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CHARACTERIZATIONS OF AXIOMATIC CATEGORIES OF MODELS CANONICALLY ISOMORPHIC TO (QUASI-)VARIETIES

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ABSTRACT. Let $\mathcal{M}_L(T)$ be the category of all homomorphisms (i.e. functions preserving satisfaction of atomic formulas) between models of a set of sentences T in a finitary first-order language L. Functors between two such categories are said to be *canonical* if they commute with the forgetful functors. The following properties are characterized syntactically and also in terms of closure of $\mathcal{M}_L(T)$ for some algebraic constructions (involving products, equalizers, factorizations and kernel pairs): There is a canonical isomorphism from $\mathcal{M}_L(T)$ to a variety (resp. quasivariety) in a finitary expansion of L which assigns to a model its (unique) expansion. This solves a problem of H. Volger.

In the case of a purely algebraic language, the properties are equivalent to: "$\mathcal{M}_L(T)$ is canonically isomorphic to a finitary variety (resp. quasivariety)" and, for the variety case, to "the forgetful functor of $\mathcal{M}_L(T)$ is monadic (tripleable)".

1. **Introduction.** Identities and quasi-identities are certainly very natural ways to express properties of classes of structures. It has been remarked early (by G. Birkhoff) that they identify classes permitting certain constructions such as substructures, products, quotients. With this in mind, one can consider more general ways of expressing properties, that is, more general types of sentences, and ask if they correspond to some specific (new or old) constructions. Keisler has probably cooked his "sandwiches" (see [2]) this way from universal-existential sentences. One can go in the other direction, looking if closure for a certain algebraic construction can be syntactically characterized. When successful, any of these attempts gives a so-called "preservation" theorem.

Ideally, the point of view of category theory focuses on the essence, on the idea behind the form. For example, "categorical" free algebras are exactly what they should be: the structures with the required universal property, even if they

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do not always correspond to the classical "freely generated" algebras. In fact, many interesting properties are preserved under isomorphisms of categories, but the usual frames are often too rigid to pass undamaged: for example, we know that an axiomatic class of algebras having a monadic (see [12]) forgetful functor can be seen as a variety, but in an expanded language (we will see that if the first language is finitary, then we can choose the second one also finitary).

Many characterizations of categories isomorphic to varieties and quasivarieties have been found (see [5], [9], [10], [11]). What we do in this paper is to start with a standard *category of models*, that is an axiomatic (in a finitary language) class of structures with their homomorphisms (in the sense of [2]), and ask when it is *canonically* isomorphic to a (finitary) variety or quasivariety. Here, "canonical" means that the underlying functions and sets are preserved by the isomorphism. Note that the above characterizations, combined with the fact that the forgetful functor for a category of models (for a finitary language) preserves filtered colimits (proved in [14]), give clues for an algebraic answer to our question. From these and with the help of several results in [15], we will find syntactical characterizations of these situations. We emphasize that the case of language with relation symbols is different from the purely algebraic case: we will have to be more specific about the nature of the isomorphism, essentially because of the too great freedom of the interpretations of the relation symbols with regard to homomorphisms. For varieties, the result will nevertheless solve a problem of [15]. In algebraic cases, this also gives a syntactical characterization of the monadicity of the forgetful functor. In the case of quasivarieties, the "unpleasant" form of the characterization will be shown to be imposed by the "non-uniformity" (to be defined later) of the property to be characterized.

2. **The characterizations.** We first recall some definitions.

Here a *language* will be a usual first-order predicate language L as defined for example in [2]. We emphasize that all relation and operation symbols will be finitary. A (L-)*structure* is what [2] calls a "model" (for L). A function $f:A \to B$ between underlying sets of structures $\mathfrak{A}$ and $\mathfrak{B}$ is a *homomorphism* if for each n-ary operation symbol F, each n-ary relation symbol R and each $a_1, \ldots, a_n \in A$, we have $f(F_{\mathfrak{A}}(a_1, \ldots, a_n)) = F_{\mathfrak{B}}(f(a_1), \ldots, f(a_n))$ and $R_{\mathfrak{A}}(a_1, \ldots, a_n)$ implies $R_{\mathfrak{B}}(f(a_1), \ldots, f(a_n))$ (not necessarily the opposite), where $R_{\mathfrak{A}}$, $R_{\mathfrak{B}}$, $F_{\mathfrak{A}}$ and $F_{\mathfrak{B}}$ are the interpretations of R and F in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. A *category of models* will be the category $\mathscr{M}(T)$ of all models of a theory (that is, a set of sentences) T in L with all the homomorphisms between them. $\mathscr{M}(\emptyset)$ is the category of all L-structures and homomorphisms.

We denote by $[At]$ (respectively $\wedge[At]$, $\exists\wedge[At]$) the set of all atomic formulas (respectively conjunctions of atomic formulas, and of the form $\exists y_1 \ldots y_n \psi$ with $\psi \in \wedge[At]$). A *variety* (respectively a *quasivariety*) is a category $\mathscr{M}(T)$ where T is equivalent to a set of sentences of the form $\forall \bar{x} \psi$ (respectively $\forall\bar{x}(\varphi \to \psi)$, called *universal strict Horn* sentences), where $\psi \in [At]$ (and $\varphi \in \wedge[At]$); here, $\bar{x}$ is a string of variables and $\forall\bar{x}$ means $\forall x_1 \ldots x_n$.

If $U:\mathcal{M}(T) \to \mathcal{S}et$ and $U':\mathcal{M}(T') \to \mathcal{S}et$ are the usual forgetful functors ($U\mathfrak{A} = A$, etc. . .), a functor $H:\mathcal{M}(T) \to \mathcal{M}(T')$ is called *canonical* if $U'H = U$. $\mathcal{M}(T)$ is said to be *closed for products* (respectively *for equalizers*) if the products (respectively equalizers) in $\mathcal{M}(\emptyset)$ of objects (respectively morphisms) in $\mathcal{M}(T)$ are in $\mathcal{M}(T)$. Note that a product in $\mathcal{M}(\emptyset)$ is the usual "direct product" (see [2]) and that if $f, g:\mathfrak{A} \to \mathfrak{B}$ are homomorphisms, the equalizer $\mathfrak{C}$ of f and g in $\mathcal{M}(\emptyset)$ is the substructure ("submodel" in the terminology of [2]) of $\mathfrak{A}$ on the set $\{a \in A \mid f(a) = g(a)\}$. We remark that if L has no relation symbol, then for $\mathcal{M}(T)$ to be closed for products (respectively equalizers) is equivalent to the condition that $\mathcal{M}(T)$ has products (equalizers) and U preserves them; the fact that it is not the case when relation symbols are present will be important. There is a natural factorization of morphisms in $\mathcal{M}(\emptyset)$: if $f:\mathfrak{A} \to \mathfrak{B}$ is a homomorphism, we will denote by $\mathfrak{Im}(f)$ the (unique) substructure of $\mathfrak{B}$ on the image of f; if for each morphism f in $\mathcal{M}(T)$ $\mathfrak{Im}(f)$ is a model of T, we will say that $\mathcal{M}(T)$ is *closed for factorizations*. $\mathcal{M}(T)$ is *closed for sandwiches* if $\mathfrak{A}$ is in $\mathcal{M}(T)$ for each diagram

$$\mathfrak{A} \overset{i}{\hookrightarrow} \mathfrak{B} \overset{j}{\hookrightarrow} \mathfrak{C} \text{ in } \mathcal{M}(\emptyset)$$

where i and j are embeddings, ji is an elementary embedding (see [2]) and $\mathfrak{B}$ is a model of T.

We will say that a theory T *satisfies condition* (*) if the following is true:
(*) For each $\psi(\bar{x}, \bar{y}) \in \wedge[At]$ such that $T \vdash \forall\bar{x}\exists\bar{y}(\psi(\bar{x}, \bar{y}))$, there exists $\psi'(\bar{x}, \bar{y}, \bar{z}) \in \wedge[At]$ such that

$$T \vdash \forall\bar{x}\exists\bar{y}\bar{z}(\psi(\bar{x}, \bar{y}) \wedge \psi'(\bar{x}, \bar{y}, \bar{z})) \wedge \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi'(\bar{x}, \bar{y}, \bar{z})).$$

Here, $\exists^1\bar{y}(\varphi(\bar{y}))$ abbreviates $\exists\bar{y}(\varphi(\bar{y})) \wedge \forall\bar{x}\forall\bar{z}(\varphi(\bar{x}) \wedge \varphi(\bar{z}) \to \bar{x} = \bar{z})$, where $\bar{x} = \bar{z}$ means $(\wedge_{i=1}^{n} (x_i = z_i))$. Note that a model of a sentence of the form $\forall\bar{x}\exists^1\bar{y}(\varphi(\bar{x}))$ has only one element.

In the context of the next theorem, condition (*) will permit us to define new operation symbols in a way to achieve our goal. The meaning of "definitional extension" can be found in [15], but will also be made clear in the proof.

The rather unpleasant form of condition (*) will be seen to reflect the "non-uniformity" of the property to be characterized: a property P concerning theories in a language L is said to be *uniform* if there exists a set T_P of sentences in L such that a theory T has property P if and only if $T \equiv T_1$ for a subset T_1 of T_P. See Remark 3 following the theorem.

THEOREM 1. *For any theory T in L, the following are equivalent*:

(a) *There is a canonical isomorphism from $\mathcal{M}(T)$ to a quasivariety in an expansion of L by operation symbols which assigns to a structure its* (*unique*) *expansion.*

(b) *There is a canonical isomorphism from* $\mathcal{M}(T)$ *to a quasivariety in an expansion of* L *which assigns to a structure its (unique) expansion.*

(c) $\mathcal{M}(T)$ *is closed for factorizations, products and equalizers.*

(d) $T \equiv T_1 \cup T_2$, *where* T_1 *is a set of universal strict Horn sentences and* T_2 *is a set of sentences of the form* $\forall \bar{x} \exists \bar{y} \psi$, $\psi \in \bigwedge[At]$, *and* T *satisfies condition* (*).

(e) T *has a definitional extension* T^* *in* L^* *such that* $\mathcal{M}(T^*)$ *is a quasivariety and* L*-homomorphisms between models of* T *are* L^**-homomorphisms.*

If L *has no relation symbol, these are equivalent to*:

(f) $\mathcal{M}(T)$ *is canonically isomorphic to a quasivariety.*

PROOF. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (c). Let T^* be a set of universal strict Horn sentences in an expansion L^* of L such that $\mathcal{M}(T^*)$ is a quasivariety as in (b). Let $f:\mathfrak{A} \to \mathfrak{B}$ in $\mathcal{M}(T)$. Then the L^*-substructure of the L^*-expansion of $\mathfrak{B}$ on the image of f must be a model of T^* (because T^* is universal) and its L-reduct is then in $\mathcal{M}(T)$. This proves the closure for factorizations. It is easy to see that the fact that $\mathcal{M}(T^*)$ is closed for products and equalizers implies the same for $\mathcal{M}(T)$, by the nature of the isomorphism.

(c) $\Rightarrow$ (d). We first prove that the closure of $\mathcal{M}(T)$ for factorizations and equalizers implies that T is equivalent to $(T_1' \cup T_2')$, where T_1' is the set of universal consequences of T and T_2' is the set of universal-existential-positive consequences of T.

Propositions 1 and 5 of [15] show that the fact that $\mathcal{M}(T)$ is axiomatic and closed for equalizers implies that it is closed for sandwiches. Let $A \models T_1' \cup T_2'$. Standard arguments involving saturated extensions and used to prove many preservation theorems show that there exist diagrams

$$\mathfrak{A} \xhookrightarrow{\;j\;} \mathfrak{B}''' \succ \mathfrak{B}_1 \qquad\qquad \begin{tikzcd} \mathfrak{A} \arrow[r, hook, "k"] & \mathfrak{B}'' \arrow[r, hook, "i"] & \mathfrak{A}' \\ \mathfrak{B}_2 \prec \mathfrak{B}' \arrow[ur, "h"'] & & \end{tikzcd}$$

where $\mathfrak{B}'$, $\mathfrak{B}''$, $\mathfrak{B}'''$ and $\mathfrak{A}'$ are L-structures, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are models of T, i, k and j are embeddings, "$\prec$" means "elementary embeddings", h is a surjective homomorphism and ik is an elementary embedding (see [2] Section 5.2 for the diagram on the left and the proof of Proposition 4 of [15] for the other diagram).

Because $\mathfrak{A} \equiv \mathfrak{A}'$, the isomorphism theorem of Keisler-Shelah (see [2] Theorem 6.1.15) says that there exists an ultrafilter K such that the ultrapowers $\prod_K \mathfrak{A}$ and $\prod_K \mathfrak{A}'$ are isomorphic. For a homomorphism $g:\mathfrak{C} \to \mathfrak{D}$, it is easy to verify that $\prod_K g:\prod_K \mathfrak{C} \to \prod_K \mathfrak{D}$, the function defined pointwise from g, is a

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homomorphism. Moreover, $\prod_K g$ is surjective (respectively an embedding) if g is (see [3] p. 115). The diagrams then induce a third one:

$$\prod_K \mathfrak{B}' \underset{\prod_K h}{\twoheadrightarrow} \prod_K \mathfrak{B}'' \underset{\prod_K i}{\hookrightarrow} \prod_K \mathfrak{A}' \leftrightarrows \prod_K \mathfrak{A} \underset{\prod_K j}{\hookrightarrow} \prod_K \mathfrak{B}'''$$

where $\prod_K \mathfrak{B}'$ and $\prod_K \mathfrak{B}'''$ are in $\mathscr{M}(T)$ because any structure is elementarily equivalent to its ultrapowers. $\mathscr{M}(T)$ being closed for factorizations, we conclude that $\prod_K \mathfrak{B}''$, and hence $\mathfrak{B}''$, is in $\mathscr{M}(T)$. But $\mathscr{M}(T)$ is also closed for sandwiches (see Propositions 4 and 14 of [15]), and the second diagram then says that $\mathfrak{A} \models T$.

This proves that $T \equiv T_1' \cup T_2'$.

To finish the proof, note first that any universal sentence is equivalent to a set of sentences of the form

$$\forall \bar{x}(\varphi \rightarrow (\varphi_1 \vee \ldots \vee \varphi_n)) \tag{1}$$

where $\varphi, \varphi_1, \ldots, \varphi_n$ are conjunctions of atomic formulas. Moreover any sentence of T_2' can be written in the form

$$\forall \bar{x}\left(\bigvee_{i=1}^{m} \exists \bar{y}\psi_i\right) \tag{2}$$

where each ψ_i is a conjunction of atomic formulas. In the present hypothesis, we can apply Proposition 12 of [15]; then, for each consequence of T of the form (1), there exists $j \in \{1, \ldots, n\}$ such that

$$T \vdash \forall \bar{x}(\varphi \rightarrow \varphi_j) \tag{3}$$

and for each consequence of T of the form (2) there exists $k \in \{1, \ldots, m\}$ such that

$$T \vdash \forall \bar{x} \exists \bar{y}(\psi_k). \tag{4}$$

The sentence in (3) being equivalent to a conjunction of universal strict Horn sentences, T is equivalent to a set of sentences of the forms described in (d). A similar argument to the one used in the proof of (2) $\Rightarrow$ (3) of Proposition 14 of [15] shows that its Propositions 9 and 10 imply condition (*).

(d) $\Rightarrow$ (e). For each sentence of the form $\forall \bar{x} \exists \bar{y}_1 \psi$, $\psi \in \wedge[At]$, which is a consequence of T, add to L strings $\bar{H}^{\psi_1}, \bar{H}^{\psi_2}, \ldots$ of operation symbols for each string $\psi_1(\bar{x}, \bar{y}_1, \bar{y}_2), \psi_2(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{y}_3), \ldots$ of formulas in $\wedge[At]$ such that $T \vdash \forall \bar{x} \exists \bar{y}_1 \ldots \bar{y}_{i+1}(\psi \wedge \psi_1 \wedge \ldots \wedge \psi_i)$ and $T \vdash \forall \bar{x} \exists^1 \bar{y}_1 \ldots \bar{y}_i \exists \bar{y}_{i+1}(\psi_i)$ for each i, where $\bar{H}^{\psi_i} = (H_i^1, \ldots, H_i^{k(i)})$ if $\bar{y}_i = (y_i^1, \ldots, y_i^{k(i)})$. Call L^* the expansion of L obtained, and add to T, seen as a theory in L^*, all sentences

$$\forall \bar{x}\left(\psi(\bar{x}, \bar{H}^{\psi_1}(\bar{x})) \wedge \left(\bigwedge_{i=1}^{n} \psi_i(\bar{x}, \bar{H}^{\psi_1}(\bar{x}), \ldots, \bar{H}^{\psi_i+1}(\bar{x}))\right)\right)$$

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for each positive integer n.

Then, for the new theory T^* obtained, we see easily that

$$T^* \vdash \forall \bar{x}\bar{y}_1 \ldots \bar{y}_i \left(\bigwedge_{j=1}^{i} \bar{y}_j = \bar{H}^{\psi_j}(\bar{x}) \leftrightarrow \exists \bar{y}_{i+1}(\psi_i(\bar{x}, \bar{y}_1, \ldots, y_{i+1})) \right).$$

For any $\mathfrak{A} \models T$, $\mathfrak{A}$ has clearly a unique expansion $\mathfrak{A}^*$ in $\mathscr{M}(T^*)$. In other words T^* is a definitional extension of T by new operations defined by formulas in $\exists\wedge[At]$. The fact that any homomorphism preserves formulas in $\exists\wedge[At]$ insures that a L-homomorphism $\mathfrak{A} \to \mathfrak{B}$ between models of T is a L^*-homomorphism $\mathfrak{A}^* \to \mathfrak{B}^*$. Clearly $\mathscr{M}(T^*)$ is a quasivariety.

(e) $\Rightarrow$ (a). Obvious.

(a) $\Rightarrow$ (f). Obvious.

(f) $\Rightarrow$ (c). The existence of a canonical isomorphism from a quasivariety to $\mathscr{M}(T)$ insures that it has products and equalizers and that its forgetful functor preserves them. Because of the absence of relation symbols, this implies that $\mathscr{M}(T)$ is closed for products and equalizers. For the factorizations, it follows easily from the fact that any quasivariety is closed for substructures.

REMARKS. (1) The proof of (c) $\Rightarrow$ (d), combined with an adaptation of condition (*) to the type of sentences considered (see Proposition 9 of [15]) leads to a preservation theorem for equalizers and factorizations.

(2) Consider, for a given theory T, the following 4 properties: (i) T is a quasivariety; (ii) T satisfies Theorem 1; (iii) $\mathscr{M}(T)$ is closed for limits (i.e. T satisfies Theorem 14 of [15]); (iv) The models of T are precisely the reducts of the objects of some quasivariety (in an extended language). Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). None of these implications can be reversed. For the first one, consider the theory of groups in the language with one nullary and one binary operations (no unary one to represent inverses). For the second one, let R and S be binary and unary relation symbols respectively,

$$T = \{\forall x(S(x) \to \exists y(R(x, y))), \forall xyz(R(x, y) \wedge R(x, z) \to y = z)\}$$

and $\mathfrak{A}$ and $\mathfrak{B}$ be the two models of T such that

$$A = \{a\}, B = \{b, b'\}, S_{\mathfrak{A}} = \emptyset = R_{\mathfrak{A}}, S_{\mathfrak{B}} = \{b\} \text{ and } R_{\mathfrak{B}} = \{(b, b')\}.$$

Then the image of the homomorphism $f:A \to B$ defined by $f(a) = b$ is not a model of T. Hence $\mathscr{M}(T)$ is not closed for factorizations, but it is easily seen to be closed for limits (i.e. for products and equalizers). Note that this also shows that Theorem 2 of [7] is false. The gap between (iii) and (iv) can come from the non-unicity of the expansions of a model as well as from the presence of homomorphisms which are no more homomorphisms in the extended language. [1] treats of that with examples. This is also related to the problem of the definability of "implicit operations", which is the subject of [8] and [6].

Theorem 3 of [8] implies in particular that (a) $\Rightarrow$ (e) (in our Theorem 1) and that the sentences defining the new operations can be chosen of the form $\exists\wedge[At]$ (see also [6]).

(3) The unpleasantness of condition (*) is unavoidable: Proposition 6 of [15] can be slightly generalized to cover the present situation by replacing its part (i) by "A theory T satisfying the equivalent conditions of (our) Theorem 1 is invariant under R". This is so because the theory Volger uses in his proof is also invariant under factorizations. Hence the property of verifying the conditions of Theorem 1 is not uniform.

The next result parallels Theorem 1 for varieties. It solves a problem of H. Volger (see [15] p. 47). For the definition of a monadic (or tripleable) functor, see for example [12]. We will say that $\mathcal{M}(T)$ is *closed for homomorphic images whose kernel pair is in* $\mathcal{M}(T)$ if in any diagram

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \overset{f}{\twoheadrightarrow} C$$

in $\mathcal{M}(\emptyset)$ where f is surjective, $\langle f_1, f_2\rangle$ is the kernel pair (in $\mathcal{M}(\emptyset)$) of f and $\mathfrak{A}$ and $\mathfrak{B}$ satisfy T, then $\mathfrak{C}$ is also in $\mathcal{M}(T)$.

Denote by $\hat{L}$ the language obtained from L by removing all its relation symbols. $[\widehat{At}]$ is the set of all atomic sentences of $\hat{L}$, and for $\psi \in \wedge[At]$, $\hat{\psi}$ is the conjunction of the atomic formulas in ψ which belong to $[\widehat{At}]$. Remark that if $\hat{\psi} = \emptyset$, then a model of $\forall\bar{x}\exists^1\bar{y}(\hat{\psi}(\bar{x}, \bar{y}))$ has only one element. Note also that a sentence $\forall\bar{x}(R(\bar{x}))$, R a relation symbol, can be written

$$\forall\bar{x}(\exists\bar{y}(\psi(\bar{x}, \bar{y})) \wedge \exists^1\bar{y}(\hat{\psi}(\bar{x}, \bar{y})))$$

for some $\psi \in \wedge[At]$ (take for ψ the formula $(R(\bar{x}) \wedge \bar{x} = \bar{y})$). We will denote by *condition* $(\hat{*})$ the one obtained from (*) above by requiring $\psi'(\bar{x}, \bar{y}, \bar{z})$ to be in $\wedge[\widehat{At}]$.

A formula in $\wedge[\widehat{At}]$ is of the form $\wedge_{j=1}^{s}(F_j(\bar{x}) = G_j(\bar{x}))$ for some terms $F_1, \ldots, G_s$ in L. We will sometimes write $\bar{F}(\bar{x}) = \bar{G}(\bar{x})$. We will say that a theory T *satisfies condition* $(+)$ if the following is true:

$(+)$ For each $\psi = \wedge_{j=1}^{s}(F_j(\bar{x}, \bar{y}, \bar{z})) = G_j(\bar{x}, \bar{y}, \bar{z}))$ in $\wedge[\widehat{At}]$ for which $T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi(\bar{x}, \bar{y}, \bar{z}))$, there exist $\psi_1, \ldots, \psi_n$ in $\wedge[\widehat{At}]$ such that

$$T \vdash \forall\bar{x}\exists^1\bar{y}_1 \ldots \bar{y}_{m(i)}\exists\bar{z}(\psi_i(\bar{x}, \bar{y}_1, \ldots, \bar{y}_{m(i)}, \bar{z}))$$

for $i = 1, \ldots, n$, and

$$T \vdash \forall\bar{x}\bar{y}'\bar{z}'\exists\{\bar{y}, \bar{z}, \bar{u}_i, \bar{u}_i', \bar{v}_i, \bar{v}_i', \bar{w}(j, i) | j = 0, \ldots, m(i);\ i = 1, \ldots, n\}$$

$$\Bigg[\psi(\bar{x}, \bar{y}, \bar{z}) \wedge \Bigg(\bigwedge_{i=1}^{n} (\psi_i(\bar{u}_i, \bar{w}(1, i), \ldots, \bar{w}(m(i), i), \bar{v}_i) \wedge$$

$$\wedge\ \psi_i(\bar{u}_i', \bar{w}(0, i), \ldots, \bar{w}(m(i) - 1, i), \bar{v}_i')) \wedge (\bar{w}(0, n)$$

$$= \bar{y} \wedge \bar{w}(m(n), n) = \bar{y}') \wedge$$

$$\wedge \left(\bigwedge_{i=1}^{n-1} (\bar{w}(0, i) = \bar{u}'_{i+1} \wedge \bar{w}(m(i), i) = \bar{u}_{i+1}) \right) \Bigg]$$

where, if $\bar{u}_1 = (u_1, \ldots, u_r)$ and $\bar{u}'_1 = (u'_1, \ldots, u'_r)$, each $\{u_i, u'_i\}$ is $\{u_i, u_i\}$ or $\{F_j(\bar{x}, \bar{y}', \bar{z}'), G_j(\bar{x}, \bar{y}', \bar{z}')\}$ for some $j \in \{1, \ldots, s\}$.

This condition is necessary to insure the unicity of $\bar{y}$ when we will replace $\forall \bar{x} \exists^1 \bar{y} \exists \bar{z}(\psi(\bar{x}, \bar{y}, \bar{z}))$ by $\forall \bar{x}(\psi(\bar{x}, \bar{H}_1(\bar{x}), \bar{H}_2(\bar{x})))$ in an appropriate expansion of L. Its form expresses the link with "congruence relations" (see [4], in particular the characterization of a congruence generated by a subset in Chapter 1), which is just another way to speak about kernel pairs. Its precise justification will appear in the proof of the next theorem.

THEOREM 2. *For any theory T in L, the following are equivalent*:

(a) *There is a canonical isomorphism from $\mathcal{M}(T)$ to a variety in an expansion of L by operation symbols which assigns to a structure its (unique) expansion.*

(b) *There is a canonical isomorphism from $\mathcal{M}(T)$ to a variety in an expansion of L which assigns to a structure its (unique) expansion.*

(c) *$\mathcal{M}(T)$ is closed for products, equalizers and homomorphic images whose kernel pairs are in $\mathcal{M}(T)$.*

(d) *T is equivalent to a set of sentences of the form*

$$\forall \bar{x}(\exists \bar{y} \bar{z} \psi \wedge \exists^1 \bar{y} \exists \bar{z} \hat{\psi}), \ \psi \in \wedge[At],$$

and satisfies conditions $(\hat{*})$ and $(+)$.

(e) *T has a definitional extension T_1^* in L^* such that $\mathcal{M}(T_1^*)$ is a variety and L-homomorphisms between models of T are L^*-homomorphisms.*

If L has no relation symbol, these are equivalent to:

(f) *$\mathcal{M}(T)$ is canonically isomorphic to a variety.*

(g) *The forgetful functor for $\mathcal{M}(T)$ is monadic.*

If L has no operation symbol, this is equivalent to:

(h) *$\mathcal{M}(T)$ is a variety.*

PROOF. (a) $\Rightarrow$ (b). Clear.

(b) $\Rightarrow$ (c). By Theorem 1, it suffices to prove that if

$$\mathfrak{A} \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathfrak{B} \overset{h}{\twoheadrightarrow} \mathfrak{C}$$

is a diagram in $\mathcal{M}(\emptyset)$ with $\langle f_1, f_2 \rangle$ = kernel pair of h, $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{M}(T)$ and h surjective, then $\mathfrak{C}$ is in $\mathcal{M}(T)$.

Let V and L^* be respectively a variety and an expansion of L as in (b), and let $\mathfrak{A}^*$ be the image of a model $\mathfrak{A}$ of T by the isomorphism. We consider the following L^*-expansion $\bar{\mathfrak{C}}$ of $\mathfrak{C}$: if F is a n-ary operation symbol of L^* not in L

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and $c_1, \ldots, c_n$ are elements of $\mathfrak{C}$, then $F_{\overline{\mathfrak{C}}}(c_1, \ldots, c_n) = h(F_{\mathfrak{B}^*}(b_1, \ldots, b_n))$ where b_i is any element of $\mathfrak{B}$ such that $h(b_i) = c_i$, $i = 1, \ldots, n$; it is the definitions of $\mathfrak{A}$, f_1 and f_2, and the fact that f_1 and f_2 are L^*-homomorphisms (of $\mathfrak{A}^*$ in $\mathfrak{B}^*$) that insure that $F_{\overline{\mathfrak{C}}}$ is a well-defined function. If R is a n-ary relation symbol of L^* not in L, define $R_{\overline{\mathfrak{C}}}$ in any way such that $R_{\mathfrak{B}^*}(b_1, \ldots, b_n) \Rightarrow R_{\overline{\mathfrak{C}}}(h(b_1), \ldots, h(b_n))$. Clearly h is a (surjective) L^*-homomorphism from $\mathfrak{B}^*$ to $\overline{\mathfrak{C}}$. But V is a variety, and then must contain $\overline{\mathfrak{C}}$. By the nature of the isomorphism, it follows that $\mathfrak{C}$ is in $\mathscr{M}(T)$.

(c) $\Rightarrow$ (d). Let T_1 be the set of consequences of T of the form $\forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi)$, $\psi \in \wedge[At]$. We first show that $T \equiv T_1$.

Let $\mathfrak{A}$ be a model of T_1. If φ is a universal-existential-positive consequence of T, Proposition 12 of [15] implies that there exists a consequence φ' of T of the form $\forall\bar{x}\exists\bar{y}(\psi)$, ψ conjunction of atomic formulas, such that $\varphi' \vdash \varphi$ (as in the proof of Theorem 1). But the form of φ' places it in T_1 and then $\mathfrak{A}$ satisfies any universal-existential-positive consequence of T. As in the proof of Theorem 1, this forces the existence of a diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, hook, "k"] & \mathfrak{B}'' \arrow[r, hook, "i"] & \mathfrak{A}' \\ \mathfrak{B}' \arrow[ur, "h"] & & \end{tikzcd}$$

with ik an elementary embedding and $\mathfrak{B}'$ in $\mathscr{M}(T)$. We prove now that the kernel pair $\langle f_1, f_2\rangle : \mathfrak{C} \rightrightarrows \mathfrak{B}'$ of h (in $\mathscr{M}(\emptyset)$) is in $\mathscr{M}(T)$.

T_1 being equivalent to a set of universal and of positive sentences, the surjectivity of h and the fact that i is an embedding imply that $\mathfrak{B}'' \models T_1$. Condition (*) being satisfied by $\mathscr{M}(T)$ (by closure for equalizers and Propositions 9 and 10 of [15]), for any sentence $\forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi)$ of T_1 there exists a formula $\psi'(\bar{x}, \bar{y}, \bar{z}, \bar{u})$, ψ' a conjunction of atomic formulas, such that

$$T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}\bar{u}(\psi \wedge \psi') \text{ and } T \vdash \forall\bar{x}\exists^1\overline{yz}\exists\bar{u}(\psi').$$

But $\forall\bar{x}\exists^1\overline{yz}\exists\bar{u}(\psi')$, being then in T_1, must be satisfied in $\mathfrak{B}''$; it is routine to check that this implies that $\forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi)$, and then any sentence of T_1, is true in $\mathfrak{C}$. But $\mathfrak{C}$ is, by construction, substructure of $\mathfrak{B}' \times \mathfrak{B}'$ (the product in $\mathscr{M}(\emptyset)$), which is in $\mathscr{M}(T)$. It follows that $\mathfrak{C}$ verifies also any sentence in the set T_2 of the universal consequences of T. Part (d) of Theorem 1 implies in particular that $T_1 \cup T_2 \equiv T$, and then $\mathfrak{C} \models T$. Applying the hypothesis, we have that $\mathfrak{B}''$ is in $\mathscr{M}(T)$, and then also $\mathfrak{A}$ because $\mathscr{M}(T)$ is closed for sandwiches (as in Theorem 1). This proves that $T \equiv T_1$.

Let $T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\alpha(\bar{x}, \bar{y}, \bar{z}) \wedge \beta(\bar{x}, \bar{y}, \bar{z}))$ with $\beta \in \wedge[At]$ and $\alpha \in [At]\backslash[\hat{A}t]$. Then α is of the form $R(\bar{F}(\bar{x}, \bar{y}, \bar{z}))$ for some string of terms $\bar{F}$ and some relation symbol R. For a given model $\mathfrak{A}$ of T, consider the L-structure $\mathfrak{A}'$ on $U\mathfrak{A}$ defined by $F_{\mathfrak{A}'} = F_{\mathfrak{A}}$ for all operation symbols F and $R_{\mathfrak{A}'} = (U\mathfrak{A})^n$ for all n-ary relation symbols R. Then the identity function is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$, and the domain of its kernel pair is (isomorphic to) $\mathfrak{A}$ itself. Hence $\mathfrak{A}' \models T$. But $\mathfrak{A}' \models \forall\bar{x}\bar{y}\bar{z}(\alpha)$, and then $\mathfrak{A}' \models \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\beta)$. This implies that $\mathfrak{A} \models \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\beta)$.

Hence we have shown that

$$T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi(\bar{x}, \bar{y}, \bar{z}))$$

implies

$$T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\hat{\psi}(\bar{x}, \bar{y}, \bar{z})).$$

In particular, T_1 (and hence T) is equivalent to a set of sentences of the form $\forall\bar{x}(\exists\bar{y}\bar{z}\psi \wedge \exists^1\bar{y}\exists\bar{z}\hat{\psi}))$, $\psi \in \wedge[At]$. This insures also that T satisfies $(\hat{*})$ (from the fact that it satisfies condition $(*)$). It remains to show that $(+)$ is also true.

If $\mathfrak{A}$ is a L-structure, an equivalence relation θ on $U\mathfrak{A}$ will here be called a *congruence on* $\mathfrak{A}$ if it is a congruence on the $\hat{L}$-reduct $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ in the usual algebraic sense (see [4]). For a congruence θ on $\mathfrak{A}$, we denote by $\mathfrak{A}/\theta$ the L-expansion of $\hat{\mathfrak{A}}/\theta$ obtained by defining $R_{\mathfrak{A}/\theta} = (\hat{U}(\hat{\mathfrak{A}}/\theta))^n$ for all n-ary relation symbols R (where $\hat{U}(\hat{\mathfrak{A}}/\theta)$ is the underlying set of $\hat{\mathfrak{A}}/\theta$). For a string $\bar{a}$ of n elements of A, we will write $|\bar{a}| = n$ and (abusively) $\bar{a} \in A$. If $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{a}' = (a'_1, \ldots, a'_n)$, the notation $\langle\bar{a}, \bar{a}'\rangle \in \theta$ will mean $\{\langle a_i, a'_i\rangle | i = 1, \ldots, n\} \subseteq \theta$.

Let $\psi \in \wedge[\hat{At}]$ as in condition $(+)$ $(\psi = (\wedge_{j=1}^s F_j = G_j) = (\bar{F} = \bar{G}))$, $\mathfrak{A}$ a model of T and $\bar{a}, \bar{b}, \bar{c}$ in A such that $\mathfrak{A} \models \psi[\bar{a}, \bar{b}, \bar{c}]$. For any $\bar{b}', \bar{c}'$ in A such that $|\bar{b}'| = |\bar{b}|$ and $|\bar{c}'| = |\bar{c}|$, we consider the relation $\theta[\psi(\bar{a}, \bar{b}', \bar{c}')] = \cup_{n=0}^{\infty} \theta_n$ on $U\mathfrak{A}$ defined by

$$\theta_0 = \{\langle\bar{d}, \bar{d}\rangle | \bar{d} \in A\} \cup \{\langle\bar{F}(\bar{a}, \bar{b}', \bar{c}'), \bar{G}(\bar{a}, \bar{b}', \bar{c}')\rangle\} \cup \{\langle\bar{G}(\bar{a}, \bar{b}', \bar{c}'), \bar{F}(\bar{a}, \bar{b}', \bar{c}')\rangle\},$$

and for each $n > 0$,

$$\begin{aligned}\theta_n = \{\langle\bar{d}_n, \bar{d}'_n\rangle | & \text{ there exist } \bar{c}_0 = \bar{d}_n, \bar{c}_1, \ldots, \bar{c}_m = \bar{d}'_n \text{ in } A \text{ and} \\ & \psi_n \in \wedge[\hat{At}] \text{ such that } T \vdash \forall\bar{x}\exists^1\bar{y}_1 \ldots \bar{y}_m\exists\bar{z}(\psi_n(\bar{x}, \bar{y}_1, \ldots, \bar{y}_m, \bar{z})) \text{ and} \\ & \mathfrak{A} \models \psi_n[\bar{d}_{n-1}, \bar{c}_0, \ldots, \bar{c}_{m-1}, \bar{e}_n] \wedge \psi_n[\bar{d}'_{n-1}, \bar{c}_1, \ldots, \bar{c}_m, \bar{e}'_n] \text{ for some} \\ & \langle\bar{d}_{n-1}, \bar{d}'_{n-1}\rangle \in \theta_{n-1} \text{ and } \bar{e}_n, \bar{e}'_n \text{ in } A\}.\end{aligned}$$

Then one can check that $\theta[\psi(\bar{a}, \bar{b}', \bar{c}')]$ is the smallest congruence θ on

$\mathfrak{A}$ containing $\langle \bar{F}(\bar{a}, \bar{b}', \bar{c}'), \bar{G}(\bar{a}, \bar{b}', \bar{c}') \rangle$ and such that for any $\psi' \in \wedge[At]$ with $T \vdash \forall\bar{x}\exists^1\bar{y}\exists\bar{z}(\psi'(\bar{x}, \bar{y}, \bar{z}))$, we have

$$[\mathfrak{A} \models \psi'[\bar{a}_1, \bar{a}_2, \bar{a}_3] \wedge \psi'[\bar{a}'_1, \bar{a}'_2, \bar{a}'_3] \text{ and } \langle \bar{a}_1, \bar{a}'_1 \rangle \in \theta] \Rightarrow \langle \bar{a}_2, \bar{a}'_2 \rangle \in \theta.$$

Let $f: A \twoheadrightarrow \mathfrak{A}/\theta[\psi(\bar{a}, \bar{b}', \bar{c}')]$ be the canonical homomorphism and $B(\subseteq A \times A)$ be the domain of the kernel pair of f. Of course $\langle a, a' \rangle \in B$ if and only if $\langle a, a' \rangle \in \theta[\psi(\bar{a}, \bar{b}', \bar{c}')]$. Let $\psi' \in \wedge[At]$ with $T \vdash \forall\bar{x}(\exists\bar{y}\bar{z}\psi' \wedge \exists^1\bar{y}\exists\bar{z}\hat{\psi}')$. For $\langle \bar{a}_1, \bar{a}_2 \rangle \in B$, there exist unique $\bar{b}_1, \bar{b}_2$ such that there exist $\bar{c}_1, \bar{c}_2$ with $\mathfrak{A} \models \psi'[\bar{a}_1, \bar{b}_1, \bar{c}_1] \wedge \psi'[\bar{a}_2, \bar{b}_2, \bar{c}_2]$. Then $\langle \bar{b}_1, \bar{b}_2 \rangle$ is in B. By ($\hat{*}$), there exist $\psi'' \in \wedge[\hat{At}]$ and $\bar{c}'_1, \bar{c}'_2, \bar{d}_1, \bar{d}_2$ in A such that

$$\mathfrak{A} \models \wedge_{i=1}^{2}(\psi'[\bar{a}_i, \bar{b}_i, \bar{c}'_i] \wedge \psi''[\bar{a}_i, \bar{b}_i, \bar{c}'_i, \bar{d}_i]) \text{ and}$$

$$T \vdash \forall\bar{x}\exists^1\bar{y}\bar{z}\exists\bar{u}(\psi''(\bar{x}, \bar{y}, \bar{z}, \bar{u})).$$

But $\langle \bar{a}_1, \bar{a}_2 \rangle$ being in B, this implies that $\langle \bar{b}_1, \bar{b}_2 \rangle$ *and* $\langle \bar{c}'_1, \bar{c}'_2 \rangle$ will also be there, by the property of $\theta[\psi(\bar{a}, \bar{b}', \bar{c}')]$ mentioned above. This shows that $\mathfrak{B} \models \forall\bar{x}(\exists\bar{y}\bar{z}\psi' \wedge \exists^1\bar{y}\exists\bar{z}\psi')$. Hence $\mathfrak{B} \models T$, and by the closure property, $\mathfrak{A}/\theta[\psi(\bar{a}, \bar{b}', \bar{c}')] \models T$.

In particular, $\mathfrak{A}/\theta[\psi(\bar{a}, \bar{b}', \bar{c}')] \models \forall\bar{x}\exists^1\bar{y}\exists\bar{z}\psi$. But $\langle \bar{F}(\bar{a}, \bar{b}, \bar{c}), \bar{G}(\bar{a}, \bar{b}, \bar{c}) \rangle$ and $\langle \bar{F}(\bar{a}, \bar{b}', \bar{c}'), \bar{G}(\bar{a}, \bar{b}', \bar{c}') \rangle$ are both in $\theta[\psi(\bar{a}, \bar{b}', \bar{c}')]$, and this forces

$$\langle \bar{b}, \bar{b}' \rangle \in \theta[\psi(\bar{a}, \bar{b}', \bar{c}')],$$

that is, $\langle \bar{b}, \bar{b}' \rangle \in \theta_n$ for some integer n. This means that there exist

$$\psi_1, \ldots, \psi_n \in \wedge[\hat{At}]$$

satisfying $T \vdash \forall\bar{x}\exists^1\bar{y}_1 \ldots \bar{y}_{m(i)}\exists\bar{z}(\psi_i)$ for $i = 1, \ldots, n$, and strings $\bar{d}_i, \bar{d}'_i, \bar{e}_i, \bar{e}'_i$, $\bar{b}(i, j)$ for $i = 1, \ldots, n$ and $j = 0, 1, \ldots, m(i)$ such that

$$\mathfrak{A} \models \psi[\bar{a}, \bar{b}, \bar{c}] \wedge \left(\bigwedge_{i=1}^{n} (\psi_i[\bar{d}_i, \bar{b}(0, i), \ldots, \bar{b}(m(i) - 1, i), \bar{e}_i] \wedge \right.$$

$$\left. \wedge\, \psi_i[\bar{d}'_i, \bar{b}(1, i), \ldots, \bar{b}(m(i), i), \bar{e}'_i])\right) \wedge \bar{b}(0, n)$$

$$= \bar{b} \wedge \bar{b}(m(n), n) = \bar{b}' \wedge \left(\bigwedge_{i=1}^{n-1} (\bar{b}(0, i) = \bar{d}'_{i+1} \wedge \bar{b}(m(i), i) = \bar{d}_{i+1})\right)$$

and where, if $\bar{d}_1 = (d_1, \ldots, d_r)$ and $\bar{d}'_1 = (d'_1, \ldots, d'_r)$, we have, for each

$$i \in \{1, \ldots, r\},\ d_i = d'_i \text{ or}$$

$$\{d_i, d'_i\} = \{F_j(\bar{a}, \bar{b}', \bar{c}'), G_j(\bar{a}, \bar{b}', \bar{c}')\} \text{ for some } j \in \{1, \ldots, s\}.$$

Expressed in infinitary logic, all this shows that

$$T \vdash \forall \bar{x}\bar{y}'\bar{z}'\exists \bar{y}\bar{z}(\bar{u})_\Gamma(\bar{u}')_\Gamma(\bar{v})_\Gamma(\bar{v}')_\Gamma\{ (\bar{w}(h, k))_\Gamma | h, k \text{ integers}\}$$

$$\Bigg(\bigvee_{\gamma \in \Gamma} (\psi(\bar{x}, \bar{y}, \bar{z}) \wedge \Bigg(\bigwedge_{i=1}^{n_\gamma} (\psi_{\gamma(i)}(\bar{u}_{\gamma(i)}, \bar{w}_{\gamma(i)}(1, i), \ldots, \bar{w}_{\gamma(i)}(m(\gamma(i)), i), \bar{v}_{\gamma(i)}) \wedge$$

$$\wedge\ \psi_{\gamma(i)}(\bar{u}'_{\gamma(i)}, \bar{w}_{\gamma(i)}(0, i), \ldots, \bar{w}_{\gamma(i)}(m(\gamma(i)) - 1, i), \bar{v}'_{\gamma(i)}))\Bigg) \wedge$$

$$\wedge\ (\bar{w}_{\gamma(n_\gamma)}(0, n_\gamma) = \bar{y} \wedge \bar{w}_{\gamma(n_\gamma)}(m(\gamma(n_\gamma)), n_\gamma) = \bar{y}') \wedge$$

$$\wedge \Bigg(\bigwedge_{i=1}^{n_\gamma - 1} (\bar{w}_{\gamma(i)}(0, i) = \bar{u}'_{\gamma(i+1)} \wedge \bar{w}_{\gamma(i)}(m(\gamma(\mathrm{i})), i) = \bar{u}_{\gamma(i+1)}))\Bigg)$$

with $(\bar{u}_{\gamma(1)}, \bar{u}'_{\gamma(1)})$ of the required form, and where $(\bar{u})_\Gamma = \{\bar{u}_\gamma\}_{\gamma\in\Gamma}$ and $\bar{u}_\gamma = \{\bar{u}_{\gamma(1)}, \ldots, \bar{u}_{\gamma(n_\gamma)}\}$, similarly for $\bar{u}'$, $\bar{v}$, $\bar{v}'$ and $\bar{w}(h, k)$, and Γ is the set of all strings $(\psi_{\gamma(1)}, \ldots, \psi_{\gamma(n_\gamma)})$ with $\psi_{\gamma(i)} \in \hat{\wedge}[At]$ and $T \vdash \forall \bar{x}\exists^1 \bar{y}_1 \ldots y_{m(\gamma(i))}\exists \bar{z}\psi_{\gamma(i)}$.

$\mathcal{M}(T)$ being closed for products, there must exist $\gamma \in \Gamma$ for which we can remove the disjunction and replace "Γ" by "γ" in this sentence. This gives condition $(+)$.

(d) $\Rightarrow$ (e). Proceed as in (d) $\Rightarrow$ (e) of Theorem 1 to obtain L^* and T^*. We need to do more to eliminate the need of the sentences of the form

$$\forall \bar{x}\forall \bar{y}\bar{y}'(\psi(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y}') \rightarrow \bar{y} = \bar{y}').$$

For each consequence of T of the form $\forall \bar{x}\exists^1\bar{y}\exists\bar{z}(\varphi(\bar{x}, \bar{y}, \bar{z}))$ with $\varphi \in \hat{\wedge}[At]$ and for each consequence of T^* of the form

$$\forall \bar{x}(\varphi(\bar{G}_1(\bar{x}), \bar{G}_2(\bar{x}), \bar{G}_3(\bar{x})) \wedge \forall \bar{y}(\varphi(\bar{y}, \bar{H}_1(\bar{y}), \bar{H}_2(\bar{y})))),$$

for some strings of terms $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_3$, $\bar{H}_1$, $\bar{H}_2$ in L^*, add to T^* the sentence $\forall \bar{x}(\bar{G}_2(\bar{x}) = \bar{H}_1(\bar{G}_1(\bar{x})))$.

Call T_1^* the new theory. Then conditions $(\hat{*})$ and $(+)$ insure that T_1^* is equivalent to its consequences of the form $\forall \bar{x}(\varphi(\bar{x}))$, $\varphi \in \wedge[At]$: if $\forall \bar{x}\exists^1\bar{y}\exists\bar{z}\psi$ is a consequence of T with $\psi \in \hat{\wedge}[At]$, then the long sentence in $(+)$ induces a sentence in T^* of the form

$$\forall \bar{x}\bar{y}'\bar{z}'\Bigg(\ldots\Bigg(\bigwedge_{i=1}^{n} \psi_i(\bar{G}_1^i(\bar{x}, \bar{y}', \bar{z}'), \bar{G}_2^i(\bar{x}, \bar{y}', \bar{z}'), \ldots)\Bigg)\ldots\Bigg);$$

but the existence of sentences of the form

$$\forall \bar{x}\bar{y}'\bar{z}'(\bar{G}_2^i(\bar{x}, \bar{y}', \bar{z}')) = \bar{H}_1^i(\bar{G}_1^i(\bar{x}, \bar{y}', \bar{z}'))$$

in T_1^* and the special form of the long sentence imply the unicity of $\bar{y}$ without the presence of $\forall \bar{x}\exists^1\bar{y}\exists\bar{z}\psi$. Finally, T^* being a definitional extension of T by new operations defined by formulas in $\exists\wedge[At]$, the same is true of T_1^*.

(e) $\Rightarrow$ (a) $\Rightarrow$ (f) $\Rightarrow$ (g). Obvious or well-known.

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(g) $\Rightarrow$ (c). Let U and U' be respectively the forgetful functors for $\mathscr{M}(T)$ and $\mathscr{M}(\emptyset)$. L having no relation symbol, the fact that U preserves limits implies that $\mathscr{M}(T)$ is closed for products and equalizers. Let

$$\mathfrak{A} \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathfrak{B} \overset{f}{\twoheadrightarrow} \mathfrak{C}$$

be a diagram in $\mathscr{M}(\emptyset)$ with f surjective, $\langle f_1, f_2 \rangle$ = kernel pair in $\mathscr{M}(\emptyset)$ of f and $\mathfrak{A}$, $\mathfrak{B}$ in $\mathscr{M}(T)$. We must show that $\mathfrak{C}$ is in $\mathscr{M}(T)$.

It is easy to verify that $U'f$ is a split coequalizer (in Set) of $\langle Uf_1, Uf_2 \rangle$ (see [12] for this concept). By Beck's Theorem (see again [12]), there is a unique homomorphism $g:\mathfrak{B} \to \mathfrak{D}$ in $\mathscr{M}(T)$ such that $Ug = U'f$ and $U\mathfrak{D} = U'\mathfrak{C}$. Again, the absence of operation symbol (and the surjectivity of f) forces $(g:\mathfrak{B} \to \mathfrak{D}) = (f:\mathfrak{B} \twoheadrightarrow \mathfrak{C})$.

(h) $\Leftrightarrow$ (d). Easy.

Remarks. (1) It is known how to construct, from the monadicity of the forgetful functor, an unbounded language in which $\mathscr{M}(T)$ is a "variety" (see for example [5]). For (g) $\Rightarrow$ (f), we had to show that there exists a finitary language permitting this. We could have deduced it from the result of Richter [14] mentioned in the introduction, but we preferred the more direct and elementary proof (of (g) $\Rightarrow$ (c)) given.

(2) Consider, for a given theory T, the following 4 properties: (i) T is a variety; (ii) $\mathscr{M}(T)$ is closed for limits and homomorphic images (i.e. T satisfies Theorem 15 of [15]); (iii) T satisfies Theorem 2; (iv) The models of T are precisely the reducts of the objects of some variety (in an extended language). Again (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and no implication can be reversed. The same example (of groups) as in Remark 2 after Theorem 1 can be used to show that (ii) $\not\Rightarrow$ (i). A counterexample, due to Diaconescu, to (iii) $\Rightarrow$ (ii) can be found on page 47 of [15] (this also invalidates Theorem 3 of [7]), and one to (iv) $\Rightarrow$ (iii) on page 166 of [1].

(3) In a language without operation symbol, then of course the property, for a theory, of verifying Theorem 2 is a uniform one, by (h). The answer in the general case is unknown to us: a proof like the one of Rabin ([13], Theorem 11), which forms the basis of the Proposition 6 of [15], creates considerable difficulties here because of the necessary presence of operation symbols.

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