J. Austral. Math. Soc. (*Series A*) **45** (1988), 169–183

A GEOMETRIC PROPERTY OF CONVEX SETS WITH APPLICATIONS TO MINIMAX TYPE INEQUALITIES AND FIXED POINT THEOREMS

MAU-HSIANG SHIH and KOK-KEONG TAN

(Received 14 April 1986; revised 31 August 1986)

Communicated by J. H. Rubinstein

Abstract

A geometric property of convex sets which is equivalent to a minimax inequality of the Ky Fan type is formulated. This property is used directly to prove minimax inequalities of the von Neumann type, minimax inequalities of the Fan-Kneser type, and fixed point theorems for inward and outward maps.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 52 A 07, 49 A 40; secondary 47 H 10.
Keywords and phrases: minimax inequality, fixed point, inward and outward maps, partition of unity, convex set, topological vector space.

1. Introduction

Properties of convex sets in topological vector spaces related to fixed point and minimax theorems were given in Fan [7, 10–15]. In 1972, Fan [13, Theorem 2] proved the following geometric theorem of convex sets which has numerous connections with other areas of mathematics and serves to unify many apparently diverse mathematical phenomena.

THEOREM 1 [KY FAN]. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space and let $B \subset X \times X$. Assume*

(a) *For each fixed $x \in X$, the section $\{y \in X\colon (x,y) \in B\}$ is open in X.*

This work was partially supported by NSERC of Canada under Grant A8096.

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

(b) *For each fixed* $y \in X$, *the section* $\{x \in X\colon (x,y) \in B\}$ *is non-empty and convex.*
Then there exists a point $x_0 \in X$ *such that* $(x_0, x_0) \in B$.

In the present paper we shall extend Theorem 1 by relaxing the compactness and convexity conditions. Direct applications to minimax type inequalities and fixed point theorems are illustrated.

2. A Geometric property of convex sets

THEOREM 2. *Let* X *be a non-empty convex subset of a Hausdorff topological vector space and let* $A, B \subset X \times X$. *Assume*

(a) *For each fixed* $x \in X$, *the section* $\{y \in X\colon (x,y) \in A\}$ *is open in* X.

(b) *For each fixed* $y \in X$, *the section* $\{x \in X\colon (x,y) \in B\}$ *contains the convex hull of the section* $\{x \in X\colon (x,y) \in A\}$.

(c) *There exist a non-empty compact convex subset* X_0 *and a non-empty compact subset* K *of* X *such that*

(c_1) *the section* $\{x \in X\colon (x,y) \in A\} \neq \emptyset$ *for all* $y \in K$ *and*

(c_2) $X_0 \cap \{x \in X\colon (x,y) \in A\} \neq \emptyset$ *for all* $y \in X \setminus K$.

Then there exists a point $x_0 \in X$ *such that* $(x_0, x_0) \in B$.

PROOF. For each $x \in X$, let $A(x) = \{y \in X\colon (x,y) \in A\}$; then by (a), $A(x)$ is open in X for each $x \in X$. By (c_1), $K \subset \bigcup_{x \in X} A(x)$. By compactness of K, there exists $\{x_1, x_2, \dots, x_n\} \subset X$ such that

$$(*) \qquad K \subset \bigcup_{i=1}^{n} A(x_i).$$

Let Ω be the convex hull of $X_0 \cup \{x_1, x_2, \dots, x_n\}$ and define

$$\tilde{A} = A \cap (\Omega \times \Omega),$$

$$\tilde{B} = B \cap (\Omega \times \Omega).$$

Then Ω is a compact convex subset of X and we have:

(i) For each fixed $x \in \Omega$, the section $\{y \in \Omega\colon (x,y) \in \tilde{A}\}$ is open in Ω by (a).

(ii) For each fixed $y \in \Omega$, the section $\{x \in \Omega\colon (x,y) \in \tilde{B}\}$ contains the convex hull of the section $\{x \in \Omega\colon (x,y) \in \tilde{A}\}$ by (b).

(iii) For each fixed $y \in \Omega$, the section $\{x \in \Omega\colon (x,y) \in \tilde{A}\} \neq \emptyset$ by (c_1), (c_2) and ($*$).

Now, for each $x \in \Omega$, let $\tilde{A}(x) = \{y \in \Omega \colon (x,y) \in \tilde{A}\}$; then by (i) $\tilde{A}(x)$ is open in Ω for each $x \in \Omega$. By (iii), $\Omega = \bigcup_{x\in\Omega} \tilde{A}(x)$. By compactness of Ω, there exists $\{y_1, y_2, \ldots, y_m\} \subset \Omega$ such that

$$\Omega = \bigcup_{j=1}^{m} \tilde{A}(y_j).$$

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a partition of unity subordinate to the covering $\{\tilde{A}(y_1), \tilde{A}(y_2), \ldots, \tilde{A}(y_m)\}$. Thus, $\alpha_1, \alpha_2, \ldots, \alpha_m$ are continuous non-negative functions on Ω such that for each $j = 1, 2, \ldots, m$, $\operatorname{supp} \alpha_j \subset \tilde{A}(y_j)$ and

$$\sum_{j=1}^{m} \alpha_j(y) = 1 \quad \text{for all } y \in \Omega.$$

Define $p \colon \Omega \to \Omega$ by setting

$$p(y) = \sum_{j=1}^{m} \alpha_j(y) y_j.$$

Then p is a continuous map which maps the convex hull $\operatorname{conv}\{y_1, y_2, \ldots, y_m\}$ of $\{y_1, y_2, \ldots, y_m\}$ into itself. By Brouwer's fixed point theorem, there exists a point $x_0 \in \operatorname{conv}\{y_1, y_2, \ldots, y_m\}$ such that $p(x_0) = x_0$. Note that for each $j = 1, 2, \ldots, m$, if $\alpha_j(x_0) > 0$, then $x_0 \in \tilde{A}(y_j)$ so that $(y_j, x_0) \in \tilde{A}$; it follows from (ii) that $(p(x_0), x_0) \in \tilde{B}$. This proves the theorem.

REMARKS. (1) When $A = B$, $X = X_0 = K$, Theorem 2 reduces to Theorem 1. (2) When $A = B$, $X_0 = K$, Theorem 2 still contains a theorem of Fan [14, Theorem 10]. (3) When $X = X_0 = K$, Theorem 2 reduces to our earlier formulation [22, Theorem 3]. (4) Theorem 2 is equivalent to the following:

THEOREM 2′. *Same hypotheses and conclusions as in Theorem 2 except that the condition* (b) *is replaced by* (b_1) $A \subset B$, *and* (b_2) *for each fixed* $y \in X$, *the section* $\{x \in X \colon (x,y) \in B\}$ *is convex.*

Theorem 2 has the following analytic form.

THEOREM 3. *Let X be a non-empty convex subset of a Hausdorff topological vector space and let f and g be two real-valued functions on $X \times X$. Assume*

(a) $g(x,x) \le 0$ *for all* $x \in X$.

(b) *For each fixed* $x \in X$, $f(x,y)$ *is a lower semi-continuous function of* y *on* X.

(c) *For each fixed* $y \in X$, *the set* $\{x \in X \colon g(x,y) > 0\}$ *contains the convex hull of the set* $\{x \in X \colon f(x,y) > 0\}$.

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

(d) *There exists a non-empty compact convex subset* X_0 *of* X *such that the set* $\{y \in X \colon f(x,y) \leq 0$ *for all* $x \in X_0\}$ *is compact.*
Then there exists a point $\tilde{y} \in X$ *such that* $f(x,\tilde{y}) \leq 0$ *for all* $x \in X$.

Indication of a proof for the equivalence of Theorems 2 and 3:
Theorem 2 $\Rightarrow$ Theorem 3. Let

$$\begin{aligned} A &= \{(x,y) \in X \times X \colon f(x,y) > 0\}, \\ B &= \{(x,y) \in X \times X \colon g(x,y) > 0\}, \\ K &= \{y \in X \colon f(x,y) \leq 0 \textit{ for all } x \in X_0\}, \end{aligned}$$

and apply Theorem 2.

Theorem 3 $\Rightarrow$ Theorem 2. Let f and g be characteristic functions of A and B, respectively, and apply Theorem 3.

REMARKS. (1) When $f \equiv g$, $X = X_0 = K$, Theorem 3 reduces to the well-known Ky Fan minimax principle [13]. (2) When $f \equiv g$, Theorem 3 reduces to Fan's theorem [15, Theorem 6]. (3) When $f \equiv g$ and $X_0 = K$, Theorem 3 reduces to Allen's theorem [1, Theorem 2]. (4) When $X_0 = K$, Theorem 3 reduces to Tan's theorem [26, Theorem 1]. (5) When $X = X_0 = K$, Theorem 3 reduces to Yen's theorem [27]. (6) Conditions (a) and (b) imply the set $\{y \in X \colon f(x,y) \leq 0$ for all $x \in X_0\}$ is non-empty. The coercive condition (d) is a unification of the two coercive conditions given in Allen [1, Theorem 2, condition (d)] and in Brézis-Nirenberg-Stampacchia [3, Theorem 1, condition (5)].

The following example shows that Allen's theorem [1, Theorem 2] is properly contained in Fan's theorem [15, Theorem 6].

EXAMPLE. Let $0 < p < 1$,

$$l_p = \left\{ x = (x(n))_{n=1}^{\infty} \colon \sum_{n=1}^{\infty} |x(n)|^p < \infty \right\},$$

$$d_p(x,y) = \sum_{n=1}^{\infty} |x(n) - y(n)|^p, \quad \text{for all } x = (x(n))_{n=1}^{\infty}, y = (y(n))_{n=1}^{\infty} \in l_p.$$

Then (l_p, d_p) is a completely metrizable topological vector space which is not locally convex. Let $(x_n)_{n=0}^{\infty}$ be a sequence in l_p defined by

$$x_0 = (0,0,\dots), \quad x_n(k) = \begin{cases} 0, & \text{if } k \neq n, \\ \dfrac{1}{n^{1-p}}, & \text{if } k = n. \end{cases}$$

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

Let $K = \{x_n : n = 0, 1, 2, \dots\}, X = \operatorname{conv}(K)$, the convex hull of K. Since $x_n \to 0$ as $n \to \infty, K$ is compact. Define $f \colon X \times X \to \mathbf{R}$ by

$$f(x, y) = 0 \quad \text{for each } (x, y) \in X \times X \text{ with } x \neq 0,$$
$$f(0, \alpha x_{2n}) = 0 \quad \text{for each } n = 0, 1, 2, \dots \text{ and for each } \alpha \in [0, 1],$$
$$f(0, \alpha x_{2n+1}) = \frac{1}{2n+1} \quad \text{for each } n = 0, 1, 2, \dots \text{ and for each } \alpha \in (0, 1],$$
$$f\left(0, \sum_{i=1}^{N} \alpha_i x_{n_i}\right) = N \quad \text{for each } N \geq 2, \text{ for each } \alpha_1, \alpha_2, \dots, \alpha_N \in (0, 1]$$
$$\text{with } 1 \leq n_1 < n_2 < \cdots < n_N \text{ such that } \sum_{i=1}^{N} \alpha_i \leq 1.$$

I. *For each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous.*

Let $\lambda \in \mathbf{R}$ be given.

Case 1. Suppose $x \neq 0$. Then the set

$$\{y \in X \colon f(x, y) \leq \lambda\} = \begin{cases} \varnothing & \text{if } \lambda < 0, \\ X & \text{if } \lambda \geq 0, \end{cases}$$

is open in X.

Case 2. Suppose $x = 0$. Then we see that

$$\{y \in X \colon f(0, y) \leq \lambda\} = \begin{cases} \varnothing, \quad \text{if } \lambda < 0, \\ A_0 = \{\alpha x_{2n} \colon n = 0, 1, 2, \dots, \alpha \in [0, 1]\}, \quad \text{if } \lambda = 0, \\ A_0 \cup \{\beta x_{2n+1} \colon n \geq N + 1, \beta \in [0, 1]\}, \\ \qquad\qquad \text{if } \dfrac{1}{2N+3} \leq \lambda < \dfrac{1}{2N+1}, \ N = 0, 1, 2, \dots, \\ \{\alpha x_n \colon n = 0, 1, 2, \dots, \alpha \in [0, 1]\}, \quad \text{if } 1 \leq \lambda < 2, \\ A_N, \quad \text{if } 2 \leq N \leq \lambda < N + 1, \end{cases}$$

where

$$A_N = \left\{ \sum_{i=1}^{N} \alpha_i x_{n_i} \colon 0 \leq n_1 < n_2 < \cdots < n_N, \alpha_1, \alpha_2, \dots, \alpha_N \in [0, 1], \sum_{i=1}^{N} \alpha_i \leq 1 \right\}.$$

Now, A_0 is compact, being the continuous image of the compact set $\{x_{2n} \colon n = 0, 1, 2, \dots\} \times [0, 1]$. Similarly $\{\beta x_{2n+1} \colon n \geq N + 1, \beta \in [0, 1]\}$ and $\{\alpha x_n \colon n = 0, 1, 2, \dots, \alpha \in [0, 1]\}$ are compact. To show $\{y \in X \colon f(0, y) \leq \lambda\}$ is closed in X, it remains to show that A_N is closed in X for $N \geq 2$; in fact, each A_N is compact, since it is the continuous image of the compact set $P_N \times \left(\prod_{i=1}^{N} K\right)$ where $P_N = \{(\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbf{R}^N \colon \lambda_i \geq 0 \text{ for all } i = 1, 2, \dots, N \text{ and } \sum_{i=1}^{N} \lambda_i \leq 1\}$.

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

II. *For each fixed* $y \in X$, $x \mapsto f(x,y)$ *is quasi-concave.*
Let $\lambda \in \mathbf{R}$ be given.

Case 1. If $\lambda < 0$ or $y = \alpha x_{2n}, n = 0,1,2,\dots,\alpha \in (0,1]$, the set $\{x \in X \colon f(x,y) > \lambda\} = X$ is convex.

Case 2. If $\lambda \geq 0$ and $y = \alpha x_{2n+1}$ for $n = 0,1,2,\dots,\alpha \in (0,1]$, the set

$$\{x \in X \colon f(x,y) > \lambda\} = \begin{cases} \{0\}, & \text{if } 0 \leq \lambda < \dfrac{1}{2n+1}, \\ \varnothing, & \text{if } \lambda \geq \dfrac{1}{2n+1}, \end{cases}$$

is also convex.

Case 3. If $\lambda \geq 0$ and $y = \sum_{i=1}^N \alpha_i x_{n_i}$ for $1 \leq n_1 < \dots < n_N, \alpha_1, \dots, \alpha_N \in (0,1)$ with $\sum_{i=1}^N \alpha_i \leq 1$ where $N \geq 2$,

$$\{x \in X \colon f(x,y) > \lambda\} = \begin{cases} \{0\}, & \text{if } 0 \leq \lambda < N, \\ \varnothing, & \text{if } \lambda \geq N, \end{cases}$$

is also convex.

III. *Allen's coercive condition is not satisfied, i.e., there does not exist a non-empty compact convex subset* M *of* X *such that for each* $y \in X \backslash M$, *there exists* $x \in M$ *with* $f(x,y) > 0$.

Suppose the contrary, that is suppose there exists a non-empty compact convex subset M of X such that for each $y \in X \backslash M$, there exists $x \in M$ with $f(x,y) > 0$. Since $f(x,y) = 0$ for each $x, y \in X$ with $x \neq 0$, we must have $0 \in M$ and for all $y \in X \backslash M$, $f(0,y) > 0$. As $f(0, x_{2n}) = 0$ for $n = 0,1,2,\dots$, $\{x_{2n} \colon n = 0,1,2,\dots\} \subset M$ and hence $\operatorname{conv}\{x_{2n} \colon n = 0,1,2,\dots\} \subset M$ since M is convex. We shall show that $\operatorname{conv}\{x_{2n} \colon n = 0,1,2,\dots\}$ is unbounded. Indeed

$$\begin{aligned} d_p\left(0, 1/N \sum_{n=1}^N x_{2n}\right) &= \sum_{n=1}^N 1/N^p \cdot 1/(2n)^{(1-p)p} \\ &\geq N \cdot 1/N^p \cdot 1/(2N)^{(1-p)p} \\ &= 1/2^{p(1-p} \cdot N^{(1-p)^2} \to \infty \quad \text{as } N \to \infty. \end{aligned}$$

Therefore $\operatorname{conv}\{x_{2n} \colon n = 0,1,2,,\dots\}$ is an unbounded subset of the compact convex set M, which is impossible.

IV. *Fan's coercive condition is satisfied, that is there exists a non-empty compact convex subset* X_0 *of* X *such that the set* $\{y \in X \colon f(x,y) \leq 0$ *for all* $x \in X_0\}$ *is compact.*

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

Indeed, take $X_0 = \{0\}$; then

$$\{y \in X\colon f(x,y) \le 0 \text{ for all } x \in X_0\} = \{\alpha x_{2n}\colon n = 0,1,2,\dots, \alpha \in [0,1]\} = A_0$$

is compact.

3. Minimax inequalities of the von Neumann type

We have shown that Theorem 2 is equivalent to a minimax inequality of the Ky Fan type. We shall now show that Theorem 2 also implies minimax inequalities of the von Neumann type directly.

THEOREM 4. *Let X and Y be non-empty convex sets, each in a Hausdorff topological vector space, and let f, u, v, g be four real-valued functions on $X \times Y$. Assume*

(a) *$u \le v$ on $X \times Y$.*

(b) *For each fixed $x \in X$, $f(x,y)$ is a lower semi-continuous function of y on Y.*

(c) *For each fixed $y \in Y$, $g(x,y)$ is an upper semi-continuous function of x on X.*

(d) *For each fixed $y \in Y$ and for each $\lambda \in \mathbf{R}$, the sectin $\{x \in X\colon u(x,y) > \lambda\}$ contains the convex hull of the section $\{x \in X\colon f(x,y) > \lambda\}$.*

(e) *For each fixed $x \in X$ and for each $\lambda \in \mathbf{R}$, the section $\{y \in Y\colon v(x,y) < \lambda\}$ contains the convex hull of the section $\{y \in Y\colon g(x,y) < \lambda\}$.*

(f) *For a given fixed $\rho \in \mathbf{R}$, suppose there exist a non-empty compact convex subset X_0 of $X \times Y$ and a non-empty compact subset K of $X \times Y$ such that*

$$X_0 \cap [\{w \in X\colon f(w,y) > \rho\} \times \{z \in Y\colon g(x,z) < \rho\}] \ne \varnothing$$

for each $(x,y) \in (X \times Y)\backslash K$.
Then there exists a point $(x_0, y_0) \in K$ such that either $f(x,y_0) \le \rho$ for all $x \in X$ or $g(x_0,y) \ge \rho$ for all $y \in Y$.

PROOF. For each $(x,y) \in X \times Y$, let

$$C(y) = \{x \in X\colon f(x,y) > \rho\}, \qquad D(y) = \{x \in X\colon u(x,y) > \rho\},$$
$$E(x) = \{y \in Y\colon v(x,y) < \rho\}, \qquad F(x) = \{y \in Y\colon g(x,y) < \rho\}.$$

Define

$$A = \bigcup_{(x,y)\in X\times Y} C(y) \times F(x) \times \{(x,y)\},$$
$$B = \bigcup_{(x,y)\in X\times Y} D(y) \times E(x) \times \{(x,y)\}.$$

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

Suppose that the assertion of the theorem were false. Then for each point $(\bar{x},\bar{y}) \in K$, there exists $(x,y) \in X \times Y$ such that $F(x,\bar{y}) > \rho$ and $g(\bar{x},y) < \rho$ so that

$$\{(x,y) \in X \times Y : ((x,y),(\bar{x},\bar{y})) \in A\} \neq \emptyset \quad \text{for each } (\bar{x},\bar{y}) \in K.$$

By (f),

$$X_0 \cap \{(x,y) \in X \times Y : ((x,y),(\bar{x},\bar{y})) \in A\} \neq \emptyset \quad \text{for each } (\bar{x},\bar{y}) \in (X \times Y)\backslash K.$$

Other conditions of Theorem 2 are easily derived from the hypotheses of Theorem 4. Thus, according to Theorem 2, there exists a point $(x_0,y_0) \in X \times Y$ such that $((x_0,y_0),(x_0,y_0)) \in B$; it follows that

$$\rho < u(x_0,y_0) \leq v(x_0,y_0) < \rho,$$

which is a contradiction. This proves the theorem.

When X and Y are compact, the condition (f) in Theorem 4 is satisfied by settng $X_0 = K = X \times Y$. Thus Theorem 4 is a generalization of a minimax inequality in [2, Theorem 5.4] by relaxing the compactness and convexity conditions.

Theorem 4 implies the following:

THEOREM 5. *Let X and Y be non-empty convex sets, each in a Hausdorff topological vector space, and let f,u,v,g be four real-valued functions on $X \times Y$. Assume the conditions* (a), (b), (c), (d), (e) *in Theorem 4 are satisfied.*

(1) *If there exists non-empty compact convex sets $M_0 \subset X, N_0 \subset Y$ and there exist non-empty compact sets $M \subset X$, $N \subset Y$ such that* $\inf_{y\in Y}\sup_{x\in M_0} f(x,y) = \inf_{y\in N}\sup_{x\in X} f(x,y)$, *and* $\sup_{x\in X}\inf_{y\in N_0} g(x,y) = \sup_{x\in M}\inf_{y\in Y} g(x,y)$, *then the following minimax inequality holds:*

Inequality I: $\inf_{y\in N}\sup_{x\in X} f(x,y) \leq \sup_{x\in M}\inf_{y\in Y} g(x,y)$.

(2) *If there exist non-empty compact convex sets $M_0 \subset X$ and $N_0 \subset Y$ such that* $\inf_{y\in Y}\sup_{x\in M_0} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y)$ *and* $\sup_{x\in X}\inf_{y\in N_0} g(x,y) = \sup_{x\in X}\inf_{y\in Y} g(x,y)$ *then the following minimax inequality holds:*

Inequality II. $\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y)$.

When $f \equiv u \equiv v \equiv g$, it is readily seen that Inequality I in Theorem 5 implies the following minimax equalities, which generalize the minimax principle of the von Neumann type due to Sion [19]:

(i) $$\min_{y\in N}\sup_{x\in X} f(x,y) = \max_{x\in M}\inf_{y\in Y} f(x,y),$$

(ii) $$\inf_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\inf_{y\in Y} f(x,y).$$

When $f \equiv u, v \equiv g$, Theorem 5 also contains a minimax inequality of Liu [19].

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

4. Systems of convex inequalities

According to Pietsch [21, page 40], a collection $\mathcal{F}$ of real-valued functions f defined on a set X is called *concave* if, given any finite subset $\{f_1, f_2, \ldots, f_n\}$ of $\mathcal{F}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, there exists $f \in \mathcal{F}$ such that $f(x) \geq \sum_{i=1}^n \alpha_i f_i(x)$ for all $x \in X$.

The following theorem given in Pietsch's book [21, page 40] concerning systems of convex inequalities is useful to study absolutely r-summing operators [21, page 232], (p,q)-dominated operators [21, page 236] and absolutely τ-summing operators [21, page 324].

THEOREM 6. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space, and let $\mathcal{F}$ be a concave collection of lower semi-continuous convex real-valued functions f on X. Suppose that for every $f \in \mathcal{F}$ there exists an $x \in X$ with $f(x) \leq \rho$. Then there exists a point $x_0 \in X$ such that $f(x_0) \leq \rho$ for all $f \in \mathcal{F}$ simultaneously.*

Observe that Theorem 6 is equivalent to a theorem of Fan [9, Theorem 1] and Pietsch referred Theorem 6 as Fan's Lemma. The proof of Theorem 6 in Pietsch's book used the well-known separation theorem on convex sets. Granas-Liu [16] obtained a result which is a generalization of Theorem 6 to three collections of functions whose proof used a minimax inequality of the von Neumann type. We shall use Theorem 2 (or equivalently, Theorem 3, which is a Ky Fan type minimax inequality) to further extend Theorem 6.

Given any two collections $\mathcal{F}$ and $\mathcal{G}$ of real-valued functions on a set X, we shall write $\mathcal{F} \leq \mathcal{G}$ if for any $f \in \mathcal{F}$ there exists $g \in \mathcal{G}$ such that $f(x) \leq g(x)$ for all $x \in X$.

THEOREM 7. *Let X be a non-empty normal closed convex set in a Hausdorff topological vector space. Let $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ be three collections of real valued functions on X such that*

(a) $\mathcal{F} \leq \mathcal{G} \leq \mathcal{H}$;

(b) *for each $f \in \mathcal{F}$, f is lower semi-continuous on X;*

(c) *for each $g \in \mathcal{G}$, g is convex on X;*

(d) *the collection $\mathcal{H}$ is concave;*

(e) *X has a non-empty compact convex subset X_0 and a non-empty compact subset K such that for any two finite sets $\{f_1, f_2, \ldots, f_n\} \subset \mathcal{F}$, $\{g_1, g_2, \ldots, g_n\} \subset \mathcal{G}$, for any $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, and for any $y \in X \backslash K$ there*

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

exists $x \in X_0$ such that

$$\sum_{i=1}^{n} \alpha_i f_i(y) > \sum_{i=1}^{n} \alpha_i g_i(x).$$

Then given any $\rho \in \mathbf{R}$ one of the following properties holds.

(i) *There is an $h \in \mathcal{H}$ such that* $\inf_{x \in X} h(x) > \rho$.

(ii) *There exists a point $\hat{y} \in K$ such that $f(\hat{y}) \le \rho$ for all $f \in \mathcal{F}$.*

PROOF. Without loss of generality we may assume that $\rho = 0$. For each $f \in \mathcal{F}$, let $Q(f) = \{x \in K : f(x) \le 0\}$; then $Q(f)$ is closed in K by lower semi-continuity of f. If the set $\{Q(f) : f \in \mathcal{F}\}$ has the finite intersection property, then by compactness of K we obtain the alternative (ii). Suppose $\{Q(f) : \in \mathcal{F}\}$ does not have the finite intersection property, then there are $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that $\bigcap_{i=1}^{n} Q(f_i) = \varnothing$. For each $i = 1, 2, \ldots, n$, let $V_i = X \backslash Q(f_j)$; then each V_i is open in X and $\{V_1, V_2, \ldots, V_n\}$ is an open covering of the normal space X. Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a continuous partition of unity subordinate to this open covering. Thus, $\beta_1, \beta_2, \ldots, \beta_n$ are continuous non-negative functions on X such that for each $i = 1, 2, \ldots, n$, $\operatorname{supp} \beta_i \subset V_i$ and $\sum_{i=1}^{n} \beta_i(x) = 1$ for $x \in X$. Choose $g_1, g_2, \ldots, g_n \in \mathcal{G}$ and $h_1, h_2, \ldots, h_n \in \mathcal{H}$ so that $f_i \le g_i \le h_i$ on X for each $i = 1, 2, \ldots, n$. Define

$$A = \left\{ (x, y) \in X \times X : \sum_{i=1}^{n} \beta_i(y) f_i(y) > \sum_{i=1}^{n} \beta_i(y) g_i(x) \right\};$$

$$B = \left\{ (x, y) \in X \times X : \sum_{i=1}^{n} \beta_i(y) g_i(y) > \sum_{i=1}^{n} \beta_i(y) g_i(x) \right\}.$$

Then the conditions (a), (b), (c_2) of Theorem 2 are satisfied. Since for each $x \in X$, $(x, x) \notin B$, by Theorem 2, there exists $\hat{y} \in K$ such that $\{x \in X : (x, \hat{y}) \in A\} = \varnothing$. Therefore

$$\sum_{i=1}^{n} \beta_i(\hat{y}) f_i(\hat{y}) \le \sum_{i=1}^{n} \beta_i(\hat{y}) g_i(x) \quad \text{for all } x \in X.$$

By concavity of $\mathcal{H}$, there is an $h \in \mathcal{H}$ satisfying $h(x) \ge \sum_{i=1}^{n} \beta_i(\hat{y}) h_i(x)$ for all $x \in X$. Consequently,

$$\begin{aligned} 0 < \sum_{i=1}^{n} \beta_i(\hat{y}) f_i(\hat{y}) &\le \sum_{i=1}^{n} \beta_i(\hat{y}) g_i(x) \\ &\le \sum_{i=1}^{n} \beta_i(\hat{y}) h_i(x) \le h(x) \quad \text{for all } x \in X. \end{aligned}$$

This proves the alternative (i). This completes the proof.

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

In the case when X is compact convex, condition (e) in Theorem 7 is satisfied by setting $X_0 = K = X$. Thus Theorem 7 generalizes Granas-Liu's result [16]. In the case when X is compact convex and $\mathcal{F} \equiv \mathcal{G} \equiv \mathcal{H}$, Theorem 7 reduces to Theorem 6.

Let h be a real-valued function defined on the product set $X \times Y$ of two arbitrary non-empty sets X, Y. According to Fan [8], h is said to be *concave* on X, if for any two elements $x_1, x_2 \in X$ and two numbers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$, there exists $x_0 \in X$ such that

$$h(x_0, y) \geq \alpha_1 h(x_1, y) + \alpha_2 h(x_2, y) \quad \text{for all } y \in Y.$$

THEOREM 8. *Let X be an arbitrary non-empty set and Y a non-empty normal closed convex subset of a Hausdorff topological vector space. Let f, g, h: $X \times Y \to \mathbf{R}$ be three functions such that*

(a) *$f \leq g \leq h$ on $X \times Y$;*

(b) *for each fixed $x \in X$, $f(x, y)$ is a lower semi-continuous function of y on Y;*

(c) *for each fixed $x \in X$, $g(x, y)$ is a convex functions of y on Y;*

(d) *h is concave on X;*

(e) *Y has a non-empty compact convex subset X_0 and a non-empty compact subset K such that for each finite subset $\{x_1, x_2, \dots, x_n\}$ of X, for any $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, and for any $y \in Y \setminus K$ there exists $x \in X_0$ such that $\sum_{i=1}^n \alpha_i f(x_i, y) > \sum_{i=1}^n \alpha_i g(x_i, x)$.*

Then

$$\min_{y \in K} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} h(x, y).$$

PROOF. Let $\rho = \sup_{x \in X} \inf_{y \in Y} h(x, y)$. Applying Theorem 7 with X being the index set, there is a $\hat{y} \in K$ such that $f(x, \hat{y}) \leq \rho$ for all $x \in X$. The conclusion follows.

In Theorem 8, X is not required to possess any topological or linear structure. When X is convex and Y is compact convex, Theorem 8 is due to Granas-Liu [16]. The connection of Fan's convex inequalities with minimax theorems was pointed out by Takahashi [25].

When $f \equiv g \equiv h$, we obtain the following new minimax theorem.

THEOREM 9. *Let X be an arbitrary non-empty set and Y a non-empty normal closed convex subset of a Hausdorff topological vector space. Let f be a real-valued function defined on $X \times Y$ such that*

(a) *For each fixed $x \in X$, $f(x, y)$ is a lower semi-continuous convex function of y on Y;*

(b) *f is concave on X;*

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

(c) *Y has a non-empty compact convex subset X_0 and a non-empty compact subset K such that for each finite subset $\{x_1, x_2, \ldots x_n\}$ of X, for any $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$, and for any $y \in Y \backslash K$ there exists $x \in X_0$ such that*

$$\sum_{i=1}^n \alpha_i f(x_i, y) > \sum_{i=1}^n \alpha_i g(x_i, x).$$

Then

$$\min_{y \in K} \sup_{x \in X} f(x,y) = \sup_{x \in X} \inf_{y \in Y} f(x,y).$$

When X is convex and Y is compact convex, Theorem 9 is a well-known minimax theorem of Kneser [18]. Another generalization of Kneser's minimax theorem was obtained by Fan [8] where both the linear structures of X and Y are eliminated.

5. Fixed point theorems

Fixed point theorems for inward or outward maps, and for single-valued or set-valued maps were studied by Browder [4, 5, 6], Fan [12, 13, 15] and Halpern and Bergman [17]. In this section, we shall apply Theorem 2 to give a generalization of Browder's recent fixed point theorem [6] to non-compact convex sets.

THEOREM 10. *Let X be a non-empty convex subset of a Hausdorff topological vector space E and let $f: X \to E$ be a continuous map. Suppose that p is a continuous real-valued function on $X \times E$ such that for all $x \in X$, $p(x, \cdot)$ is a convex function on E. Assume that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that*

(a) *For each $y \in K$ with $y \neq f(y)$, there exists $x \in y + \bigcup_{\lambda > 0} \lambda(X - y)$ such that $p(y, x - f(y)) < p(y, y - f(y))$.*

(b) *For each $y \in X \backslash K$ with $y \neq f(y)$, there exists $x \in y + \bigcup_{\lambda \geq 1} \lambda(X_0 - y)$ such that $p(y, x - f(y)) < p(y, y - f(y))$.*

Then f has a fixed point in X.

PROOF. Suppose that f has no fixed point in X. Define $A = \{(x,y) \in X \times X : p(y, x - f(y)) < p(y, y - f(y))\}$. Then (i) For each fixed $x \in X$, the section $\{y \in X : (x,y) \in A\}$ is open in X by continuities of f and p. (ii) For each fixed $y \in X$, the section $\{x \in X : (x,y) \in A\}$ is convex since $p(y, \cdot)$ is a convex function. (iii) By (a), for each $y \in K$ there exists $x_0 \in y + \bigcup_{\lambda > 0} \lambda(X - y)$ such that $p(y, x_0 - f(y)) < p(y, y - f(y))$. If $x_0 \in X$, then the

https://doi.org/10.1017/S144678870003007X Published online by Cambridge University Press

section $\{x \in X\colon (x,y) \in A\} \neq \varnothing$. If $x_0 \notin X$, by convexity of X, there exist $\bar{x} \in X$ and $\lambda > 1$ such that $x_0 = y + \lambda(\bar{x} - y)$, so that $\bar{x} = ((\lambda-1)/\lambda)y + (1/\lambda)x_0$. As $p(y, \cdot)$ is convex, we have

$$\begin{aligned} p(y, \bar{x} - f(y)) &\le \frac{\lambda-1}{\lambda} p(y, y - f(y)) + \frac{1}{\lambda} p(y, x_0 - f(y)) \\ &< p(y, y - f(y)); \end{aligned}$$

thus $(\bar{x}, y) \in A$ and hence the section $\{x \in X\colon (x,y) \in A\} \neq \varnothing$. (iv) By (b), for each $y \in X \backslash K$, there exists $\bar{x} \in X_0$ and $\lambda \ge 1$ such that $x = y + \lambda(\bar{x} - y)$ and $p(y, x - f(y)) < p(y, y - f(y))$. If $\lambda = 1$, then $x = \bar{x}$, so that $(\bar{x}, y) \in A$. If $\lambda > 1$ by the same argument as in (iii), we also have $(\bar{x}, y) \in A$. In both cases, we conclude that

$$X_0 \cap \{x \in X\colon (x,y) \in A\} \neq \varnothing.$$

Applying Theorem 2 with $A \equiv B$, there exists $\hat{x} \in X$ such that $(\hat{x}, \hat{x}) \in A$, which is impossible. Thus f has a fixed point X, completing the proof.

Theorem 10 generalizes Browder's fixed point theorem [6, Theorem 1] to non-compact convex sets. By setting $p(x,y) = \|y\|$ in Theorem 5 if the underlying space is a normed linear space, we have the following generalization of the Browder fixed point theorem [6, Corollary 1] and therefore a new generalization of the classical Schauder fixed point theorem.

COROLLARY 1. *Let X be a non-empty convex subset of a normed linear space E and let $f\colon X \to E$ be a continuous map. Suppose that there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that*

(a) *For each $y \in K$, $f(y)$ lies in the closure of $y + \bigcup_{\lambda > 0} \lambda(X - y)$.*

(b) *For each $y \in X \backslash K$, $f(y)$ lies in the closure of $y + \bigcup_{\lambda \ge 1} \lambda(X_0 - y)$.*
Then f has a fixed point.

REMARKS. (1) Theorem 10 and Corollary 1 remain valid if in the unions $\bigcup_{\lambda > 0}$ and $\bigcup_{\lambda \ge 1}$ in conditions (a) and (b) are replaced by $\bigcup_{\lambda < 0}$ and $\bigcup_{\lambda \le -1}$, respectively. (2) A more general version of Corollary 1 has been obtained in our recent paper [23].

References

[1] G. Allen, 'Variational inequalities, complementarity problems, and duality theorems', *J. Math. Anal. Appl.* **58** (1977), 1–10.

[2] H. Ben-El-Mechaiekh, P. Deguire and A. Granas, 'Points fixes et coincidences pour les fonctions multivoques II (Applications de type ϕ et ϕ^*)', *C.R. Acad. Sci. Paris Sér. I Math.* **295** (1982), 381–384.

[3] H. Brézis, L. Nirenberg and G. Stampacchia, 'A remark on Ky Fan's minimax principle', *Boll. Un. Mat. Ital.* **6** (1972), 293–300.

[4] F. E. Browder, 'A new generalization of the Schauder fixed point theorem', *Math Ann.* **174** (1967), 285–290.

[5] F. E. Browder, 'The fixed point theory of multi-valued mappings in topological vector spaces', *Math. Ann.* **177** (1968), 283–301.

[6] F. E. Browder, 'On a sharpened form of the Schauder fixed-point theorem', *Proc. Nat. Acad. Sci. U.S.A* **74** (1977), 4749–4751.

[7] K. Fan, 'Fixed-point and minimax theorems in locally convex topological linear spaces', *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121–126.

[8] K. Fan, 'Minimax theorems', *Proc. Nat. Acad. Sci. U.S.A.* **39** (1953), 42–47.

[9] K. Fan, 'Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations', *Math. Z.* **68** (1957), 205–216.

[10] K. Fan, 'A generalization of Tychonoff's fixed point theorem', *Math. Ann.* **142** (1961), 305–310.

[11] K. Fan, 'Sur un théorème minimax', *C.R. Acad. Sci. Paris Sér I* **259** (1964), 3925–3928.

[12] K. Fan, 'Extensions of two fixed point theorems of F. E. Browder', *Math. Z.* **112** (1969), 234–240.

[13] K. Fan, 'A minimax inequality and applications', *Inequalities* III, Ed. O. Shisha, pp. 103–113 (Academic Press, 1972).

[14] K. Fan, 'Fixed-point and related theorems for non-compact convex sets', *Game theory and related topics*, Eds. O. Moeschlin and D. Pallaschke, pp. 151–156 (North-Holland, 1979).

[15] K. Fan, 'Some properties of convex sets related to fixed point theorems', *Math. Ann.* **266** (1984), 519–537.

[16] A. Granas and F.-C. Liu, 'Remark on a theorem of Ky Fan concerning systems of inequalities', *Bull. Inst. Math. Acad. Sinica* **11** (1983), 639–643.

[17] B. Halpern and G. Bergman, 'A fixed point theorem for inward and outward maps', *Trans. Amer. Math. Soc.* **130** (1968), 353–358.

[18] H. Kneser, 'Sur un théorèmé fondamental de la théorie des jeux', *C.R. Acad. Sci. Paris* **234** (1952), 2418–2420.

[19] F. C. Liu, 'A note on the von Neumann-Sion minimax principle', *Bull. Inst. Math. Acad. Sinica* **6** (1978), 517–524.

[20] J. von Neumann, 'Zur theorie der gesellschaftsspiele', *Math. Ann.* **100** (1928), 295–320.

[21] A. Pietsch, *Operator ideals* (North-Holland, Amsterdam, 1980).

[22] M.-H. Shih and K-K. Tan, 'The Ky Fan minimax principle, sets with convex sections, and variational inequalities', *Differential geometry–calculus of variations and their applications* Eds. M. Rassias and T. Rassias, pp. 471–481, a volume dedicated to the memory of L. Euler on the occasion of the 200th anniversary since his death, (Dekker, New York, 1985).

[23] M.-H. Shih and K.-K. Tan, 'Covering theorems of convex sets related to fixed-point theorems', *Nonlinear and convex analysis: Proceedings in honor of Ky Fan*, Eds. B.-L. Lin and S. Simons, pp. 235–244, (Dekker, 1987).

[24] M. Sion, 'On general minimax theorems', *Pacific J. Math.* **8** (1958), 171–176.

[25] W. Takahashi, Nonlinear variational inequalities and fixed point theorems, *J. Math. Soc. Japan* **28** (1976), 168–181.

[26] K.-K. Tan, 'Comparison theorems on minimax inequalities, variational inequalities, and fixed point theorems', *J. London Math. Soc.* **23** (1983), 555–562.
[27] C. L. Yen, 'A minimax inequality and its applications to variational inequalities', *Pacific J. Math.* **97** (1981), 477–481.

Department of Mathematics
Chung Yuan University
Chung-Li, Taiwan
Republic of China

Department of Mathematics,
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia B3H 3J5
Canada