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A bilinear transformation

By G. N. Watson.

1. *Introduction.*

The problem which I enunciate and solve in this paper seems to have originated in the study of properties of polyhedral functions. It is a problem of elementary analytical geometry of three dimensions, and the solution which I give, though somewhat tedious, is both elementary and direct. There are several comments which I have to make about current solutions, but I reserve these until the end of the paper since they will be more easily appreciated when it is possible to compare the current solutions with my solution.

Complex numbers are represented on an Argand diagram. The plane of the diagram is in contact with a Neumann sphere, the point of contact S being the origin in the Argand diagram. The centre and the radius of the sphere are O and a; we regard S as the south pole of the sphere, the north pole being N. We take Ox, Oy, ON as axes of reference, Ox and Oy being parallel to the real and imaginary axes in the Argand diagram. A fixed point A is taken on the sphere and, referred to these axes, the direction cosines of OA are denoted by (l, m, n).

A (variable) complex number w is represented by a point P in the Argand diagram; we take Q to be that point of the sphere whose stereographic projection from N on the plane of the diagram is P. We define Q' as the point to which Q is carried by the rotation of the plane OAQ about OA through a constant angle α; it is naturally

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supposed that rotation about OA is taken to be positive when it is of the same screw nature as rotation from Ox to Oy about ON. The stereographic projection of Q' from N on the Argand diagram is P', and w' is the complex number represented by P'.

The problem which I propose is the determination of the simplest form of the relation which connects the complex numbers w and w'. I anticipate the working of my solution of the problem by stating that the relation in question is

$$ww'(l - im) \sin \tfrac{1}{2}\alpha - 2aw'(n \sin \tfrac{1}{2}\alpha + i \cos \tfrac{1}{2}\alpha) \\ - 2aw(n \sin \tfrac{1}{2}\alpha - i \cos \tfrac{1}{2}\alpha) - 4a^2(l + im) \sin \tfrac{1}{2}\alpha = 0.$$

2. *The solution of the rotational problem.*

As a preliminary step, we express the coordinates (x', y', z') of Q', in terms of the coordinates (x, y, z) of Q, the axes in both cases being Ox, Oy, ON.

We take an auxiliary set of axes (without change of origin) the direction cosines of the new set referred to the old being (l_1, m_1, n_1), (l_2, m_2, n_2), (l, m, n); it is supposed that the new set is a rectangular system of the same screw nature as the old. For the sake of uniformity, we temporarily write (l_3, m_3, n_3) in place of (l, m, n).

Let the coordinates of Q and Q' referred to the new set of axes be (ξ, η, ζ) and (ξ', η', ζ'). The standard transformation formulæ are

$$\begin{aligned} \xi &= l_1 x + m_1 y + n_1 z, & x' &= l_1 \xi' + l_2 \eta' + l_3 \zeta', \\ \eta &= l_2 x + m_2 y + n_2 z, & y' &= m_1 \xi' + m_2 \eta' + m_3 \zeta', \\ \zeta &= l_3 x + m_3 y + n_3 z, & z' &= n_1 \xi' + n_2 \eta' + n_3 \zeta', \end{aligned}$$

while the formulæ for rotation about the new third axis are

$$\xi' = \xi \cos \alpha - \eta \sin \alpha, \qquad \eta' = \xi \sin \alpha + \eta \cos \alpha, \qquad \zeta' = \zeta.$$

When we express x' in terms of x, y, z from these formulæ, we find that

$$\begin{aligned} x' &= \{l_1(l_1 x + m_1 y + n_1 z) + l_2(l_2 x + m_2 y + n_2 z)\} \cos \alpha \\ &+ \{l_2(l_1 x + m_1 y + n_1 z) - l_1(l_2 x + m_2 y + n_2 z)\} \sin \alpha \\ &+ l_3(l_3 x + m_3 y + n_3 z). \end{aligned}$$

This formula can be simplified by means of standard relations connecting direction cosines, namely

$$l_1^2 + l_2^2 = 1 - l_3^2 = 1 - l^2,$$

$$\begin{aligned} l_1 m_1 + l_2 m_2 &= -l_3 m_3 = -lm, & l_1 n_1 + l_2 n_2 &= -ln, \\ l_2 m_1 - l_1 m_2 &= -n_3 = -n, & l_2 n_1 - l_1 n_2 &= +m. \end{aligned}$$

The simplified form of the formula for x' and the corresponding results for y' and z' are as follows:

$$x' = x \cos \alpha + l(1 - \cos \alpha)(lx + my + nz) + (mz - ny) \sin \alpha,$$

$$y' = y \cos \alpha + m(1 - \cos \alpha)(lx + my + nz) + (nx - lz) \sin \alpha,$$

$$z' = z \cos \alpha + n(1 - \cos \alpha)(lx + my + nz) + (ly - mx) \sin \alpha.$$

It is not surprising that these formulæ should have found their way into a Cambridge examination paper. In fact they formed part of question 5 in the paper set on the afternoon of Tuesday, June 7, 1904, in the Trinity College May Examination. I do not know who composed the paper, but the style of this question is much more characteristic of R. A. Herman than of any of his colleagues at that time. I have not seen the formulæ in print elsewhere, but I can attribute that to my lack of familiarity with text-books on analytical geometry of three dimensions.

3. *The formulæ for stereographic projection.*

The formulæ connecting the coordinates of P and Q are fairly well known, but I work them out for the sake of completeness. Take the colatitude (north polar distance) and the longitude of Q to be θ and ϕ, so that ϕ is the phase of w. With SP and SN as coordinate axes, the coordinates of N, Q, P are $(0, 2a)$, $(a \sin \theta, a + a \cos \theta)$, $(|w|, 0)$, so that, since these points are collinear, we have

$$\frac{a \sin \theta}{|w|} + \frac{a(1 + \cos \theta)}{2a} = 1,$$

and hence immediately

$$|w| = \frac{2a \sin \theta}{1 - \cos \theta},$$

$$w = 2ae^{i\phi} \cot \tfrac{1}{2}\theta = \frac{2a(x + iy)}{a - z};$$

similarly we have

$$w' = \frac{2a(x' + iy')}{a - z'}.$$

4. *The relation connecting w' with w.*

As a preliminary to calculating w' in terms of w, we begin by expressing $x' + iy'$ and $a - z'$ in terms of x, y, z and thence in terms of θ and ϕ. The results of these calculations do not appear to be expressible in any simple way as functions of w; the explanation of this

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is that they contain a common factor which is not an analytic function of w. Accordingly, when we have calculated $x'+iy'$ and $a-z'$ in terms of x, y, z, we proceed to factorise the results. For the most part, it is simpler to work with x, y, z than with θ and ϕ, provided that we make a discreet use of the formulæ

$$x^2+y^2+z^2=a^2, \qquad l^2+m^2+n^2=1.$$

After expressing $a-z'$ as a homogeneous quadratic function of $\cos\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\alpha$, thus

$$\begin{aligned} a-z' &= (a-z)\cos^2\tfrac{1}{2}\alpha+(a+z)\sin^2\tfrac{1}{2}\alpha \\ &\quad -2n(lx+my+nz)\sin^2\tfrac{1}{2}\alpha-2(ly-mx)\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha, \end{aligned}$$

we proceed to factorise the expression on the right by evaluating its determinant. We have

$$\begin{aligned} &(a-z)\{(a+z)-2n(lx+my+nz)\}-(ly-mx)^2 \\ &\quad=(a^2-z^2)(l^2+m^2+n^2)-2n(a-z)(lx+my+nz)-(ly-mx)^2 \\ &\quad=(x^2+y^2)(l^2+m^2)+n^2(a^2-z^2)-2n(a-z)(lx+my) \\ &\qquad\qquad -2n^2z(a-z)-(ly-mx)^2 \\ &\quad=(lx+my)^2-2n(a-z)(lx+my)+n^2(a-z)^2 \\ &\quad=(lx+my+nz-na)^2. \end{aligned}$$

The last expression shews that $(a-z')(a-z)$ can be written in the form of the product of the pair of factors

$$(a-z)\cos\tfrac{1}{2}\alpha-(ly-mx)\sin\tfrac{1}{2}\alpha\pm i(lx+my+nz-na)\sin\tfrac{1}{2}\alpha,$$

and hence we have

$$\begin{aligned} \frac{a-z'}{a-z} &= \{\cos\tfrac{1}{2}\alpha-ni\sin\tfrac{1}{2}\alpha+(m+il)e^{i\phi}\cot\tfrac{1}{2}\theta\sin\tfrac{1}{2}\alpha\} \\ &\qquad\times\{\cos\tfrac{1}{2}\alpha+ni\sin\tfrac{1}{2}\alpha+(m-il)e^{-i\phi}\cot\tfrac{1}{2}\theta\sin\tfrac{1}{2}\alpha\}. \end{aligned}$$

We now turn to $x'+iy'$. Evidently

$$\begin{aligned} x'+iy' &= (x+iy)(\cos^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\alpha)+2(l+im)(lx+my+nz)\sin^2\tfrac{1}{2}\alpha \\ &\qquad +2\{z(m-il)-n(y-ix)\}\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha; \end{aligned}$$

and the determinant of this quadratic function of $\cos\frac{1}{2}\alpha$ and $\sin\frac{1}{2}\alpha$ is equal to

$$\begin{aligned} &-(x+iy)^2+2(l+im)(x+iy)(lx+my+nz) \\ &\qquad\qquad +\{z(l+im)-n(x+iy)\}^2 \\ &=(n^2-1)(x+iy)^2+2(l+im)(x+iy)(lx+my)+z^2(l+im)^2 \\ &=(l+im)(x+iy)\{2(lx+my)-(l-im)(x+iy)\}+z^2(l+im)^2 \\ &=(l+im)^2(x^2+y^2)+z^2(l+im)^2=a^2(l+im)^2. \end{aligned}$$

The last expression shews that $(x' + iy')(x + iy)$ can be written in the form of the product of the pair of factors

$$(x + iy) \cos \tfrac{1}{2}\alpha + \{z(m - il) - n(y - ix)\} \sin \tfrac{1}{2}\alpha \mp a(m - il) \sin \tfrac{1}{2}\alpha,$$

and hence we have

$$\frac{x' + iy'}{x + iy} = \{\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha - (m - il)\, e^{-i\phi} \tan \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha\}$$
$$\times \{\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha + (m - il) e^{-i\phi} \cot \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha\}.$$

The products which have now been obtained for

$$\frac{a - z'}{a - z} \qquad \text{and} \qquad \frac{x' + iy'}{x + iy}$$

have in common their second factor

$$\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha + (m - il) e^{-i\phi} \cot \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha,$$

so that by division we have

$$\frac{w'}{w} = \frac{\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha - (m - il) e^{-i\phi} \tan \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha}{\cos \tfrac{1}{2}\alpha - ni \sin \tfrac{1}{2}\alpha + (m + il) e^{-i\phi} \cot \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha}.$$

Hence

$$\frac{w'}{2a} = \frac{w(\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha) - 2a(m - il) \sin \tfrac{1}{2}\alpha}{2a(\cos \tfrac{1}{2}\alpha - ni \sin \tfrac{1}{2}\alpha) + w(m + il) \sin \tfrac{1}{2}\alpha}.$$

This is the formula which expresses w' in terms of w. When we clear of fractions, we obtain the bilinear relation

$$ww'(l - im) \sin \tfrac{1}{2}\alpha - 2aw'(n \sin \tfrac{1}{2}\alpha + i \cos \tfrac{1}{2}\alpha)$$
$$- 2aw(n \sin \tfrac{1}{2}\alpha - i \cos \tfrac{1}{2}\alpha) - 4a^2(l + im) \sin \tfrac{1}{2}\alpha = 0.$$

This relation holds for all values of the constants and variables with the customary convention that w' is "the point at infinity" when w is equal to $-2a(\cot \tfrac{1}{2}\alpha - ni)/(m + il)$, provided that l and m are not both zero and that α is not an integral multiple of 2π, with a similar convention about w being "the point at infinity." The reader should find no difficulty in examining how the relation simplifies in the special cases (i) $l = m = 0$, $n = \pm 1$, (ii) $\sin \tfrac{1}{2}\alpha = 0$.

The significance of the discarded factor

$$\cos \tfrac{1}{2}\alpha + ni \sin \tfrac{1}{2}\alpha + (m - il)\, e^{-i\phi} \cot \tfrac{1}{2}\theta \sin \tfrac{1}{2}\alpha,$$

is worth mention. When it vanishes, the conjugate expression, which may be written in the form

$$\cos \tfrac{1}{2}\alpha - ni \sin \tfrac{1}{2}\alpha + (m + il) \frac{w}{2a} \sin \tfrac{1}{2}\alpha$$

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must also vanish. This means that w' must be infinite, provided that $n \neq \pm 1$.

5. *The self-corresponding points of the transformation.*

We obtain the self-corresponding points of the transformation by taking Q (and Q') to be either at A or at the point antipodal to A. It is therefore evident that they are

$$2a\frac{l+im}{1-n}, \qquad -2a\frac{l+im}{1+n},$$

provided that n is not equal to ± 1 (in these exceptional cases, l and m are both zero).

These considerations shew that the bilinear transformation can be written in the alternative form

$$\frac{(1+n)\,w'+2a\,(l+im)}{(1-n)\,w'-2a\,(l+im)} = K\frac{(1+n)\,w+2a\,(l+im)}{(1-n)\,w-2a\,(l+im)},$$

where K is the multiplier of the transformation. Substitute on the left for w' its value in terms of w; it is then found after some fairly straightforward algebra that

$$K = e^{ia}.$$

The bilinear transformation can consequently be written in the alternative forms

$$\frac{(1+n)\,w'+2a\,(l+im)}{(1-n)\,w'-2a\,(l+im)} = e^{ia}\frac{(1+n)\,w+2a\,(l+im)}{(1-n)\,w-2a\,(l+im)},$$

$$\frac{w'+2a\,(l+im)/(1+n)}{w'-2a\,(l+im)/(1-n)} = e^{ia}\frac{w+2a\,(l+im)/(1+n)}{w-2a\,(l+im)/(1-n)}.$$

These forms of the bilinear transformation are of less value than the form obtained in §4, since one is nugatory and the other is meaningless in the special cases $n = \pm 1$.

6. *Conclusion.*

There are two solutions of tbe problem which have come to my notice. The older, given by Cayley (1), seems to me to be fundamentally the same as mine, except that he makes use of a number of formulæ in the theory of rotations instead of formulæ involving direction cosines; his work, moreover, is partially in the nature of a verification rather than a direct proof, and this enables him to evade the introduction of a factor corresponding to the factor which was discarded in §4. This factor seems to me to have its uses, because it indicates that non-analytic functions are liable to obtrude themselves incidentally in work about analytic functions. I mention also that Cayley raises the question of specifying signs of rotations only to dismiss it.

The newer solution is due to Forsyth (2). This is on the lines of the work of §5 *supra*. It appears to me to be divisible into three parts, thus: (i) It is remarked that the representation of the P'-plane on the P-plane is conformal, by reason of simple properties of stereographic projection. Hence w' is a monogenic function of w, and conversely (a minor point is that due allowance has to be made for the possibility of w' or w becoming infinite). Moreover the correspondence connecting P' with P is (1, 1).

(ii) It is inferred that the monogeneity and (1, 1) correspondence obtained in (i) are sufficient to ensure that w' and w are connected by a bilinear relation.

(iii) The self-corresponding points of the bilinear transformation are obtained by elementary geometry (as in §5), and the multiplier K is then determined by taking w (and therefore also w') to be "near" a self-corresponding point. If the self-corresponding point is called w_0, then

$$| K | = \lim \left| \frac{w' - w_0}{w - w_0} \right|, \qquad \arg K = \lim \arg \left(\frac{w' - w_0}{w - w_0} \right),$$

and hence we have

$$| K | = 1, \qquad \arg K = a.$$

It is not explained by Forsyth why the phase has to be $+ a$ and not $- a$, nor does he explain why he takes w_0 to be $-2a\,(l+im)/(1+n)$, and not the other self-corresponding point; there is no difficulty in verifying that each of the incorrect choices yields the same incorrect result.

It is, of course, quite natural to use complex differentiation to deal with (i) and (iii) near the end of a work on theory of functions, even in the discussion of a problem of elementary geometry. The proposition which is the subject of (ii) is probably regarded as "obvious," this term being used to indicate that a really convincing proof of the proposition is much too troublesome to construct.

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2. A. R. Forsyth, *Theory of Functions* (Cambridge, 1900), 717.

46 Warwick New Road,

Leamington, Warwickshire.