorphic to S^3 by Lemma 2, M is a lens space or S^3. Thus $T^2 \times I \# T^2 \times I$ and $T^2 \times I \# S^1 \times D^2$ are excluded. Since M is irreducible, we also see that $S^1 \times D^2 \# S^1 \times D^2$ and $P^3 \# T^2 \times I$ are excluded. If a basic block B is homeomorphic to (two hole disk) $\times S^1$, one component of boundary tori of this basic block B bounds a solid torus. Otherwise, M admits an incompressible torus. If $\{x\} \times S^1$ in boundary of B bounds a disk in this solid torus, then M is homeomorphic to a lens space. Thus, the Seifert structure of (two hole disk) $\times S^1$ extends to this solid torus. Thus, it yields that each of the boundary components of (two hole disk) $\times S^1$ bounds a solid torus. Otherwise, M is homeomorphic to S^3 or a lens space. Then we see that a Seifert space whose orbit manifold is S^2 with three exceptional fibers can be constructed from the basic block (two hole disk) $\times S^1$. Also a singular vertex type of V_0 which corresponds to the closed orbit of (two hole disk) $\times S^1$ is $e^+_{V_0} = 2$ and $e^-_{V_0} = 1$ or $e^+_{V_0} = 1$ and $e^-_{V_0} = 2$, where V_0 is untwisted. Each closed orbit corresponding to the remaining saddle type vertices is contained in a solid torus. They are not singular vertices by Lemma 3. Clearly, they satisfy condition (2) of the case of the lens space in Theorem B. They are S^3-type vertices. Then we see that it satisfies conditions of case (2). When a basic block is either an orientable S^1-bundle over a punctured Möbius band or a Seifert space whose orbit manifold is an annulus with one exceptional fiber of index 2, we refer to the assertions in the proof of Theorem 1 in [10], and the proof of Theorem 4 in [10]. Then we see that the Seifert structure of these basic blocks are extended to the outside of these. Thus in the first case, M has a Seifert structure whose orbit space is P^2 with at most one exceptional fiber. Also, we see that the type of the singular vertex V_0 is $e^+_{V_0} = e^-_{V_0} = 1$, where V_0 is untwisted. Since remaining saddle orbits are contained in solid tori (see assertions in Theorem 1 of [10]), the remaining vertices are of S^3-type by Lemma 3. It satisfies the condition (1). In the last case, M has a Seifert structure whose orbit space is S^2 with three exceptional fibers, and the basic block yields singular vertex whose vertex type is $e^+_{V_0} = e^-_{V_0} = 1$, where the vertex V_0 is twisted. The remaining vertices are of S^3-type. Thus it satisfies the condition (3).

Proof of Theorem B.

First suppose M is a lens space.

Necessity. As a neighbourhood of an attractor or a repeller is a solid torus, and as the graph is a tree by Proposition 1, the condition (1) is satisfied. Also, if V is a closed orbit, then $k_V = 1$. Thus $e^+_V = 1$ or 2 and $e^-_V = 1$ or 2 can be obtained. If V is twisted, then $e^+_V = e^-_V = 1$ (see the proof of Theorem 2 in [6]).

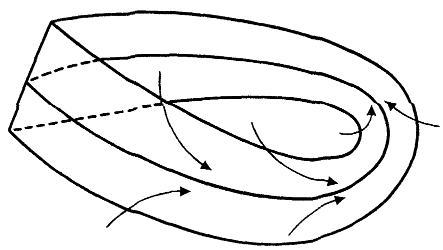

Figure (4)

Sufficiency. By using (1), (2) and (3) in Figure (3) and Figure (4), it is easy to construct an NMS flow on S^3 which has a basic block corre-

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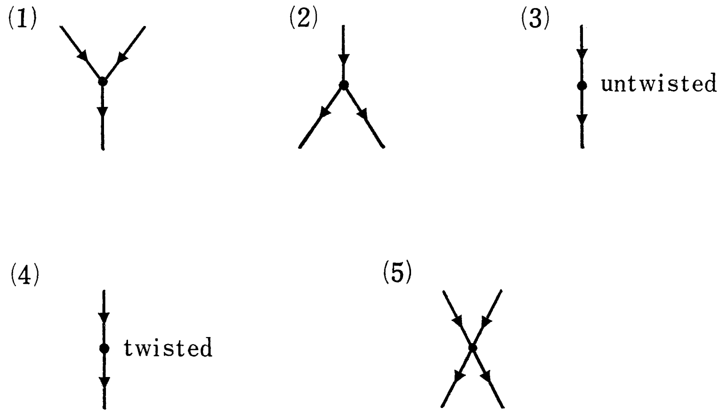

Figure (5)

sponding with the vertex type (1), (2), (3), (4) and (5) in Figure (5). Each boundary component bounds a solid torus in the both side in S^3. Therefore, by using a similar method in Theorem A, we can construct an NMS flow associated to a Lyapunov graph Γ on any lens space. Suppose, M is not a lens space.

Necessity. Since a Seifert structure of a manifold is not always unique (see Jaco [8]), under the assumption the following three possibilities occur by Proposition 4:

(1) M has a structure of a Seifert manifold whose orbit space is S^2 with three exceptional fibers and also has a structure of a Seifert manifold whose orbit space is P^2 with at most one exceptional fiber.

(2) M has a structure of a Seifert manifold whose orbit space is S^2 with three exceptional fibers but does not have a structure of a Seifert manifold whose orbit space is P^2 with at most one exceptional fiber.

(3) M has a structure of a Seifert manifold whose orbit space is P^2 with at most one exceptional fiber but does not admit a structure of a Seifert manifold whose orbit space is S^2 with three exceptional fibers.

For example, a prism manifold corresponds to case (1), and if a Seifert fibered manifold whose orbit space is S^2 with three exceptional fibers has infinite fundamental group, then it can not have a Seifert structure whose orbit space is P^2 with at most one exceptional fiber (see Jaco [8]). Then such a manifold satisfies case (2). And under our assumptions, the fact that there is no manifold which satisfies the condi-

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tion (3) is known (see Jaco [8]). The rest of our proof is a direct consequence from Proposition 4

Sufficiency. For a construction of a basic block containing a closed orbit corresponding with a singular vertex, we can use a basic block in Figure (3). Since we can adopt a standard solid torus as the outside of the basic block, we can reduce the construction of the remaining basic blocks to the case of a lens space.

§ 5. Special Lyapunov graphs

By Theorem 2 in Kim [9], we see that a lens space which contains a Klein bottle can be constructed from a basic block, an orientable S^1-bundle over a punctured Möbius band. Because a solid torus or a knot complement $S^3 - N(h)$ can not contain a Klein bottle, we see that the vertex corresponding with a closed orbit of the above basic block is a singular vertex.

On the other hand, in Theorem B we can construct any associated graph on a lens space without using a basic block which is an orientable S^1-bundle over a punctured Möbius band. This means that we can not find a singular vertex of a Lyapunov graph of an NMS flow on lens space by just looking at the ordinary Lyapunov graph. Therefore, we define a special Lyapunov graph which can distinguish a singular vertex from other vertices, by putting a mark "$*$" on the singular vertex.

DEFINITION 10. If a special Lyapunov graph has no singular vertex and satisfies the conditions (1) and (2) in the case of the lens space of Theorem B, or conditions (0), (1) and (2) of Theorem A, we call this graph a *Lyapunov graph of S^3-type.* We will provide the following propositions about a special Lyapunov graph.

PROPOSITION 5. *Suppose that ϕ_t is an NMS flow on a lens space M which does not contain a Klein bottle and suppose that M is not a projective space P^3. Let g be a Lyapunov function. Then the special Lyapunov graph Γ associated with the flow ϕ_t and the function g does not have singular vertices. In other words, the graph Γ is of S^3-type.*

Proof. Suppose that the associated Lyapunov graph Γ has a singular vertex V_0. Since we consider a Lyapunov graph on a lens space which does not contain a Klein bottle and is not homeomorphic to P^3, a basic block which has one closed orbit corresponding to V_0 is homeomorphic to one of the followings; $S^1 \times D^2 \# S^1 \times D^2$, $S^1 \times D^2 \# T^2 \times I$, $T^2 \times I \# T^2 \times I$, (two hole disk) $\times S^1$ and a Seifert manifold whose orbit space is an annulus with one exceptional fiber of index 2. Suppose that such a basic block is (two hole disk) $\times S^1$. Each connected component of $M -$ (two hole disk) $\times S^1$ is a solid torus as in Theorem B. Then if a simple closed curve $\{x\} \times S^1$ of (two hole disk) $\times S^1 \cap$ (solid torus) bounds a disk in this solid torus, the closed orbit which corresponds to the singular vertex V_0 is included in a solid torus. By Lemma 3, it is not a singular vertex. This is a contradiction. Thus the Seifert structure of (two hole disk) $\times S^1$ must extend to the outside of this basic block. Here, if a lens space has a Seifert structure whose orbit manifold is S^2, then the number of exceptional fibers is at most two. Thus (two hole disk) $\times S^1$ is excluded. For the remaining case, using the assertions in the proof of Theorem 1 in [10], we can show that a closed orbit which corresponds to a singular vertex V_0 is also contained in a solid torus. It is a contradiction. It means that there is no singular vertex in a Lyapunov graph associated with the ϕ_t and the Lyapunov function g on such a lens space.

COROLLARY 1. *Suppose that ϕ_t is an NS flow with a Lyapunov function g on a lens space which does not contain a Klein bottle and is not a projective space P^3. Let Γ be the special Lyapunov graph associated with the flow ϕ_t and the Lyapunov function g. If Γ has a vertex labelled with a saddle type closed orbit, then the vertex can not be a singular vertex.*

This corollary is shown by the above proof.

PROPOSITION 6. *Suppose that ϕ_t is an NMS flow on $L(4p, 2p-1)$ or P^3 with a Lyapunov function g. Then the special Lyapunov graph Γ associated with the flow ϕ_t and the function g satisfies followings:*

(1) *A special Lyapunov graph on $L(4p, 2p-1)$ associated with a flow ϕ_t and a function g is a graph of S^3-type or a graph which satisfies the followings; there is one singular vertex V_0 satisfying $e^+_{V_0} = e^-_{V_0} = 1$, and it is untwisted. The remaining vertices are of S^3-type.*

(2) *Special Lyapunov graph Γ on P^3 associated with a flow ϕ_t and a function g is a graph of S^3-type or a graph which satisfies the followings; there is one singular vertex V_0 satisfying $e^+_{V_0} = e^-_{V_0} = 1$ and it is twisted. The remaining vertices are of S^3-type.*

Proof. If a flow ϕ_t on $L(4p, 2p-1)$ or P^3 has no closed orbit of saddle type, the graph associated with a flow ϕ_t and Lyapunov function g must consist of only one sink vertex and one source vertex. It means that the graph is of S^3-type. Thus we may suppose that a flow ϕ_t on $L(4p, 2p-1)$ or P^3 has at least one closed orbit of saddle type. Let V_0 be a vertex which corresponds to a saddle orbit, and also we assume that the vertex V_0 is a singular vertex. If there is no such a vertex, the graph Γ is of S^3 type. Thus if we cut Γ at any cut point C_{V_0} on any edge of V_{V_0}, a component of $\Gamma - \{C_{V_0}\}$ which does not contain V_0 corresponds to a submanifold of M which is homeomorphic to a solid torus or a knot-complement $S^3 - N(h)$, where the boundary of a compressing disk of $\partial(S^3 - N(h))$ is a meridian of this knot h.

Case (1). Suppose that a basic block which contains a closed orbit of saddle type corresponding to the singular vertex V_0 is one of followings; $S^1 \times D^2 \# S^1 \times D^2$, $S^1 \times D^2 \# T^2 \times I$, $T^2 \times I \# T^2 \times I$, (two hole disk) $\times S^1$, and a Seifert space whose orbit space is an annulus with one exceptional fiber of index 2. Then, we can show that if these basic blocks contain a closed orbit of saddle type corresponding to the vertex V_0, the closed orbit is contained in a solid torus or a knot-complement as follows. This means the vertex V_0 is not a singular vertex. Thus we can omit all basic blocks in the above cases. For example let a basic block be a Seifert space whose orbit manifold is an annulus with one exceptional fiber of index 2. Since $L(4p, 2p-1)$ has no incompressible tori, one component of $L(4p, 2p-1)$ − (the above basic block B) must be a solid torus. By the assertions 3 in the proof of Theorem 1 of [10], we see that a fiber of this basic block does not bound a disk in this solid torus. Otherwise, $L(4p, 2p-1)$ is homeomorphic to P^3. This means that a Seifert structure extends to this soild torus. Then another component of $L(4p, 2p-1) - B$ is a solid torus, and also this Seifert structure extends to this solid torus. If $L(4p, 2p-1)$ has a Seifert structure whose orbit space is S^2, the number of singular fibers is at most two. Then the vertex V_0 is not a singular vertex. Therefore, suppose that a basic block which contains a closed orbit of saddle type is homeomorphic to an orientable S^1-bundle over a punctured Möbius band. Since a solid torus and a non-trivial knot-complement $S^3 - N(h)$ can not contain a Klein bottle and since the vertex V_0 corresponds a closed orbit saddle type of an orientable S^1-bundle over a punctured Möbius band, there is one singular vertex V_0 such that $e^+_{V_0} = e^-_{V_0} = 1$ and it is untwisted. For the remaining vertices, they are contained in a solid torus or a non-trivial knot-complement by Proposition 3. Thus they are of S^3-type by Lemma 3.

Case (2). Suppose that a basic block which contains a closed orbit of saddle type corresponding to a singular vertex V_0 is one of followings; $S^1 \times D^2 \# S^1 \times D^2$, $S^1 \times D^2 \# T^2 \times I$, $T^2 \times I \# T^2 \times I$ and (two hole disk) $\times S^1$. Then we can show that the closed orbit of saddle type corresponding to the vertex V_0 is contained in a solid torus or a knot-complement by a proof similar to case (1). This is a contradiction. Then we can omit these basic blocks. Suppose that a basic block B which has a closed orbit of saddle type is a Seifert space whose orbit manifold is an annulus with one exceptional fiber of index 2. Since P^3 has no incompressible torus, one component of $P^3 - B$ is a solid torus. If a Seifert structure of this basic block B extends to this solid torus, another component of $P^3 - B$ must be a solid torus. Otherwise, P^3 admits an incompressible torus, or it is homeomorphic to S^3 by Lemma 2. If a fiber of this basic block B bounds a disk D^2 in this solid torus, and if another component of $P^3 - B$ is a non-trivial knot-complement $S^3 - N(h)$, then P^3 is homeomorphic to S^3. Because the above disk D^2 is a meridian of h, P^3 is homeomorphic to (solid torus) $\cup_T$ (non-trivial knot-complement). Thus P^3 is homeomorphic to S^3 by Lemma 2. Then each component of $P^3 - B$ is a solid torus. Using the proof of Theorem 2 in [10], we see that a closed orbit saddle type of the above basic block is contained in a solid torus. Then, in this case, the vertex V_0 is not a singular vertex. This is a contradiction. Thus, this basic block is also omitted. A solid torus and a knot-complement $S^3 - N(h)$ can not contain a projective plane P^2. Therefore, if such a basic block is homeomorphic to $P^3 \# T^2 \times I$, the vertex V_0 corresponding to the closed orbit of saddle type of $P^3 \# T^2 \times I$ is a singular vertex. This vertex type is $e^+_{V_0} = e^-_{V_0} = 1$, where V_0 is twisted. Another vertex is of S^3-type, because another closed orbit is contained in a solid torus or a knot-complement. Then, Proposition 6 is proved.

Remark 9. For the case (1), since $L(4p, 2p-1)$ is not homeomorphic to P^3, a basic block is not homeomorphic to $P^3 \# T^2 \times I$, For the case (2), a basic block is not homeomorphic to an orientable S^1-bundle over a punctured Möbius band, since P^3 does not contain a Klein bottle.

§ 6. Examples of special Lyapunov graphs of NS flows

By using a construction in [5], we can show that there exist some examples concerning an NS flow which is not an NMS flow and whose associated Lyapunov graph has one singular vertex on $L(4p, 2p-1)$ and a Seirfert space whose orbit manifold is S^2 with three exceptional fibers.

Case (1). *A Seifert space whose orbit manifold is* S^2 *with three exceptional fibers*; A graph associated with a flow ϕ_t and a Lyapunov function g has one singular vertex V_0 such that $e^+_{V_0} \leqq k_{V_0} + 1$, $e^-_{V_0} \leqq k_{V_0} + 1$ and $k_{V_0} + 2 = e^+_{V_0} + e^-_{V_0}$. Any other vertex V which is neither an attractor nor a repeller satisfies the following conditions; $e^+_V \leqq k_V + 1$, $e^-_V \leqq k_V + 1$ and $e^+_V + e^-_V \geqq k_V + 1$.

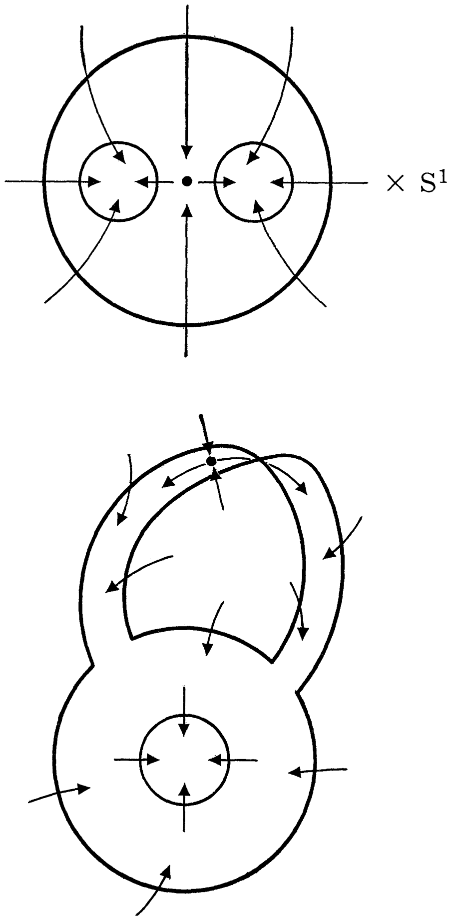

Figure (6)

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Case (2). *A prism manifold and a lens space* $L(4p, 2p-1)$; A Lyapunov graph associated with a flow ϕ_t and a Lyapunov function g has a singular vertex V_0 such that $e^+_{V_0} \leqq k_{V_0} + 1$, $e^-_{V_0} \leqq k_{V_0} + 1$ and $k_{V_0} + 1 = e^+_{V_0} + e^-_{V_0}$, and also any other vertex V which is neither attracting nor repelling satisfies the following conditions; $e^+_V \leqq k_V + 1$, $e^-_V \leqq k_V + 1$ and $k_V + 1 \leqq e^+_V + e^-_V$. For our construction of examples of case (1) (resp. case (2)) we use a two hole disk (resp. a punctured Möbius band) on a basic block (two hole disk) $\times S^1$ (resp. an orientable S^1-bundle over a punctured Möbius band) as a cross section of the flow ϕ_t first return map has one saddle point (see Figure (6)). Next, we add new fixed points on this cross section as follows.

Case (1). We add $(k_{V_0} - e^+_{V_0})$ pairs of a sink and a saddle, and also add $(k_{V_0} + 1 - e^-_{V_0})$ pairs of a source and a saddle. Finally we add $(n - k_{V_0})$ nilpotent handles.

Case (2). We add $(k_{V_0} - e^+_{V_0})$ pairs of a sink and a saddle, and also add $(k_{V_0} - e^-_{V_0})$ pairs of a source and a saddle. Finally we add $(n - k_{V_0})$ nilpotent handles.

Then a basic block which contains a basic set corresponding to a singular vertex V_0 in case (1) and case (2) is obtained by a method similar to the proof of Theorem 1 of [6]. Now, we will construct a fitted NS flow whose Lyapunov graph satisfies case (1) on a Seifert fibered manifold M whose orbit space is S^2 with three exceptional fibers. For case (2), we can construct a fitted NS flow similary. Let B be a basic block which has a basic set corresponding to a singular vertex V_0. And let $X^i_1, X^i_2, \cdots, X^i_{N_i}$ be basic blocks contained in $M - B$ $(1 \leqq i \leqq k_{V_0} + 2)$. Because we can regard each component of $M - B$ as a standard solid torus, we can adopt basic block as in Theorem 1 of [6] for $X^i_1, X^i_2, \cdots, X^i_{N_i}$ $(1 \leqq i \leqq k_{V_0} + 2)$. Thus they can be embedded in S^3 and their boundary components are all standard tori. Then by choosing attaching maps $g^i_1, g^i_2, \cdots, g^i_{N_i-1}$ $(1 \leqq i \leqq k_{V_0} + 2)$, we can construct $X^i_1 \bigcup_{g^i_1} X^i_2 \bigcup_{g^i_2} \cdots \bigcup_{g^i_{N_i-1}} X^i_{N_i}$ as a standard solid torus $(1 \leqq i \leqq k_{V_0} + 2)$. Next, choosing attaching maps $h_1, h_2, \cdots, h_{k_{V_0}+2}$, we glue these basic blocks $X^i_1 \bigcup_{g^i_1} X^i_2 \bigcup_{g^i_2} \cdots \bigcup_{g^i_{N_i-1}} X^i_{N_i}$ $(1 \leqq i \leqq k_{V_0} + 2)$ along the boundaries of the basic block B. Then we can construct a non-singular flow ϕ_t on a Seifert fibered manifold whose orbit space is S^2 with three exceptional fibers. By changing each g^i_j $(1 \leqq j \leqq N_i - 1)$ $(1 \leqq i \leqq k_{V_0} + 2)$, into $g^{i'}_j$ which is isotopic to g^i_j, and also changing each h_s into h'_s $(1 \leqq s \leqq k_{V_0} + 2)$ which is isotopic to h_s, we can reconstruct a fitted flow ϕ'_t from the flow ϕ_t by Proposition 2. Let $X^i_{K_i}$ $(1 \leqq K_i \leqq N_i)$ be a basic block which attaches to the basic block B $(1 \leqq i \leqq k_{V_0} + 2)$. Since each boundary of these basic blocks is a standard torus in S^3, then we can assume that $X^i_1 \cup_{g^i_1} \cdots \cup_{g^i_{K_i-2}} X_{K_i-1}$, $X^i_1 \cup_{g^i_1} \cdots \cup_{g^i_{K_i-2}} X^i_{K_i-1} \cup_{g^i_{K_i-1}} X^i_{K_i} \cup_{g^i_{K_i}} \cdots \cup_{g^i_{N_i-1}} X^i_{N_i}$, $X^i_{K_i+1} \cup_{g^i_{K_i+1}} \cdots \cup_{g^i_{L_i-1}} X_{L_i}$, $X^i_{K_i+2} \cup_{g^i_{K_i+2}} \cdots \cup_{g^i_{L_i-1}} X_{L_i}, \cdots$, and X_{N_i} are solid tori, where the basic blocks X_{L_i} and X_{N_i} $(1 \leqq k_i \leqq L_i \leqq N_i)$ contain a repelling closed orbit or an attracting closed orbit. (If necessary, we suitably change the lower indexes of basic blocks $X^i_1, X^i_2, \cdots$, and $X^i_{N_i}$). By using Alexander trick, we see that $B \cup_{h'_s} (X^s_1 \cup_{g^s_1} X^s_2 \cup_{g^s_2} \cdots \cup_{g^s_{N_s-1}} X^s_{N_s})$ is homeomorphic to $B \cup_{h_s} (X^s_1 \cup_{g^s_1} X^s_2 \cup_{g^s_2} \cdots \cup_{g^s_{N_s-1}} X_{N_s})$, $B \cup_{h'_s} (X^s_1 \cup_{g^{s\prime}_1} \cdots \cup_{g^{s\prime}_{K_s-2}} X^s_{K_s} \cup_{g^{s\prime}_{K_s-1}} \cdots \cup_{g^s_{N_s-1}} X^s_{N_s})$ is homeomorphic to $B \cup_{h'_s} (X^s_1 \cup_{g^{s\prime}_1} \cdots \cup_{g^{s\prime}_{K_s-2}} X^s_{K_s-1} \cup_{g^s_{K_s-1}} \cdots \cup_{g^s_{N_s-1}} X_{N_s})$, $B \cup_{h'_s} (X^s_1 \cup_{g^{s\prime}_1} \cup_{g^{s\prime}_{K_s-1}} X^s_{K_s} \cup_{g^{s\prime}_{K_s}} X^s_{K_s+1} \cup_{g^s_{K_s+1}} \cdots \cup_{g^s_{N_s-1}} X_{N_s})$ is homeomorphic to $B \cup_{h'_s} (X^s_1 \cup_{g^{s\prime}_1} \cdots \cup_{{}^{s\prime}_{K_s-2}} X^s_{K_s-1} \cup_{g^{s\prime}_{K_s-1}} X^s_{K_s} \cup_{g^s_{K_s}} \cdots \cup_{g^s_{N_s-1}} X_{N_s})$, etc. Thus we see that a homeomorphism type of M does not change.

Remark 10. Of course, if a singular vertes V_0 corresponds to a closed orbit of saddle type, we can construct an NS flow which is not an NMS flow on P^3, $L(4p, 2p-1)$ and a Seifert fiberd space whose orbit manifold is S^2 with three exceptional fibers by similar construction.

QUESTION. Does any Lyapunov graph on $L(2p-1, q)$ associated with an NS flow ϕ_t and a Lyapunov function g admit no singular vertex?

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