THE IMMERSIBILITY OF A SEMIGROUP INTO A GROUP

J. LAMBEK

1. Introduction. A *semigroup* is a set of elements which is closed under an associative operation, usually called multiplication. When can a semigroup be embedded in a group, i.e., under what condition is it isomorphic to a subset of a group? A necessary condition for immersibility is clearly the so-called *cancellation law*:

(C) *If* $ax = ay$ *or* $xb = yb$, *then* $x = y$.

It is well known that a finite semigroup with cancellation law is a group, also that an Abelian semigroup (one in which multiplication is commutative) can be embedded in a group if and only if the cancellation law holds. In general however the cancellation law is not sufficient for immersibility, as was shown by A. Malcev in 1936.

In a second paper of 1939, Malcev stated necessary and sufficient conditions. They assert that certain chains of equations imply further equations. The number of these conditions is infinite, and the chains are of unbounded length. He proved in a third paper of 1940 that no finite part of these conditions will suffice.

Malcev offers a rather complicated construction for obtaining such chains of equations. In the present paper I have tried to give a simpler construction, by the device of using parts of polyhedra, rather than natural numbers, for labelling the equations and variables contained in these conditions. Acquaintance with Malcev's work will not be expected from the reader. In defining the term "polyhedron", I shall roughly follow the book on topology by Seifert and Threlfall.

A *face* (abstract polygon) is a division of a topological circle into two or more arcs, called *sides*, by an equal number of points, which we will term *angles*. An abstract polyhedron is a system of F faces, containing together $2E$ sides, such that every side is mapped topologically on exactly one other. A pair of sides thus mapped into each other is called an *edge*. Hence every edge has two sides. We may speak about the *midpoint* (some interior point) of an edge, which divides the edge into two *half-edges*. A set of angles which correspond to one another under the mapping is called a *vertex*. Every edge has two vertices. To every edge there belong four angles, which may be classified by pairs in two different ways: angles at the same vertex of the edge, and angles on the same side of the edge. Every half-edge has one vertex, two sides, and two angles. The polyhedron is called *Eulerian* if the total number of

Received February 22, 1950.

vertices is V such that $V + F - E = 2$. This is also a necessary and sufficient condition for the surface defined by the polyhedron to be homeomorphic (i.e. topologically equivalent) to the sphere. Throughout the present paper we shall always mean "abstract Eulerian polyhedron" when we say "polyhedron."

Given a semigroup $\mathfrak{H}$, we shall understand by *polyhedral condition* the following statement:

(P) *If the elements of* $\mathfrak{H}$ *are assigned to all sides and angles of any Eulerian polyhedron, such that to each half-edge there corresponds an equation* $xa = yb$, *where* x *and* y *have been assigned to the sides,* a *and* b *to the corresponding angles of the half-edge, then these* $2E$ *equations are interdependent, i.e. any one of them can be derived from the totality of all others.* (See Figure 1.)

The application of this condition to two polyhedra which are topologically equivalent will of course give the same result. We shall prove that (P) is a necessary and sufficient condition for immersibility of a semigroup $\mathfrak{H}$ with cancellation law into a group. This will establish the following

THEOREM: *A semigroup can be embedded in a group if and only if the cancellation law* (C) *and the polyhedral condition* (P) *are satisfied.*

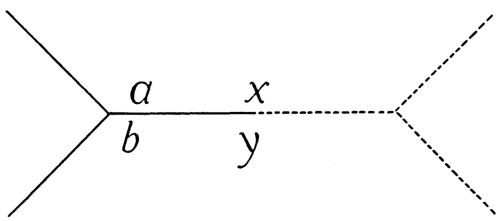

Fig. 1

2. Necessity of polyhedral condition. Let the semigroup $\mathfrak{H}$ be contained in a group $\mathfrak{G}$, so that the elements of $\mathfrak{H}$ possess inverses in $\mathfrak{G}$. Assign elements of $\mathfrak{H}$ to the sides and angles of a given polyhedron. Assume that the equations belonging to all but one half-edge are true, the remaining equation is to be deduced.

An *oriented triangulation* of the polyhedron is obtained as follows: Directed radii are drawn from the centre (some interior point) of each face to the angles of the face. (It will be remembered that faces are circles, and that by angles we understand points on the circumference.) Directed radii are drawn from the midpoints of all sides to the centre of the face. Each half-edge of the original polyhedron is given a direction from the midpoint of the edge towards the vertex. The half-edges thus oriented as well as the directed radii will be the oriented edges of the triangulation.

The equation $xa = yb$, corresponding to any half-edge of the polyhedron, can be replaced by the two equations $xa = p$ and $yb = p$, corresponding to

https://doi.org/10.4153/CJM-1951-005-8 Published online by Cambridge University Press

two triangles of the triangulation. The variables occurring in these equations are assigned to the edges of the triangulation: namely x and y to the radii from the midpoint, a and b to the radii towards the vertex, and p to the half-edge itself. (See Fig. 2.)

If the equation $xa = yb$ was to be inferred, we may now add $yb = p$ to the given equations, and leave only $xa = p$ to be deduced. Thus, corresponding to each oriented triangle, we have an equation; for instance $xa = p$ and $yb = p$ correspond to two of the four triangles on Fig. 2. Of these $4E$ equations all but one are given, and one is to be derived.

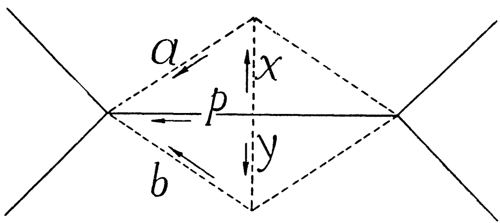

Fig. 2

Consider any path made up entirely of edges of the triangulation. Corresponding to such a path we form a product in the following way: If the element x of $\mathfrak{H}$ has been assigned to the nth edge of the path, then the nth term of the product is x or x^{-1}, depending on whether this edge has been traversed in the right or in the wrong direction. For instance, there are six different closed paths by means of which the perimeter of the upper left triangle of Fig. 2 can be once described. Correspondingly we may obtain one of the six products:

$$(\dagger) \qquad xap^{-1},\ ap^{-1}x,\ p^{-1}xa;\quad pa^{-1}x^{-1},\ a^{-1}x^{-1}p,\ x^{-1}pa^{-1}.$$

As long as $xa = p$, each of these products has the value 1: and conversely, if any one of the six products (†) is 1, then $xa = p$.

Consider now a closed path consisting of edges of the triangulation, and surrounding only triangles for which the corresponding equations are given. We prove by induction that the product corresponding to this path will be unity. We have shown above that this is indeed so, if the path surrounds only one triangle. Otherwise we may decompose the closed path into two paths Q and R traversed in succession such that there will exist a path P, lying entirely inside the closed path, and joining the endpoint of Q to the endpoint of R. Let P' be the path P traversed in opposite direction. If $f(P)$ denotes the product associated with P, then $f(P)\,f(P') = 1$. Hence $f(Q)\ f(R) = f(Q)\,f(P)\,f(P')\,f(R) = 1$, since $f(Q)\,f(P) = f(P')\,f(R) = 1$, by induction hypothesis.

Suppose now the upper left triangle of Fig. 2 is the one for which the corresponding equation is to be derived. Since the surface defined by our poly-

https://doi.org/10.4153/CJM-1951-005-8 Published online by Cambridge University Press

hedron is homeomorphic to the sphere, the perimeter of this triangle divides the whole surface into two simply connected regions. Let us describe a closed path along this perimeter, and call the outside of the triangle the inside of the path. Corresponding to this path we obtain one of the six products (†), whose value will be unity, in virtue of the above. Hence $xa = p$, as was to be deduced. We have thus shown the necessity of the polyhedral condition.

3. Ratios. In preparation for the sufficiency proof of the polyhedral condition, we shall consider a semigroup $\mathfrak{H}$ satisfying (C) and (P). It is our intention to introduce ratios, in more or less the same way as is usually done when $\mathfrak{H}$ is the set of natural numbers.

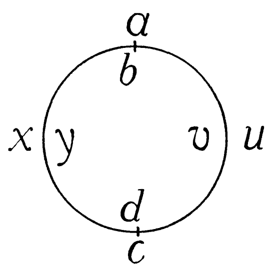

Fig. 3

Let a and b be any two elements of $\mathfrak{H}$. We shall designate by a/b the set of pairs of elements x,y such that $xa = yb$. If a/b is not the empty set we shall call it a *ratio*. Similarly we define I, (a), $(a)^{-1}$ as sets of pairs x,y such that $x = y$, $xa = y$, $x = ya$, respectively. With the help of the cancellation law, we can easily show that they are also ratios. In fact

(1) $$I = t/t, \qquad (a) = at/t, \qquad (a)^{-1} = t/at,$$

where t is an arbitrary element of $\mathfrak{H}$. We also note that

(2) $$(a) = (b) \textit{ if and only if } a = b.$$

When can we say that two ratios are equal? We will prove:

(3) *$a/b = c/d$ if and only if there exist x and y belonging to $\mathfrak{H}$ such that $xa = yb$ and $xc = yd$.* (It is assumed here that a/b is in fact a ratio, and therefore not empty.)

The necessity of this condition follows directly from the definition of ratios. To prove its sufficiency, let us assume that the condition of the theorem holds, and also that $ua = vb$. Let us now apply the polyhedral condition to a simple polyhedron consisting of two edges, two faces, and two vertices (Fig. 3). The equations corresponding to three of the half-edges of this polyhedron are true by assumption, consequently the fourth must hold, viz. $uc = vd$. This argument works both ways, hence $ua = vb$ if and only if $uc = vd$, and therefore $a/b = c/d$, by definition.

We define multiplication of ratios as follows:

$$(4) \qquad (a/d)(d/b) = a/b.$$

Thus two ratios may be multiplied to give another ratio, provided they can be written in the form a/d and d/b respectively such that a/b is a ratio. Is the product of the two ratios unique, if it exists? To answer this in the affirmative we must show:

(5) *If* $a/d = a'/d'$ *and* $d/b = d'/b'$, *then* $a/b = a'/b'$.

We may assume that $xa = yd$, $xa' = yd'$, $zd = wb$, $zd' = wb'$, $ua' = vb'$, and wish to prove that $ua = vb$. The result follows immediately if we consider the polyhedron consisting of two vertices, three faces, and three edges. (See Fig. 4.)

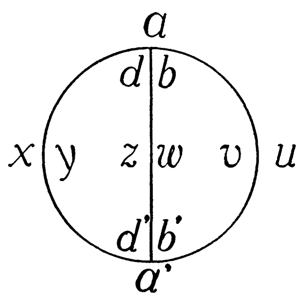

Fig. 4

I is the unit element under multiplication. For

$$(a/b)I = (a/b)(b/b) = a/b$$

by (1) and (4). The same applies to multiplication by I on the left. We note the existence of inverses; thus by (1) and (4),

$$(a/b)(b/a) = a/a = I.$$

In particular we see from (1) that $(a)^{-1}$ is the inverse of (a), as was anticipated by our notation. We find that

$$(a)(b) = (abt/bt)(bt/t) = abt/t = (ab).$$

Hence, in view of (2), the correspondence $a \rightarrow (a)$ maps $\mathfrak{H}$ isomorphically on a subset of the set of ratios.

We have embedded $\mathfrak{H}$ in the set of ratios. The latter has all properties of a group, except that it is not closed under multiplication, and associativity has not yet been shown to hold. We shall embed it in a larger set, in which multiplication is always defined and associative. It may be worth noting that, if $\mathfrak{H}$ is an Abelian semigroup, the ratios do form a group already.

4. Associativity. The set of ratios almost form a group, except that it is not closed under multiplication, so that the associative law, as usually stated, has no meaning. With some care however it is possible to enunciate an associative law even here. If we can bracket a sequence of ratios in such a way that they can be multiplied out to give a single ratio, then this "product" will be unique. To be more precise: we shall say that a finite sequence of ratios $(\ldots, a/b, b/c, \ldots)$ *contracts* into the sequence $(\ldots, a/c, \ldots)$. If a sequence of ratios reduces to a single ratio by iterated contraction, we shall call this ratio its *product*. The associative law then states:

(6) *If a sequence of ratios has a product, then this is unique.*

To prove (6), consider a sequence $S(0)$ of $n + 1$ ratios. This contracts to $S(\pm 1)$, consisting of n ratios, which in turn contracts to $S(\pm 2)$, and so on, until we obtain a single ratio $S(\pm n)$. The choice of the plus sign refers to one method of iterated contraction, the minus to another. We must show that $S(+n) = S(-n)$.

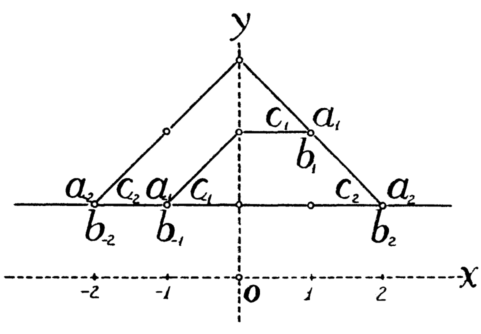

Fig. 5

If k is an integer between 0 and n, we write $i = \pm k$, and note that $S(i)$ has $n + 1 \mp k$ places or terms. Represent the jth place of $S(i)$ by the point (i,j) in the Cartesian plane. If $k \neq 0$, all but two terms of $S(\pm k \mp 1)$ reappear in $S(\pm k)$; join the corresponding points by straight lines. But two consecutive terms, say a_i/c_i and c_i/b_i are contracted into a_i/b_i. Join the two former points by straight lines to the latter, which will be called a ***vertex***. Also join the two points (or vertices) $(\pm n, 1)$ to the point at infinity, along the line $y = 1$. A broken line joining two vertices, even if it passes through the point at infinity, will be called an *edge*. There are three edges meeting at every vertex. The simply connected regions into which the edges divide the plane will be called *faces*. Since the plane can be mapped on a sphere by an inverse stereographic projection, we obtain a concrete representation of an Eulerian polyhedron. We may also verify independently that $V = 2n$, $F = n + 2$, and $E = 3n$, so that $V + F - E = 2$. A simple case, for which $n = 2$, is illustrated by Fig. 5.

Consider the vertex corresponding to the contraction of

$$S(\pm k \mp 1) = (\ldots, a_i/c_i, c_i/b_i, \ldots) \text{ into } S(\pm k) = (\ldots, a_i/b_i, \ldots).$$

To the three angles formed at this vertex we assign a_i,c_i, and b_i, in this order, going from top to bottom, as shown in Fig. 5. By our construction, if a and b have been assigned to the upper, respectively lower angle at one end of any finite edge, and if this edge passes through any integral lattice point (i, j), then the jth term of $S(i)$ is a/b. But in the same way, we find that this term is c/d, where c and d correspond to the two angles at the other end of the edge. Thus, for any finite edge, we have a "proportion" $a/b = c/d$. We will prove that such a proportion also holds for the edge passing through the point at infinity.

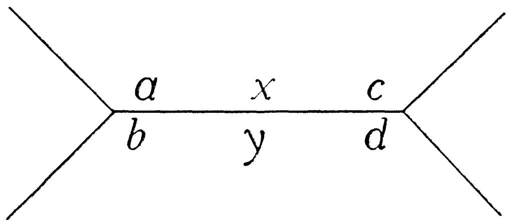

Fig. 6

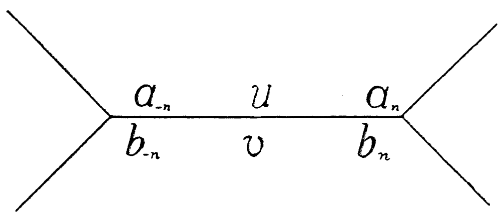

Fig. 7

In view of (3), the above proportion may be replaced by the two equations $xa = yb$ and $xc = yd$. Here x and y may be conveniently assigned to the two sides of the edge (see Fig. 6), and the two equations may be said to correspond to the two half-edges. Consider now the edge joining $(n, 1)$ and $(-n, 1)$ through the point at infinity. An appropriate transformation will bring the point at infinity into the finite part of the plane. Since a_{-n}/b_{-n} is a ratio, by definition, there exist elements u and v of $\mathfrak{H}$ such that $ua_{-n} = vb_{-n}$. We may assign u and v to the upper, respectively lower side of the edge depicted in Fig. 7, and the given equation will correspond to the left half of this edge. With the help of the polyhedral condition, we deduce the remaining equation $ua_n = vb_n$. It follows from (3) that $a_{-n}/b_{-n} = a_n/b_n$, i.e., $S(-n) = S(+n)$. This concludes the proof of the associative law.

https://doi.org/10.4153/CJM-1951-005-8 Published online by Cambridge University Press

5. Sufficiency of polyhedral condition. Two finite sequences of ratios, U and V, will be called *similar*, if there is a sequence W from which both can be obtained by repeated contraction. We will prove the following result:

(7) *If both U and V reduce to S by iterated contraction, then they are similar.*

First, suppose U contracts to S, so that $U = (P, a/c, c/b, Q)$ and $S = (P, a/b, Q)$, where P and Q may be empty sequences. Since V reduces to S, we may put $V = (X, Y, Z)$, where X, Y, and Z reduce to P, a/b, and Q respectively, by iterated contraction. It is easily seen that $W = (X, Y, b/c, c/b, Z)$ can be reduced to both U and V by repeated contraction, so that U and V are similar. Hence (7) holds when U reduces to S in one step.

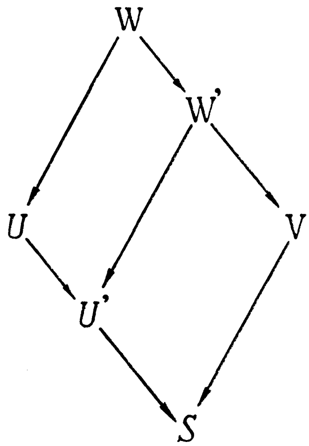

Fig. 8

Next, suppose U reduces to S in n steps, $n > 1$. Then U contracts to U' which reduces to S in $n - 1$ steps. By induction hypothesis, there exists a sequence W' which reduces to U' and V by iterated contraction. Since U reduces to U' in only one step, by the above, there exists a sequence W which reduces to both U and W' and therefore V. Hence U and V are similar, as was to be proved. (See Fig. 8 for an illustration of the second part of this proof.)

We are now in a position to show that similarity of sequences of ratios is an equivalence relation in the usual sense.

(8) *Similarity is symmetric, reflexive, and transitive.*

Its symmetry is obvious. Reflexivity follows from the fact that (S, I) contracts to S. To prove transitivity, let us assume that R is similar to S, and S is similar to T. Hence there exists a sequence U which reduces to both R and S, and a sequence V which reduces to both S and T (see Fig. 9). Since both U and V reduce to S, by (7) they can both be obtained from a sequence W

by repeated contraction. Now W reduces to R via U and to T via V, hence R is similar to T, as was to be proved.

In this connection we may also state:

(9) *If S is similar to S' and T is similar to T', then (S,T) is similar to (S',T').*

For, by repeated contraction, we obtain S and S' from U, T and T' from V, hence (S,T) and (S', T') from (U,V).

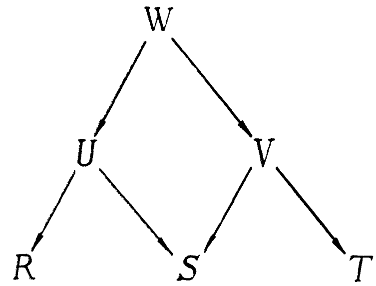

Fig. 9

A ratio may be regarded as a sequence of ratios with one term. When are two ratios similar?

(10) *Two ratios are similar if and only if they are equal.*

Because of reflexivity we know that equal ratios are similar. Conversely, let two ratios be similar. By definition, this means that both can be derived from the same sequence W by repeated contraction. From (6) we deduce that they are equal.

Let us denote by S^* the class of all sequences which are similar to S, so that $S^* = T^*$ if and only if S and T are similar, in view of (8). We define *multiplication* of similarity classes as follows:

$$S^*T^* = (S,T)^*. \tag{11}$$

From (9) we know that the product thus defined is unique.

Associativity becomes apparent if we write both $(S^*T^*)U^*$ and $S^*(T^*U^*)$ as $(S,T,U)^*$. The unit element under multiplication is I^*, since both (S,I) and (I,S) are similar to S; for both are obtained by contracting (I,S,I). If T contains the reciprocals of the ratios of S in reverse order, then both (S,T) and (T,S) reduce to and are therefore similar to I; thus T^* may be regarded as the inverse of S^* under multiplication. We have thus proved that the similarity classes form a group $\mathfrak{G}$, with multiplication defined by (11).

The correspondence $a/b \to (a/b)^*$ is a homomorphic mapping of the set of ratios on a subset of $\mathfrak{G}$. For, by (11) and (4),

https://doi.org/10.4153/CJM-1951-005-8 Published online by Cambridge University Press

$$(a/b)^*(b/c)^* = (a/b,b/c)^* = (a/c)^* = ((a/b)(b/c))^*.$$

More than this, the mapping is isomorphic. For if $(a/b)^* = (c/d)^*$ then a/b and c/d are similar, hence $a/b = c/d$, by (10). The correspondence $a/b \rightarrow (a/b)^*$ therefore embeds the set of ratios in $\mathfrak{G}$. But the correspondence $a \rightarrow (a)$ embeds the semigroup $\mathfrak{H}$ in the set of ratios, as we have shown in §**3**. Hence the correspondence $a \rightarrow (a)^*$ embeds $\mathfrak{H}$ in $\mathfrak{G}$. This establishes the sufficiency of the polyhedral condition.

6. Application to Abelian semigroups. Let $\mathfrak{H}$ be an Abelian semigroup with cancellation law. Although it is not difficult to show directly that $\mathfrak{H}$ is immersible in a group (namely the set of ratios), we shall test the usefulness of the polyhedral condition, by showing independently that the latter holds in $\mathfrak{H}$.

Let elements of $\mathfrak{H}$ be assigned to all angles and sides of any given polyhedron. Of the equations corresponding to the half-edges we will assume that all but one hold, and we wish to deduce the remaining equation. As in the necessity proof of the polyhedral condition, we introduce a triangulation and replace each equation $xa = yb$ by two equations $xa = p$ and $yb = p$ corresponding to triangles. We may assume then that all but two of these latter equations are given.

If we reverse the direction assigned to all half-edges, we obtain a cyclic orientation for each triangle, enabling us to distinguish clockwise and counter-clockwise triangles. The triangles are thus divided into two classes, so that triangles with a common edge do not belong to the same class. We will write the equation corresponding to a triangle of the first class as $xa = p$, and the equation corresponding to a triangle of the second class as $p = yb$, making a careful distinction between the two sides of each equation. Now multiply all $4E - 2$ given equations together, after their sides have been thus arranged. It will be observed that the four variables belonging to the half-edge whose equation is to be deduced occur once in the product equation. All other variables occur twice, once on each side of the product equation, and may therefore be cancelled, by (C). There results an equation containing four variables, and it is easily seen that this is in fact the equation we wish to deduce. Hence the polyhedral condition is satisfied, as was to be proved.

References

[1] A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann., vol. 113 (1937), 687.

[2] ——— *Über die Einbettung von assoziativen Systemen in Gruppen*, Mat. Sbornik, vol. 6 (48) (1939), 331-336.

[3] ——— *Über die Einbettung von assoziativen Systemen in Gruppen, II*, Mat. Sbornik, vol. 8 (50) (1940), 251-264.

[4] H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (New York, 1947).

McGill University

https://doi.org/10.4153/CJM-1951-005-8 Published online by Cambridge University Press