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THE DIRICHLET PROBLEM FOR BAIRE-TWO FUNCTIONS ON SIMPLICES

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Abstract

We show that solvability of the abstract Dirichlet problem for Baire-two functions on a simplex X cannot be characterized by topological properties of the set of extreme points of X.

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1. Introduction

Let X be a compact convex subset of a locally convex space, let $\mathfrak{A}(X)$ stand for the space of all continuous affine functions on X and let ext X denote the set of all extreme points of X. If f is a bounded function on ext X, we may ask under what conditions f admits an affine extension that preserves as many properties of f as possible. This question is called the *abstract Dirichlet problem* (see [5, Theorem 3.17]).

The question of solvability of the abstract Dirichlet problem naturally leads to a geometric notion of a simplex (see [5, Section 3]). If X is a simplex, every bounded continuous function defined on ext X can be extended to an affine continuous function on X if and only if ext X is closed (see [5, p. 615] or [1, Satz 2]).

An analogous problem for Baire-one functions on simplices was solved in [16, Theorem 1], namely, every bounded Baire-one function defined on ext X is extendible to an affine Baire-one function on X if and only if ext X is a Lindelöf H-set.

Both these conditions characterize solvability of the abstract Dirichlet problem for certain classes of functions purely by a topological condition imposed on ext X. In particular, if X_1, X_2 are simplices whose sets of extreme points are homeomorphic, the abstract Dirichlet problem for continuous (or Baire-one) functions is always solvable on X_1 if and only if it is always solvable on X_2.

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These results prompt a natural question whether it is possible to provide such a characterization for functions of higher Baire classes. Since affine functions of Baire class two need not satisfy the barycentric formula, it is more reasonable to look for Baire-two strongly affine extensions. (We recall that a universally measurable function $f \in \mathcal{U}(X)$ on a compact convex set X satisfies the *barycentric formula* (or is *strongly affine*) if $\mu(f) = f(r(\mu))$, $\mu \in \mathcal{M}^1(X)$, where $r(\mu)$ is the barycenter of a probability measure μ on X. It is easy to see that any strongly affine function is bounded; see, for example, [8, Satz 2.1].)

The aim of our paper is to show that simplices, whose sets of extreme points are homeomorphic, may behave quite differently from the point of view of the abstract Dirichlet problem for Baire-two functions. Indeed, we obtain a stronger result in the following theorem.

THEOREM 1.1. *There exist metrizable simplices X_1, X_2 and a homeomorphism $\varphi : \overline{\operatorname{ext} X_1} \to \overline{\operatorname{ext} X_2}$ such that:*

(a) $\varphi(\operatorname{ext} X_1) = \operatorname{ext} X_2$*;*
(b) *there exists a bounded Baire-two function on* $\operatorname{ext} X_1$ *that cannot be extended to a Baire-two affine function on X_1;*
(c) *if $\alpha \in [2, \omega_1)$, any bounded Baire-α function on* $\operatorname{ext} X_2$ *can be extended to a function of affine class α on X_2.*

If $\mathcal{F}$ is a set of real-valued functions, we inductively define the following sets of functions: we set $\mathcal{F}_0 = \mathcal{F}$ and, with $\mathcal{F}_\beta$, $\beta < \alpha$, already defined for an ordinal number $\alpha \in (0, \omega_1)$, we define $\mathcal{F}_\alpha$ to be the space of all pointwise limits of bounded sequences of functions from $\bigcup_{\beta<\alpha} \mathcal{F}_\beta$. If X is a topological space, we write $\mathcal{B}^b_\alpha(X) = (\mathcal{C}(X))_\alpha$ for the space of all bounded Baire functions of class α, $\alpha \in [0, \omega_1)$. If $\mathcal{F} = \mathfrak{A}(X)$, the space $\mathfrak{A}_\alpha(X) = (\mathfrak{A}(X))_\alpha$ is called the *functions of affine class α*.

The proof of Theorem 1.1 is a modification of the construction used in [14], where a simplex with peculiar properties was presented. The main tool was to find a suitable *function space* and transfer its properties to a compact convex set. (By a function space $\mathcal{H}$ on a compact space K we mean a linear subspace of the space $\mathcal{C}(K)$ of all continuous functions on K such that $\mathcal{H}$ contains constants and separates points of K.) The idea of the construction used in [14] was to start with a simple function space and inductively increase its complexity. At the end the projective limit of the constructed function spaces was taken.

It turns out that a variant of this construction can be used to produce examples required by Theorem 1.1. The inductive construction goes as follows: we start with a simple function space $\mathcal{H}_0$ on the unit interval $[0, 1]$ and a set $A \subset [0, 1]$ and increase the complexity of $\mathcal{H}_0$ in two different ways. Roughly speaking, the first modification ensures that points of A are split up infinitely many times, whereas the second modification splits the points up only once. But both procedures provide function spaces with the same Choquet boundaries. At the end we take the projective limits of constructed spaces to get a pair of function spaces on a compact space that give rise to the required examples.

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Since a rather detailed survey of function spaces and their properties is presented in [14], for the sake of brevity we shall follow the notation and definitions in [14].

We recall that $\mathcal{U}^b(K)$ stands for the space of all *bounded universally measurable* functions on a compact space K (that is, functions that are $\overline{\mu}$-measurable with respect to the completion $\overline{\mu}$ of any Radon measure $\mu \in \mathcal{M}^+(K)$). If $\mathcal{F} \subset \mathcal{U}^b(K)$, we write $\mathcal{F}^\perp$ for the space of all measures $\mu \in \mathcal{M}(K)$ with $\mu(f) = 0$ for each $f \in \mathcal{F}$, and $\mathcal{F}^{\perp\perp}$ for the space of all bounded universally measurable functions f satisfying $\mu(f) = 0$ for each $\mu \in \mathcal{F}^\perp$.

2. Auxiliary results

The following notion will be useful in the main construction.

DEFINITION 2.1. We say that a function space $\mathcal{H}$ on a compact space K is *Baire-one complemented* if there exists a mapping $x \mapsto \mu_x$, $x \in K$, such that:

(i) $\mu_x \in \mathcal{M}(K)$ and $\sup\{\|\mu_x\| : x \in K\} < \infty$;
(ii) $\mu_x(h) = h(x)$ for each $x \in K$ and $h \in \mathcal{H}$;
(iii) if $f \in \mathcal{B}_1^b(K)$ and $h(x) = \mu_x(f)$, $x \in K$, then $h \in \mathcal{B}_1^b(K) \cap \mathcal{H}^{\perp\perp}$.

REMARK 2.2. If $x \mapsto \mu_x$, $x \in K$, is the mapping from Definition 2.1, the mapping $P : \mathcal{B}_1(K) \to \mathcal{B}_1^b(K) \cap \mathcal{H}^{\perp\perp}$ defined as $Pf(x) = \mu_x(f)$, $x \in K$, $f \in \mathcal{B}_1^b(K)$, is a projection of $\mathcal{B}_1^b(K)$ onto $\mathcal{B}_1^b(K) \cap \mathcal{H}^{\perp\perp}$. Since it follows from [9, Theorem 5.1] that $\mathcal{B}_1^b(K) \cap \mathcal{H}^{\perp\perp} = \mathcal{H}_1$, the projection P maps $\mathcal{B}_1^b(K)$ onto $\mathcal{H}_1$.

As in [14, Lemma 3.3], we start with the following classical family of sets (see [11, pp. 82–86] or [6, Lemma 2.3]).

2.1. Family of sets Let $\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ be a family of subsets of $[0, 1]$ such that:

(a) $F_\emptyset = [0, 1]$;
(b) $\{F_{s^\wedge n} : n \in \mathbb{N}\}$ is a disjoint family of nonempty nowhere dense perfect subsets of F_s;
(c) $\bigcup\{F_{s^\wedge n} : n \in \mathbb{N}\}$ is dense in F_s;
(d) $\operatorname{diam} F_s < 2^{-(s_1+\cdots+s_{|s|})}$, $s \in \mathbb{N}^{<\mathbb{N}}$.

We remark that the set $\bigcap_{n=1}^\infty \bigcup_{|s|=n} F_s \in \mathbf{\Pi}_3^0([0, 1]) \setminus \mathbf{\Sigma}_3^0([0, 1])$ (we refer the reader to [7, Ch. II, Section 11.a] for the notation concerning Borel classes of sets).

LEMMA 2.3. *Let $\mathcal{H}$ be a Baire-one complemented function space on a compact space K. Then $\mathcal{B}_2^b(K) \cap \mathcal{H}^{\perp\perp} = \mathcal{H}_2$.*

PROOF. Assume that $P : \mathcal{B}_1^b(K) \to \mathcal{H}_1$ is the projection given by a mapping $x \mapsto \mu_x$, $x \in K$, that satisfies the properties from Definition 2.1.

Given $f \in \mathcal{B}_2^b(K) \cap \mathcal{H}^{\perp\perp}$, let $\{f_n\}$ be a bounded sequence of functions from $\mathcal{B}_1^b(K)$ pointwise converging to f. Then $Pf_n \in \mathcal{H}_1$, $n \in \mathbb{N}$, and $Pf_n \to Pf$ by the Lebesgue dominated convergence theorem. Thus $f = Pf \in \mathcal{H}_2$. $\square$

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Before proceeding, we recall that a probability measure μ on a compact space K is termed *discrete* if $\mu = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$, where the sum is either finite or infinite, numbers a_n are positive, $\sum_{n=1}^{\infty} a_n = 1$ and points x_n lie in K. We mention the following well-known easy observation.

LEMMA 2.4. *Let f be an affine bounded function on a compact convex set X and $\mu \in \mathcal{M}^1(X)$ be discrete. Then $\mu(f) = f(r(\mu))$.*

3. Construction of function spaces

The construction of suitable simplicial function spaces will be done by a modification of the method used in [14]. Assume that $\mathcal{H}$ is a simplicial function space on a metrizable compact space K such that $\mathcal{A}_c(\mathcal{H}) = \mathcal{H}$. Let T be the *kernel* associated with the mapping $x \mapsto \delta_x$, $x \in K$. (We recall that δ_x is the unique $\mathcal{H}$-maximal measure $\mathcal{H}$-representing a point $x \in K$. We refer the reader to [2, p. 38] for the definition of a kernel.) Assume that $Tf \in \mathcal{B}_2(K)$ for each bounded Baire-two function f on K.

Let $\{F_k : k \in \mathbb{N}\}$ be a pairwise disjoint family of compact subsets of $\mathrm{Ch}_{\mathcal{H}} K$ and let $\eta \in (0, 1)$.

Let $\mathcal{H}$ be Baire-one complemented by a projection P with $\|P\| \le 3$ such that $Pf = f$ on $\bigcup_{k=1}^{\infty} F_k$.

We define sets $L_1, L_2, L \subset K \times \mathbb{R}$ as

$$\begin{aligned} L_1 &= \bigcup_{k=1}^{\infty} (F_k \times \{1/k\}) \cup (F_k \times \{-1/k\}), \\ L_2 &= \bigcup_{k=1}^{\infty} (F_k \times \{2/k\}) \cup (F_k \times \{-2/k\}), \\ L &= (K \times \{0\}) \cup L_1 \cup L_2. \end{aligned}$$

Let $p : L \to K$ denote the natural projection. Then L is a metrizable compact space with the topology inherited from $K \times \mathbb{R}$ and K can be considered as a subspace of L via the mapping $x \mapsto (x, 0)$, $x \in K$. Let

$$\begin{aligned} \mathcal{H}^1 = \{f \in \mathcal{C}(L) :\ & f|_K \in \mathcal{H} \text{ and} \\ & f(x, 0) = cf(x, 1/k) + (1 - c) f(x, -1/k),\ x \in F_k,\ k \in \mathbb{N}\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \mathcal{H}^2 = \{f \in \mathcal{C}(L) :\ & f|_K \in \mathcal{H} \text{ and} \\ & 2f(x, 0) = f(x, 2/k) + f(x, -2/k),\ x \in F_k,\ k \in \mathbb{N}\}. \end{aligned} \tag{3.2}$$

Let S denote the kernel on L associated with the mapping

$$x \mapsto \begin{cases} \varepsilon_x & \text{if } x \in L \setminus \bigcup_{k=1}^{\infty} F_k, \\ \frac{1}{2}(\varepsilon_{(u,2/k)} + \varepsilon_{(u,-2/k)}) & \text{if } x = (u, 0),\ u \in F_k,\ k \in \mathbb{N}. \end{cases}$$

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LEMMA 3.1. *The following assertions hold:*

(a) *both $\mathcal{H}^1$ and $\mathcal{H}^2$ are simplicial function spaces (for $i = 1, 2$, let δ_x^i denote the unique $\mathcal{H}^i$-maximal measure for $x \in L$ and let T^i be the kernel associated with the mapping $x \mapsto \delta_x^i$, $x \in L$);*
(b) *$\mathcal{H}^i = \mathcal{A}_c(\mathcal{H}^i)$, $i = 1, 2$;*
(c) $\mathrm{Ch}_{\mathcal{H}^1} L = \mathrm{Ch}_{\mathcal{H}^2} L = (L \setminus K) \cup (\mathrm{Ch}_{\mathcal{H}} K \setminus \bigcup_{k=1}^{\infty} F_k)$;
(d) *the mapping $h \mapsto h \circ p$, $h \in \mathcal{H}$, provides an isometric embedding of $\mathcal{H}$ into $\mathcal{H}^1 \cap \mathcal{H}^2$;*
(e) $S\delta_x = \delta_x^2$, $x \in K$;
(f) *$T^2 f \in \mathcal{B}_2^b(L)$ for each $f \in \mathcal{B}_2^b(L)$;*
(g) *if $f \in \mathcal{U}^b(L)$ satisfies equations (3.2) and $f|_K \in \mathcal{A}(\mathcal{H})$, then $f \in \mathcal{A}(\mathcal{H}^2)$;*
(h) *$\mathcal{H}^2$ is Baire-one complemented by a projection Q with $\|Q\| \leq 3$ such that, for each $f \in \mathcal{B}_1^b(L)$,*
    (h1) *$(Qf)|_K = P(f|_K)$, and*
    (h2) *$(Qf)|_{L_1} = f|_{L_1}$.*

PROOF. Since the proof is a slight modification of [14, Lemma 5.1], we point out only the changes that have to be made.

First we observe that (a), (b), (c) and (d) can be proved in exactly the same way as in [14, Lemma 5.1].

If $x \in K$, then $S\delta_x$ is carried by $\mathrm{Ch}_{\mathcal{H}^2} L$ and $S\delta_x \in \mathcal{M}_x(\mathcal{H}^2)$. Thus

$$S\delta_x = \delta_x^2.$$

This proves (e).

Next we verify (f). Assuming that $Tf \in \mathcal{B}_2^b(K)$ for each $f \in \mathcal{B}_2^b(K)$, let f be a bounded Baire-two function on L. We need to show that $T^2 f \in \mathcal{B}_2^b(L)$. We notice that $Sf \in \mathcal{B}_2^b(L)$.

By (e), for each $x \in K$ we get

$$\begin{aligned}(T^2 f)(x) &= \delta_x^2(f) = (S\delta_x)(f)\\ &= \delta_x(Sf) = T((Sf)|_K)(x).\end{aligned}$$

Since $Tf \in \mathcal{B}_2^b(K)$ for each $f \in \mathcal{B}_2^b(K)$ by our assumption, $T^2 f$ is a Baire-two function on K. Since $T^2 f = f$ on the open set $L \setminus K$, $T^2 f \in \mathcal{B}_2^b(L)$.

To verify (g), let $f \in \mathcal{U}^b(L)$ satisfy the hypothesis. Given $x \in K$, (e) implies that

$$\delta_x^2(f) = (S\delta_x)(f) = \delta_x(Sf) = \delta_x(f) = f(x).$$

Obviously, $\delta_x^2(f) = f(x)$ for every $x \in L \setminus K$. Using [14, Lemma 2.7], we conclude that $f \in \mathcal{A}(\mathcal{H}^2)$.

For the proof of (h), let $x \mapsto \mu_x$, $x \in K$, be the mapping that generates the projection P guaranteed by the assumption. By our hypothesis, $\mu_x = \varepsilon_x$ for every

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$x \in \bigcup_{k=1}^{\infty} F_k$. We extend this mapping on the whole space L by setting

$$\mu_x^2 = \begin{cases} \mu_x & \text{if } x \in K, \\ \varepsilon_x & \text{if } x \in L_1 \cup (L_2 \cap (K \times (-\infty, 0))), \\ 2\varepsilon_{(u,0)} - \varepsilon_{(u,-2/k)} & \text{if } x = (u, 2/k),\ u \in F_k,\ k \in \mathbb{N}. \end{cases} \tag{3.3}$$

Then

$$Qf(x) = \mu_x^2(f), \quad x \in L,\ f \in \mathcal{B}_1^b(L),$$

is the required projection. Indeed, it is easy to verify that $Qf \in \mathcal{B}_1^b(L)$ for $f \in \mathcal{B}_1^b(L)$ and $\|Q\| \le 3$. Also conditions (h1) and (h2) are satisfied. To show that $Qf \in \mathcal{A}(\mathcal{H}^2)$, we realize that Qf satisfies the assumptions of (g). Indeed, $Qf|_K \in \mathcal{A}(\mathcal{H})$ by (h1) and (3.3) yields validity of equations (3.2) for Qf. As $\mathcal{H}^2$ is simplicial and $\mathcal{H}^2 = \mathcal{A}_c(\mathcal{H}^2)$, [14, Theorem 2.6(b2)] yields $Qf \in (\mathcal{H}^2)^{\perp\perp}$. This concludes the proof. □

3.1. Inductive construction Let $\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ be the family of perfect sets in $[0, 1]$ provided by Lemma 2.1 and let $A = \bigcap_{n=1}^{\infty} \bigcup_{|s|=n} F_s$. Let $\{\eta_n\}$ be a sequence of numbers in $(0, 1)$ such that

$$\sum_{i=1}^{\infty} (1 - \eta_i) < \infty. \tag{3.4}$$

For every $n \ge 0$, we construct by induction:

(i) simplicial function spaces $\mathcal{H}_n^1$, $\mathcal{H}_n^2$ on a metrizable compact space $K_n \subset \mathbb{R}^{n+1}$ such that $\mathrm{Ch}_{\mathcal{H}_n^1} K_n = \mathrm{Ch}_{\mathcal{H}_n^2} K_n$ and $\mathcal{H}_n^2$ is Baire-one complemented by a projection P_n of norm at most 3;

(ii) closed subsets L_n^1, L_n^2 of K_n;

(iii) a countable family $\mathcal{F}_n = \{F_n(k) : k \in \mathbb{N}\}$ of pairwise disjoint compact sets in $\mathrm{Ch}_{\mathcal{H}_n^1} K_n$; and

(iv) a continuous surjection $p_{n+1} : K_{n+1} \to K_n$ as follows.

In the first step, let $K_0 = L_0^1 = L_0^2 = [0, 1]$, $\mathcal{H}_0^1 = \mathcal{H}_0^2 = \mathcal{C}([0, 1])$, P_0 be the identity mapping and $\mathcal{F}_0 = \{F_s : |s| = 1\}$. Assume that the objects have been defined for each $k = 0, \dots, n$. To construct $\mathcal{H}_{n+1}^2$, we use Lemma 3.1 for K_n, $\mathcal{F}_n$, η_n and $\mathcal{H}_n^1$ to get $K_{n+1}, L_{n+1}^1, L_{n+1}^2, p_{n+1} : K_{n+1} \to K_n$ and new simplicial function spaces $\widehat{\mathcal{H}}_1, \widehat{\mathcal{H}}_2$ on K_{n+1}. We set $\mathcal{H}_{n+1}^1 = \widehat{\mathcal{H}}_1$. Since $\mathrm{Ch}_{\mathcal{H}_n^1} K_n = \mathrm{Ch}_{\mathcal{H}_n^2} K_n$, we can use Lemma 3.1 again for the same objects; we only replace $\mathcal{H}_n^1$ by $\mathcal{H}_n^2$ and get another pair of simplicial function spaces $\widetilde{\mathcal{H}}_1, \widetilde{\mathcal{H}}_2$ on K_{n+1}. In this case we set $\mathcal{H}_{n+1}^2 = \widetilde{\mathcal{H}}_2$.

If the family $\mathcal{F}_n$ was enumerated as $\mathcal{F}_n = \{F(k) : k \in \mathbb{N}\}$, for each $k \in \mathbb{N}$ and a sequence $s \in \mathbb{N}^{n+2}$ of length $n + 2$, we consider the following couple of sets:

$$\begin{aligned} F(s, k, +) = \{&x = (x(0), \dots, x(n+1)) \in K_{n+1} : \\ &p_{n+1}(x) = (x(0), \dots, x(n)) \in F(k),\ x(0) \in F_s,\ x(n+1) = 1/k\}, \\ F(s, k, -) = \{&x = (x(0), \dots, x(n+1)) \in K_{n+1} : \\ &p_{n+1}(x) = (x(0), \dots, x(n)) \in F(k),\ x(0) \in F_s,\ x(n+1) = -1/k\}. \end{aligned}$$

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We set

$$\mathcal{F}_{n+1} = \{F(s, k, +),\ F(s, k, -) : s \in \mathbb{N}^{n+2}, k \in \mathbb{N}\}.$$

Let

$$P_{n+1} : \mathcal{B}_1^b(K_{n+1}) \to \mathcal{B}_1^b(K_{n+1}) \cap (\mathcal{H}_{n+1}^2)^{\perp\perp}$$

be the projection from Lemma 3.1(h). This finishes the inductive step.

3.2. Definition of function spaces We define the function spaces similarly as in [14, Section 5.2]. We have obtained sequences $\{\mathcal{H}_n^i\}$, $i = 1, 2$, of simplicial spaces on compact metrizable spaces $\{K_n\}$ together with surjective mappings p_n – in short,

$$K_0 \overset{p_1}{\leftarrow} K_1 \overset{p_2}{\leftarrow} K_2 \leftarrow \cdots . \tag{3.5}$$

Let $K = \varprojlim K_n$ be the limit of the inverse system (3.5) (see [4, Chapter 2.5]) of the sequence $\{K_n\}$, that is,

$$K = \left\{x = \{x_n\} \in \prod_{n=0}^{\infty} K_n : p_{n+1}(x_{n+1}) = x_n,\ n \ge 0\right\}$$

with the product topology. Then K is a metrizable compact space and we can consider each compact space K_n homeomorphically embedded in K via the mapping

$$\begin{aligned} e_n :\ & K_n \to K : \\ & x \mapsto ((p_1 \circ \cdots \circ p_n)(x), \ldots, (p_{n-1} \circ p_n)(x), p_n(x), \overset{n\text{th}}{x}, x, \ldots). \end{aligned}$$

Conversely, we can define retractions of K onto K_n as

$$r_n : K \to K_n : \{x_n\} \mapsto (x_0, \ldots, x_{n-1}, x_n, x_n, x_n \ldots).$$

Using these mappings, we can regard each function space $\mathcal{H}_n^i$, $i = 1, 2$, as a subspace of $\mathcal{C}(K)$; specifically, we use the mapping

$$h \mapsto h \circ r_n, \quad h \in \mathcal{H}_n^i, \quad i = 1, 2.$$

In what follows we shall use these identifications implicitly.

We fix $n \ge 0$. For $x = (x_0, x_1, x_2, \ldots) \in K$, we write $x_n \in K_n \subset \mathbb{R}^{n+1}$ in coordinates as

$$x_n = (x_n(0), x_n(1), \ldots, x_n(n)).$$

We define a 'coordinate' function $c_n : K \to \mathbb{R}$ as

$$c_n(x) = x_n(n), \quad x \in K. \tag{3.6}$$

We define function spaces $\mathcal{H}_i$, $i = 1, 2$, on K as

$$\mathcal{H}^i = \overline{\bigcup_{n=1}^{\infty} \mathcal{H}_n^i}, \quad i = 1, 2.$$

As in [14, Lemma 6.1], we get the following properties.

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LEMMA 3.2. *Let $\mathcal{H}^1$, $\mathcal{H}^2$ be the spaces defined above. Then:*

(a) $\mathcal{H}^i$, $i = 1, 2$ *are well-defined simplicial function spaces on* K*;*

(b) $\mathcal{H}^i = \mathcal{A}_c(\mathcal{H}^i)$, $i = 1, 2$*;*

(c) $\mathrm{Ch}_{\mathcal{H}^1} K = \mathrm{Ch}_{\mathcal{H}^2} K = K \setminus \bigcup_{n=0}^{\infty} \bigcup \mathcal{F}_n$.

3.3. Maximal measures Given $n \geq 0$, $x \in K_n$ and $i = 0, 1$, let $\delta^i_{x,n}$ denote the unique $\mathcal{H}^i_n$-maximal measure representing x. For $x \in K$ and $i = 1, 2$, let δ^i_x denote the $\mathcal{H}^i$-maximal measure representing x.

3.4. Cantor set As in [14, Section 6.1, Lemmas 6.2 and 6.3], for every point $a \in A$ we get a homeomorphic copy

$$C_a = \left\{ x = (x_0, x_1, x_2, \ldots) \in K \setminus \bigcup_{n=0}^{\infty} K_n : x_0 = a \right\}$$

of the Cantor set $\{0, 1\}^{\mathbb{N}}$. The homeomorphism $\varphi_a : \{0, 1\}^{\mathbb{N}} \to C_a$ is provided by the mapping

$$\varphi_a : \quad \{0, 1\}^{\mathbb{N}} \to C_a : (\tau_1, \tau_2, \ldots,) \mapsto x = (a, x_1, x_2, \ldots),$$

defined as

$$c_n(x) = x_n(n) \begin{cases} > 0 & \text{if } \tau_n = 1, \\ < 0 & \text{if } \tau_n = 0, \end{cases} \qquad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, $a \in \bigcup\{F_s : |s| = n\}$ and $t \in \{0, 1\}^n$ we define a point

$$x_{a,t} = (x_{a,t}(0), x_{a,t}(1), \ldots, x_{a,t}(n)) \in K_n \subset \mathbb{R}^{n+1}$$

by setting

$$x_{a,t}(0) = a \quad \text{and} \quad x_{a,t}(i) \begin{cases} > 0 & \text{if } t_i = 1, \\ < 0 & \text{if } t_i = 0, \end{cases} \qquad i = 1, \ldots, n.$$

Let S be a countable subset of $\{0, 1\}^{\mathbb{N}}$ defined as

$$S = \{\tau \in \{0, 1\}^{\mathbb{N}} : \tau_n = 0 \text{ for at most finitely many natural numbers } n\}.$$

Let μ_n, $n \in \mathbb{N}$, be measures on $\{0, 1\}$ defined as

$$\mu_n(\{0\}) = 1 - \eta_n \quad \text{and} \quad \mu_n(\{1\}) = \eta_n, \qquad n \in \mathbb{N}.$$

Let $\mu \in \mathcal{M}^1(\{0, 1\}^{\mathbb{N}})$ denote the product measure $\prod_{n=1}^{\infty} \mu_n$.

For each $t \in \{0, 1\}^n$, let

$$a_t = \prod_{i=1}^{n} b_i \quad \text{where } b_i = \begin{cases} \eta_i & \text{if } t_i = 1, \\ 1 - \eta_i & \text{if } t_i = 0. \end{cases}$$

If $t \in \{0, 1\}^n$, let

$$U_t = \{\tau \in \{0, 1\}^{\mathbb{N}} : \tau|_{|t|} = t\}$$

denotes the standard clopen set in $\{0, 1\}^{\mathbb{N}}$ determined by t. Then $\mu(U_t) = a_t$.

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LEMMA 3.3. *Let a be a point in A. Then:*

(a) $\varphi_a\mu = \delta_a^1$ *(here $\varphi_a\mu$ denotes the image of the measure μ);*
(b) $\delta_a^1(\varphi_a(S)) = 1$.

PROOF. For the proof of (a), we notice that the measure $\varphi_a\mu$ is carried by $\mathrm{Ch}_{\mathcal{H}^1} K$ (see Lemma 3.2(c)). We claim that $\varphi_a\mu \in \mathcal{M}_a(\mathcal{H}^1)$. Indeed, let h be a function in $\mathcal{H}_m^1$ for some $m \geq 0$. If $t \in \{0, 1\}^m$, then $h = h(x_{a,t})$ on $\varphi_a(U_t)$, and thus

$$\begin{aligned}
(\varphi_a\mu)(h) &= \mu(h \circ \varphi_a) \\
&= \sum_{t\in\{0,1\}^m} \int_{U_t} h \circ \varphi_a \, d\mu \\
&= \sum_{t\in\{0,1\}^m} \int_{U_t} h(x_{a,t}) \, d\Big(\prod_{i=1}^m \mu_i\Big) \\
&= \sum_{t\in\{0,1\}^m} \mu(U_t) h(x_{a,t}) \\
&= \sum_{t\in\{0,1\}^m} a_t h(x_{a,t}) \\
&= h(a),
\end{aligned}$$

where the last equality follows from the construction (see Equations (3.1)). Thus $\varphi_a\mu \in \mathcal{M}_a(\mathcal{H}^1)$. Since $\varphi_a\mu$ is carried by $\mathrm{Ch}_{\mathcal{H}^1} K$ and $\mathcal{H}^1$ is simplicial, $\varphi_a\mu = \delta_a^1$.

To verify (b), we notice that (a) yields

$$\delta_a^1(\varphi_a(S)) = (\varphi_a\mu)(\varphi_a(S)) = \mu(S).$$

Hence, it is enough to show that $\mu(\{0, 1\}^{\mathbb{N}} \setminus S) = 0$. But this follows from (3.4), since

$$\begin{aligned}
\mu(\{0, 1\}^{\mathbb{N}} \setminus S) &= \mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\tau \in \{0, 1\}^{\mathbb{N}} : \tau_n = 0\}\Big) \\
&= \lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} \{\tau \in \{0, 1\}^{\mathbb{N}} : \tau_n = 0\}\Big) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu\big(\{\tau \in \{0, 1\}^{\mathbb{N}} : \tau_n = 0\}\big) \\
&= \lim_{n\to\infty} \sum_{k=n}^{\infty} (1 - \eta_k) \\
&= 0.
\end{aligned}$$

This finishes the proof. □

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LEMMA 3.4. *Let x be a point of K_n. Then:*

(a) $\delta^2_{x,n+1} = \delta^2_x$;

(b) *if $f \in \mathcal{U}^b(K)$ satisfies $f|_{K_n} \in \mathcal{A}(\mathcal{H}^2_n)$, $n \geq 0$, then $f \in \mathcal{A}(\mathcal{H}^2)$.*

PROOF. It is easy to see that $\delta^2_{x,n+1}$ is an $\mathcal{H}^2$-representing measure for x. Further, by virtue of Lemmas 3.1(c) and 3.2(c), $\delta^2_{x,n+1}$ is supported by

$$L^2_{n+1} \cup \Big(\mathrm{Ch}_{\mathcal{H}^2_n} K_n \setminus \bigcup \mathcal{F}_n\Big) \subset \mathrm{Ch}_{\mathcal{H}^2} K.$$

Hence, $\delta^2_{x,n+1}$ is $\mathcal{H}^2$-maximal and $\delta^2_{x,n+1} = \delta^2_x$. This proves (a).

For the proof of (b), let $f \in \mathcal{U}^b(K)$ be as in the premise. By (a),

$$\delta^2_x(f) = f(x) \quad \text{for each } x \in \bigcup_{n=1}^{\infty} K_n. \tag{3.7}$$

As $K \setminus \bigcup_{n=1}^{\infty} K_n \subset \mathrm{Ch}_{\mathcal{H}^2} K$, (3.7) holds for every $x \in K \setminus \bigcup_{n=1}^{\infty} K_n$ as well. By [14, Lemma 2.7], $f \in \mathcal{A}(\mathcal{H}^2)$. □

LEMMA 3.5. *For any $f \in \mathcal{B}^b_2(K)$, the function*

$$x \mapsto \delta^2_x(f), \quad x \in K,$$

is a Baire-two function on K.

PROOF. Let f be a bounded Baire-two function on K. By Lemma 3.4(a),

$$\delta^2_x(f) = \delta^2_{x,n+1}(f), \quad x \in K_n.$$

Thus the function $x \mapsto \delta^2_x(f)$, $x \in K$, is Baire-two on each K_n by virtue of Lemma 3.1(f).

By Lemma 3.2(c), $f(x) = \delta^2_x(f)$ for $x \in K \setminus \bigcup_{n=1}^{\infty} K_n$. It follows from [14, Lemma 3.4] that f is a Baire-two function on K. □

LEMMA 3.6. *The space $\mathcal{H}^2$ is Baire-one complemented by a projection of norm at most* 3.

PROOF. According to the inductive construction, for each $n \in \mathbb{N}$,

$$(P_n f)|_{K_{n-1}} = P_{n-1}(f|_{K_{n-1}}) \quad \text{and} \quad (P_n f)|_{L^1_n} = f|_{L^1_n}, \quad f \in \mathcal{B}^b_1(K_n). \tag{3.8}$$

Further, $\widehat{L} = \bigcup_{n=1}^{\infty} L^2_n$ is an open subset of K.

By (3.8), the mapping

$$Pf(x) = \begin{cases} P_n(f|_{K_n})(x) & \text{if } x \in K_n,\ n \geq 0, \\ f(x) & \text{if } x \in K \setminus \bigcup_{n=0}^{\infty} K_n, \end{cases} \qquad f \in \mathcal{B}^b_1(K),$$

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is well defined and $Pf \in \mathcal{B}_1^b(K)$ for each $f \in \mathcal{B}_1^b(K)$. Indeed, it follows from Lemma 3.1(h) that $Pf|_{L_n^2} \in \mathcal{B}_1^b(L_n^2)$, $n \in \mathbb{N}$. Thus Pf is a Baire-one function on $\widehat{L}$. As $Pf = f$ on $K \setminus \widehat{L}$, $Pf \in \mathcal{B}_1^b(K)$.

Given a function $f \in \mathcal{B}_1^b(K)$, then $Pf|_{K_n}$ is $\mathcal{H}_n^2$-affine for each $n \geq 0$. By Lemma 3.4(b), $Pf \in \mathcal{A}(\mathcal{H}^2)$.

Finally, as $\|P_n\| \leq 3$, we get $\|P\| \leq 3$ by definition. This concludes the proof. □

4. Proof of Theorem 1.1

We are now in a position to prove the main result. Let $\mathcal{H}^1$, $\mathcal{H}^2$ be the simplicial function spaces on the metrizable space K constructed in Section 3. For $i = 1, 2$, let X_i be the state space of $\mathcal{H}^i$ and $\phi_i : K \to X_i$ be the standard homeomorphic embeddings from [14, Section 2.5]. Then X_1, X_2 are metrizable simplices and $\phi_2 \circ \phi_1^{-1}$ restricted to $\overline{\operatorname{ext} X_1}$ is the homeomorphism required by Theorem 1.1(a).

If $i = 1, 2$ and $s \in X_i$, let $\widehat{\delta_s^i}$ stand for the unique $\mathfrak{A}(X_i)$-maximal measure $\mathfrak{A}(X_i)$-representing s.

For the proof of Theorem 1.1(b), let $f = \chi_{K \setminus \bigcup_{n=0}^{\infty} K_n}$ and

$$\widehat{f}(s) = f(\phi_1^{-1}(s)), \quad s \in \operatorname{ext} X_1.$$

Then $\widehat{f} \in \mathcal{B}_2^b(\operatorname{ext} X_1)$ and there is no affine Baire-two function on X_1 extending $\widehat{f}$.

Indeed, assume that $\widehat{h}$ is such a function. By [12, Théorème 3] or [3, Proposition 9], $\widehat{h}$ is bounded. We pick a point $a \in A$. By [10, Proposition 3.2],

$$\phi_1 \delta_a^1 = \widehat{\delta}^1_{\phi_1(a)}.$$

Thus $\widehat{\delta}^1_{\phi_1(a)}$ is a discrete measure (see Lemma 3.3). According to Lemma 2.4 and [15, Lemma 4.2],

$$\begin{aligned}\widehat{h}(\phi_1(a)) &= \widehat{\delta}^1_{\phi_1(a)}(\widehat{h}) = (\phi_1 \delta_a^1)(\widehat{h})\\ &= \delta_a^1(\widehat{h} \circ \phi_1) = \delta_a^1(\widehat{f} \circ \phi_1)\\ &= \delta_a^1(f).\end{aligned}$$

As δ_a^1 is carried by $K \setminus \bigcup_{n=0}^{\infty} K_n$ (see Lemma 3.3),

$$\delta_a^1(f) = 1.$$

On the other hand, if $a \in K_0 \setminus A$, then

$$\widehat{h}(\phi_1(a)) = \widehat{f}(\phi_1(a)) = f(a) = 0.$$

Thus

$$\widehat{h} = \begin{cases} 0 & \text{on } \phi_1(K_0 \setminus A) \text{ or} \\ 1 & \text{on } \phi_1(A). \end{cases}$$

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By choice of A, $\widehat{h}$ is not a Baire-two function.

For the proof of (c), let T^2 be the kernel on X_2 associated with the mapping $s \mapsto \widehat{\delta}_s^2$, $s \in X_2$. We remark that

$$T^2 g \in \mathfrak{A}_{\mathrm{bf}}(X_2), \quad g \in \mathcal{B}^b(\operatorname{ext} X_2). \tag{4.1}$$

(Since any function from $\mathcal{B}^b(\operatorname{ext} X_2)$ is the restriction of some function from $\mathcal{B}^b(X_2)$, claim (4.1) follows from [9, Corollary 6.2].)

Let $\widehat{f}$ be a bounded Baire-two function on $\operatorname{ext} X_2$. By extending $\widehat{f}$ by 0 on $X_2 \setminus \operatorname{ext} X_2$, we may assume that $\widehat{f}$ is defined on the whole of X_2. We claim that

$$T^2 \widehat{f} \in \mathfrak{A}_2(X_2). \tag{4.2}$$

To this end, we notice that

$$h(x) = \delta_x^2(\widehat{f} \circ \phi_2), \quad x \in K,$$

is a Baire-two function on K (see Lemma 3.5). Using [9, Corollary 6.2], [14, Theorem 2.6(c)] and Lemma 3.2(b), we get that $h \in (\mathcal{H}^2)^{\perp\perp}$. As $\mathcal{H}^2$ is Baire-one complemented, it follows from Lemmas 3.6 and 2.3 and Remark 2.2 that h is a pointwise limit of a bounded sequence $\{h_n\}$ of functions from $\mathcal{B}_1^b(K) \cap (\mathcal{H}^2)^{\perp\perp} = (\mathcal{H}^2)_1$.

Let $I : \mathcal{U}^b(K) \cap (\mathcal{H}^2)^{\perp\perp} \to \mathfrak{A}_{\mathrm{bf}}(X_2)$ be the isometry from [14, Section 2.6]. By [14, Theorem 2.5(e)], $Ih_n \to Ih$ and $Ih \in \mathfrak{A}_2(X_2)$.

Since

$$T^2 \widehat{f} = Ih \quad \text{on } \operatorname{ext} X_2,$$

$T^2 \widehat{f} = Ih$ on X_2 (we use the minimum principle [13, Proposition 3.6]). Hence $T^2 \widehat{f} \in \mathfrak{A}_2(X_2)$.

If $\alpha \in (2, \omega_1)$, we observe that $T^2 \widehat{f_n} \to T^2 \widehat{f}$ whenever $\{\widehat{f_n}\}$ is a bounded sequence of Borel functions on $\operatorname{ext} X_2$ pointwise converging to f and use (4.2) as the starting point for a straightforward transfinite induction. Hence, given $\widehat{f} \in \mathcal{B}_\alpha^b(\operatorname{ext} X_2)$, the function $T^2 \widehat{f}$ is the required extension of affine class α.

This concludes the proof.

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