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A NOTE ON MONOTONE LINDELÖFNESS OF COUNTABLE SPACES

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Abstract

In this note, we give an example of a Hausdorff, countable monotone Lindelöf space which is not metrizable, which gives a negative answer to a question raised by Levy and Matveev.

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1. Introduction

By a space, we mean a topological space. A space X is *monotonically Lindelöf* (mL) [1, 3, 4] if for every open cover $\mathcal{U}$ of X there is a countable open cover $r(\mathcal{U})$ of X that refines $\mathcal{U}$ and has the property that if an open cover $\mathcal{U}$ refines an open cover $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. In this case, r will be called a *monotone Lindelöf operator* for the space X. Levy and Matveev [4] showed that every second countable space is monotonically Lindelöf and discussed the monotone Lindelöfness of countable spaces, and asked the following question.

QUESTION [4, Question 3]. Is it consistent that every countable mL space is metrizable?

The purpose of this note is to construct an example stated in the abstract which gives a negative answer to the question in the class of Hausdorff spaces.

Let ω be the first infinite cardinal. Other terms and symbols that we do not define will be used as in [2].

2. An example on a monotonically Lindelöf space

EXAMPLE 2.1. There exists a Hausdorff, countable mL space which is not metrizable.

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PROOF. Let

$$A = \{a_n \mid n \in \omega\} \quad \text{and} \quad B = \{b_m \mid m \in \omega\}$$
$$Y = \{\langle a_n, b_m \rangle \mid n \in \omega, m \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \quad \text{where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_n \in A$ for each $n \in \omega$ takes the form

$$U_{a_n}(m) = \{a_n\} \cup \{\langle a_n, b_i \rangle \mid i > m\} \quad \text{for } m \in \omega$$

and a basic neighborhood of a takes the form

$$U_a(F) = \{a\} \cup \bigcup\{\langle a_n, b_m \rangle \mid a_n \in A \setminus F, m \in \omega\} \text{ for a finite subset } F \text{ of } A.$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular, since the point a can not be separated from the closed subset A by disjoint open subsets of X. Thus, X is not metrizable, since it is not regular. By the construction of the topology of X, it is not difficult to see that X is second countable. Thus, X is mL, so we complete the proof. □

REMARK 2.2. The author does not know if it is consistent that every countable, Tychonoff mL space is metrizable.

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