108.32 Point masses and polygons

Introduction

The mid-points of the four sides of a quadrilateral form the vertices of a parallelogram. This was first proved by the French academician Pierre Varignon (1654–1722), and published posthumously in 1731, and details of his proof, his work and his life, can be found in [1]. However, to place Varignon's result in the context of this discussion, we shall view the midpoint of a segment as the centre of gravity of two equal point masses, one at each end of the segment.

In his Article [2], Nick Lord begins with the following result taken from [3]:

> *"Take any hexagon, and find the centres of gravity of each set of three consecutive vertices. These immediately form a hexagon whose opposite sides are equal and parallel in pairs."*

He then follows this with the observation, and proof, that Varignon's result ($n = 2$), and this result ($n = 3$), are the first two in a chain of similar results in which his general result is, for $n = 2, 3, 4, \dots$:

> *"Take any 2n-sided polygon, and find the centres of gravity of each set of n consecutive vertices. These form a 2n-sided polygon whose opposite sides are equal and parallel in pairs."*

Here we shall show that this in itself is a special case of a larger collection of similar results which we describe below.

Point masses at the vertices of a polygon

We begin with a $2n$-sided polygon, and we label its vertices $\mathbf{a}_1, \dots, \mathbf{a}_{2n}$ in this order around the polygon. We are going to distribute a unit mass over the set $\{\mathbf{a}_1, \dots, \mathbf{a}_{2n}\}$ of vertices, and to motivate this, we return to Nick Lord's result [2]. He begins with a mass of $1/n$ at each point $\mathbf{a}_1, \dots, \mathbf{a}_n$; then he considers a mass of $1/n$ placed at each point $\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{n+1}$; then at each point $\mathbf{a}_3 \dots, \mathbf{a}_{n+2}$, and so on. We can describe this in terms of a cyclic permutation if we start with the initial mass distribution of a mass $1/n$ at each point $\mathbf{a}_1, \dots, \mathbf{a}_n$, and then create the second, third, ... , mass distributions by cyclically moving the masses to the 'next' set of n vertices. For example, the second mass distribution (that is, $1/n$ at each point $\mathbf{a}_2, \dots, \mathbf{a}_{n+1}$) is created by moving the masses $1/n$ from $\mathbf{a}_1, \dots, \mathbf{a}_n$ to the points $\mathbf{a}_2, \dots, \mathbf{a}_{n+1}$, respectively. This process is then repeated until we return to the initial distribution.

To obtain a more general result we start with a unit mass which is distributed (not necessarily uniformly) over $\{\mathbf{a}_1, \dots, \mathbf{a}_{2n}\}$ by placing a mass m_i at the point $\mathbf{a}_i$ for $i = 1, \dots, 2n$, where $m_i \geqslant 0$ and $\sum_i m_i = 1$. We then define $\mathbf{b}_1$ as the centre of gravity of this mass distribution; thus $\mathbf{b}_1 = m_1\mathbf{a}_1 + \dots + m_{2n}\mathbf{a}_{2n}$. Next, $\mathbf{b}_2$ is the centre of gravity of the mass distribution obtained by cyclically permuting the masses, then $\mathbf{b}_3$ and so on; thus

$$\mathbf{b}_2 = m_1\mathbf{a}_2 + \dots + m_{2n-1}\mathbf{a}_{2n} + m_{2n}\mathbf{a}_1,$$

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$$\mathbf{b}_3 = m_1\mathbf{a}_3 + \dots + m_{2n-1}\mathbf{a}_1 + m_{2n}\mathbf{a}_2,$$

$$\dots$$

$$\mathbf{b}_{2n} = m_1\mathbf{a}_{2n} + \dots + m_{2n-1}\mathbf{a}_{2n-2} + m_{2n}\mathbf{a}_{2n-1}.$$

Throughout we shall use the notation $\mathbf{a}_k$, and m_k, where k is taken modulo $2n$ (so $\mathbf{a}_{2n+1} = \mathbf{a}_1$, and $m_{2n+1} = m_1$, and so on). Thus, for each integer p,

$$\mathbf{b}_k = \sum_{j=1}^{2n} m_j\mathbf{a}_{j+(k-1)} = \sum_{j=1}^{2n} m_{j+p}\mathbf{a}_{j+p+k-1}. \quad (1)$$

Not every initial mass distribution yields the conclusion in [2] so we shall seek a condition on the initial mass distribution which does. We shall say that the initial distribution (of a mass m_i at the point $\mathbf{a}_i$) is *balanced* if the sum of the masses at each pair of opposite vertices is $1/n$; explicitly, if

$$m_1 + m_{n+1} = m_2 + m_{n+2} = \dots = m_n + m_{2n} = \tfrac{1}{n}. \quad (2)$$

Since the initial mass distribution in [2] is defined by $m_1 = \dots = m_n = \frac{1}{n}$ and $m_{n+1} = \dots = m_{2n} = 0$, it is indeed a balanced distribution. Our more general result now follows.

Theorem 1: Let $\mathbf{a}_1, \dots, \mathbf{a}_{2n}$ be the vertices of a $2n$-sided polygon, and place a mass m_i at the vertex $\mathbf{a}_i$, where $m_i \geqslant 0$ and $\sum_j m_j = 1$, in such a way that this is a balanced distribution (that is, so that (2) holds). Then the points $\mathbf{b}_1, \dots, \mathbf{b}_{2n}$, defined by $\mathbf{b}_k = m_1\mathbf{a}_k + \dots + m_{2n}\mathbf{a}_{k+2n-1}$ are the consecutive vertices of a $2n$-sided polygon whose opposite sides are parallel and of equal length.

Proof: We wish to show that $\mathbf{b}_{k+1} - \mathbf{b}_k = -(\mathbf{b}_{k+1+n} - \mathbf{b}_{k+n})$ or, equivalently, $\mathbf{b}_k + \mathbf{b}_{k+n} = \mathbf{b}_{k+1} + \mathbf{b}_{k+1+n}$, so it is enough to show that $\mathbf{b}_{n+k} + \mathbf{b}_k$ is independent of k. Now (1) and (2) show that

$$\begin{aligned}
\mathbf{b}_{n+k} + \mathbf{b}_k &= \sum_r m_r\mathbf{a}_{n+k+r-1} + \sum_r m_r\mathbf{a}_{k+r-1} \\
&= \sum_r m_r\mathbf{a}_{n+k+r-1} + \sum_r m_{r+n}\mathbf{a}_{n+k+r-1} \\
&= \sum_r (m_r + m_{r+n})\,\mathbf{a}_{n+k+r-1} \\
&= \frac{1}{n}\sum_r \mathbf{a}_{n+k+r-1} \\
&= \frac{1}{n}\sum_r \mathbf{a}_r,
\end{aligned}$$

which is independent of k as required. We mention, in passing, that this shows that $\sum_k \mathbf{b}_k = \sum_r \mathbf{a}_r$, so that the centroid of unit masses placed at the points $\mathbf{a}_i$ is the same as that of the unit masses placed at the $\mathbf{b}_k$.

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Some examples

The process described above starts with a polygon P, whose vertices are the $\mathbf{a}_i$, and a unit mass distribution m_i which, for brevity, we denote by μ, on $\{\mathbf{a}_1, \ldots, \mathbf{a}_{2n}\}$, and uses these to create a new polygon $P(\mu)$ whose vertices are the $\mathbf{b}_j$. Let us consider the case of a quadrilateral. First, consider the balanced, uniform, mass distribution $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. In this case rotating the point masses has no effect, and each $\mathbf{b}_k$ is $\frac{1}{4}\left(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4\right)$; thus the parallelogram $P(\mu)$ is just the single point $\frac{1}{4}\left(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4\right)$, and all sides of $P(\mu)$ have zero length.

Next, Varignon's result corresponds to the initial, balanced, mass distribution $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$. This gives the initial distribution as a mass of $\frac{1}{2}$ at $\mathbf{a}_1$ and $\mathbf{a}_2$, so $\mathbf{b}_1 = \frac{1}{2}\left(\mathbf{a}_1 + \mathbf{a}_2\right)$. Next, $\mathbf{b}_2$ is the centre of gravity of a mass of $\frac{1}{2}$ placed at each of $\mathbf{a}_2$ and $\mathbf{a}_3$, and so on. For $2n$-gons, Nick Lord's development begins with the initial mass distribution $\left(\frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots, 0\right)$. As another example, if we consider the balanced mass distribution $\left(\frac{3}{8}, \frac{4}{8}, \frac{1}{8}, 0\right)$ we obtain the parallelogram illustrated in Figure 1.

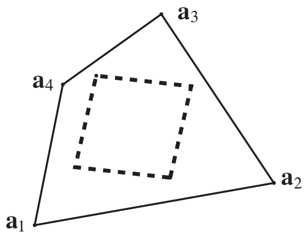

FIGURE 1: The mass distribution $\left(\frac{3}{8}, \frac{4}{8}, \frac{1}{8}, 0\right)$

Finally, if we start with an arbitrary quadrilateral with vertices $\mathbf{a}_i$ and then apply Varignon's theorem, we obtain a parallelogram with vertices $\mathbf{b}_j$. Another application to this second parallelogram yields a parallelogram with vertices $\mathbf{c}_k$, and so on: see Figure 2. We leave the reader to calculate the vectors $\mathbf{c}_k$ in terms of the $\mathbf{a}_i$ and hence the mass distribution on the $\mathbf{a}_i$ which produces the vectors $\mathbf{c}_k$ directly.

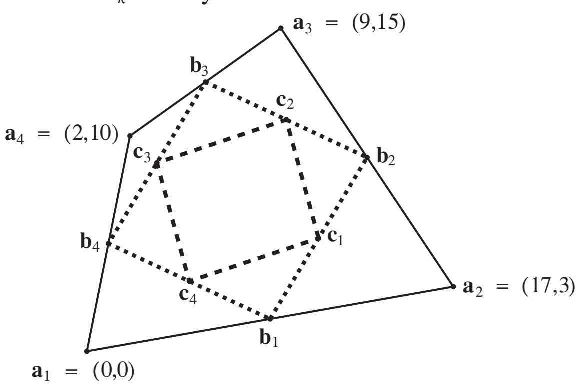

FIGURE 2: The initial mass distribution $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0\right)$ gives the $\mathbf{c}_j$

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108.33 Some inequalities for a triangle

In a recent Article [1] an upper bound was derived for $h_a + h_b + h_c$, the sum of the (lengths of the) altitudes of a triangle. In this Note we find a different upper bound in terms of R, the radius of the circumcircle. We also derive several other inequalities for a triangle which we have been unable to find in the literature, despite the fact that they follow quickly from known results.

Our notation is standard – for a triangle ABC, a, b and c are the side-lengths, $2s = a + b + c$ and r is the radius of the incircle. R is the radius of the circumcircle and r_a, r_b and r_c are the radii of the excircles, while h_a, h_b and h_c are the altitudes. The shorthand [WEIFFTTIE]. will indicate the phrase, “With equality if and only if the triangle is equilateral”, throughout.

We need these known preliminary results, all easily proved and widely available in [2] and [3], for example.

Lemma 1: We have $h_a + h_b + h_c \leqslant \frac{\sqrt{3}}{2}(a + b + c)$. [WEIFFTTIE]. See [3, p. 274].

Lemma 2: We have $a = 2R \sin A$; $b = 2R \sin B$; $c = 2R \sin C$. See [2, p. 200].

Lemma 3: We have $\sin A + \sin B + \sin C \leqslant \frac{3}{2}\sqrt{3}$. [WEIFFTTIE]. See [2, p. 315}.

Lemma 4: We have $r_a + r_b + r_c - r = 4R$. See [2, p. 207].

Lemma 5 (Euler 1767): We have $R \geqslant 2r$. [WEIFFTTIE]. See [2, p. 216].

Euler's proof of this result was very beautiful. He showed that the distance d between the incentre and the circumcentre is given by $d^2 = R(R - 2r)$ and since $d^2 \geqslant 0$, we have $R \geqslant 2r$.