Some notes on a Nomogram giving the relation between price and yield of terminable bonds bearing a fixed rate of interest

Consider a bond of 100 redeemable at par after a period of n years, bearing a nominal rate of interest I payable half-yearly. Let $i/2$ be the yield rate per half-year, and P be the price after deducting accrued interest. Assume interest is compounded half-yearly.

Then $$P = 100v^{2n} + \frac{I}{2} a_{\overline{2n}|} \qquad \ldots\ldots(1).$$

This equation involves four variables P, I, n, i. If the values of any three of these are given then the fourth is fixed. But the algebraic determination of a particular variable is sometimes laborious. The equation can be completely solved by a nomogram, which will show the relation between all four variables simultaneously. The general form of a suitable nomogram is shown in Fig. 1. Results are read by placing the cursor LL′ across the graduated lines A and B and the grid system C.

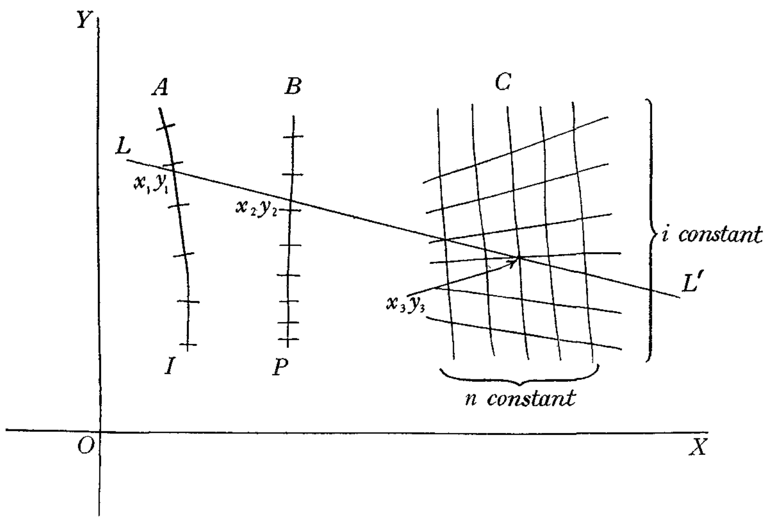

Fig. 1.

The cursor LL′ consists of a straight line ruled on a strip of glass or other transparent substance.

The line A is graduated for values of I.

The line B is graduated for values of P.

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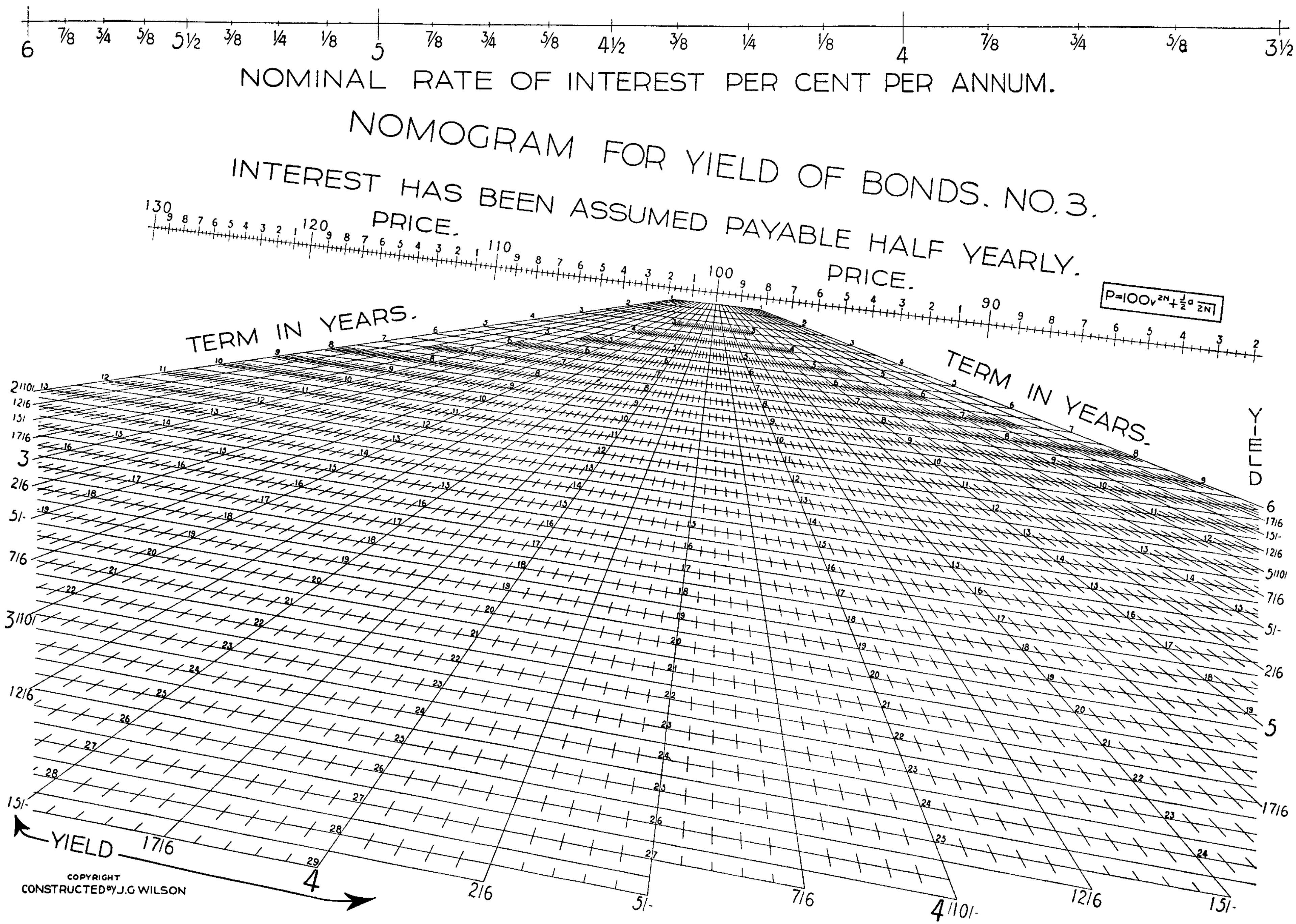

Between pp. 314 and 315

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The grid system consists of lines along which either n or i is constant.

The values at the intersections of LL′ with the graduated lines A and B and a point in the grid show a numerical relation between the variables.

Let the coordinates of these points of intersection be $x_1, y_1 : x_2, y_2 : x_3, y_3$.

Then the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ ......(2),

since $x_1, y_1 : x_2, y_2 : x_3, y_3$ are collinear.

x_1 and y_1 are functions of I.

x_2 and y_2 are functions of P.

x_3 and y_3 are functions of i and n.

To determine these functions we transform the original equation into the form of the determinant above.

Equation (1) may be written:

$$\frac{I}{2}\left(\frac{a_{\overline{2n}|}}{100v^{2n}}\right) - P\left(\frac{1}{100v^{2n}}\right) + 1 = 0.$$

Put $x = \frac{I}{2}$, and $y = P$, then:

$$\left.\begin{array}{l} x + 0.y - \frac{I}{2} = 0, \\ 0.x + y - P = 0, \\ \frac{a_{\overline{2n}|}}{100v^{2n}}\,x - \frac{\left(1+\frac{i}{2}\right)^{2n}}{100}\,y + 1 = 0. \end{array}\right\}$$

Therefore the determinant

$$\begin{vmatrix} 1 & 0 & -\frac{I}{2} \\ 0 & 1 & -P \\ \frac{s_{\overline{2n}|}}{100} & -\frac{\left(1+\frac{i}{2}\right)^{2n}}{100} & 1 \end{vmatrix} = 0.$$

Dividing first row by $-\frac{\mathrm{I}}{2}$ and second row by $-\mathrm{P}$ and multiplying middle column by -1, we get

$$\begin{vmatrix} -\frac{2}{\mathrm{I}} & 0 & 1 \\ 0 & \frac{1}{\mathrm{P}} & 1 \\ \frac{s_{\overline{2n}|}}{100} & \frac{\left(1+\frac{i}{2}\right)^{2n}}{100} & 1 \end{vmatrix} = 0 \qquad \ldots\ldots(3).$$

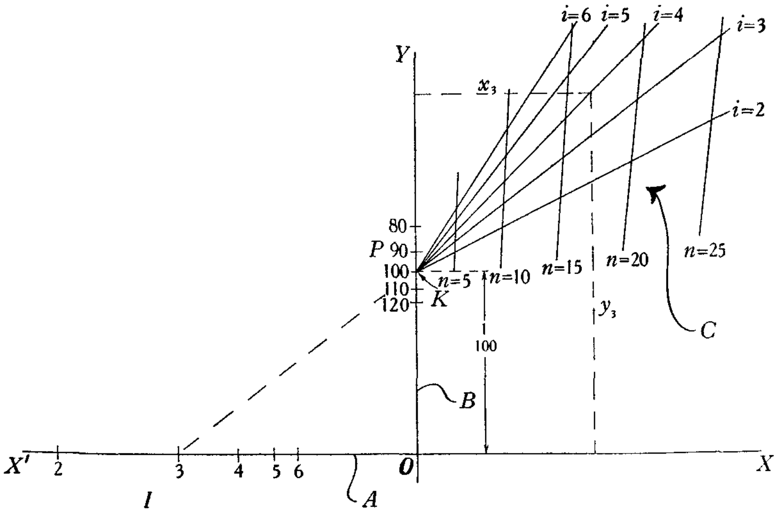

Fig. 2.

In order that determinants (2) and (3) may be identically equal the following relations must be true:

$$x_1 = -\frac{2}{\mathrm{I}}, \quad y_1 = 0,$$

$$x_2 = 0, \qquad y_2 = \frac{1}{\mathrm{P}},$$

$$x_3 = \frac{s_{\overline{2n}|}}{100}, \quad y_3 = \frac{\left(1+\frac{i}{2}\right)^{2n}}{100}.$$

These relations are expressed graphically in Fig. 2.

The locus of line A is given in terms of the parameter I and is obviously a straight line coinciding with the X axis, along which the distance to any graduation of $I = -2/I$.

The locus of line B is similarly given in terms of the parameter P and is a straight line coinciding with the Y axis, along which the distance to any graduation of $P = 1/P$.

The grid C is expressed in terms of the two parameters i and n. If $n = 0$ then $x_3 = 0$ and $y_3 = \frac{1}{100}$, therefore all the lines along which i is constant pass through the point K, whose coordinates are $(0, \frac{1}{100})$.

And
$$\frac{y_3 - \frac{1}{100}}{x_3} = \frac{\left(1+\frac{i}{2}\right)^{2n} - 1}{s_{\overline{2n}|}} = \frac{i}{2}.$$

Therefore the constant i lines are straight and their slope is proportional to i. Further it follows that the i lines when produced pass through points on OX′ such that $I = i$. This is to be expected since if $I = i$ the price of the bond will be 100 for all values of n. It is found by actual drawing that the lines with n constant are curves that are nearly straight, and approximately parallel to OY.

A more convenient shape and greater accuracy of reading can be obtained by using oblique axes. By shearing the whole diagram, we cause the angle YOX′ to become acute, which brings the useful part of the I line opposite the useful part of the P line. The result is shown in Fig. 3. The shearing does not alter the value of the x coordinate of any point, and does not affect the collinearity of points on the original diagram. Line XOX′ is sheared into position, GOG′ and other lines are distorted correspondingly. For the sake of clearness Fig. 3 is not drawn truly to scale.

The effect of shearing is shown by transforming determinant (3) as follows:

Subtract from the second column, $\cot\theta$ times the first column, where $\theta =$ angle YOG′ of Fig. 3 and we get:

$$\begin{vmatrix} -\frac{2}{I} & \frac{2}{I}\cot\theta & 1 \\ 0 & \frac{1}{P} & 1 \\ \frac{s_{\overline{2n}|}}{100} & \frac{\left(1+\frac{i}{2}\right)^{2n} - s_{\overline{2n}|}\cot\theta}{100} & 1 \end{vmatrix} = 0 \qquad \text{......(4).}$$

The rows of the above determinant represent exactly the drawing in Fig. 3. It is interesting to show that the determinant reduces to the original equation (1).

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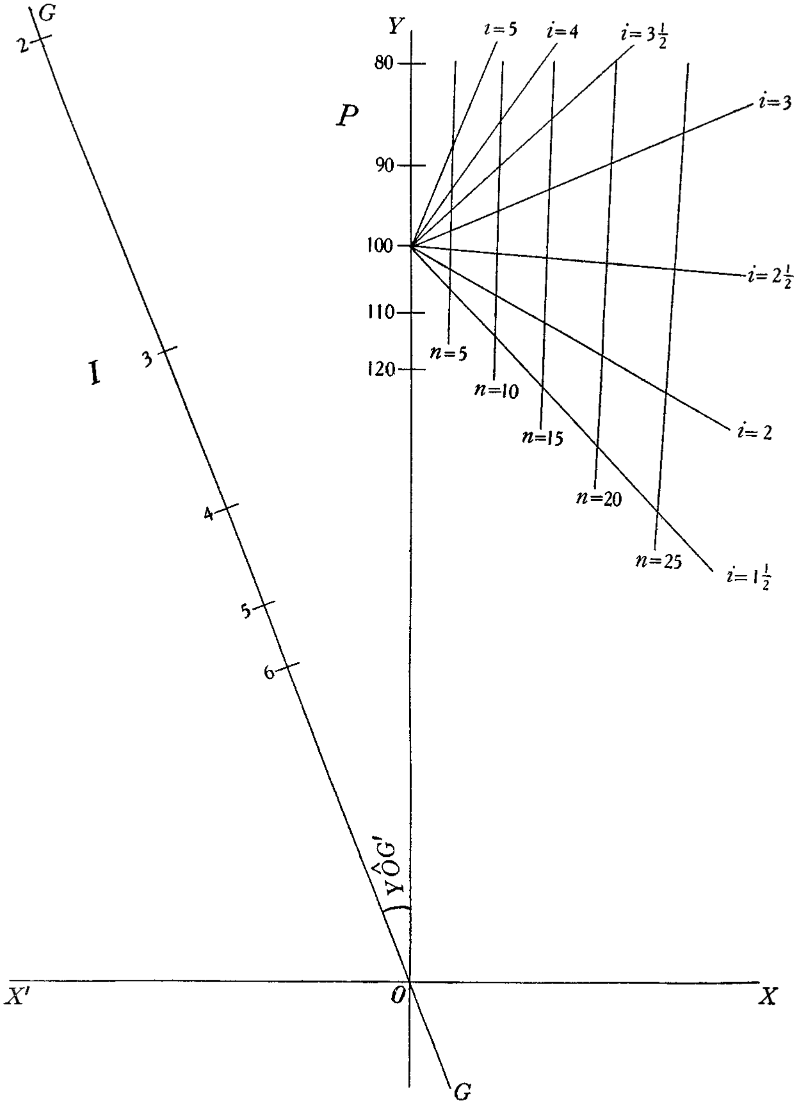

Fig. 3.

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By ordinary algebraic rules we obtain the expression represented by determinant (4).

$$\left(-\frac{2}{I}\right)\left(\frac{1}{P}\right)+\left(\frac{2}{I}\cot\theta\right)\left(\frac{s_{\overline{2n}|}}{100}\right)-\left(\frac{s_{\overline{2n}|}}{100}\right)\left(\frac{1}{P}\right)$$

$$-\left(\frac{\left(1+\frac{i}{2}\right)^{2n}-s_{\overline{2n}|}\cot\theta}{100}\right)\left(-\frac{2}{I}\right)=0,$$

therefore

$$-100+P,\quad s_{\overline{2n}|}\cot\theta-\frac{I}{2}s_{\overline{2n}|}+\left(1+\frac{i}{2}\right)^{2n}P-P,\quad s_{\overline{2n}|}\cot\theta=0,$$

therefore
$$P=100v^{2n}+\frac{I}{2}a_{\overline{2n}|},$$

which is equation (1).

In the construction of a particular chart, the ranges of I, i, and n are first decided, then the horizontal and vertical scales and the angle YOG′ are determined so as to fit the nomogram on the drawing sheet. It is not necessary to show the origin on the sheet. The shearing does not cause any difficulty in drawing. The only lines presenting any trouble are the n lines, which are best drawn after the i lines have been fixed.

In Fig. 4 KP is any i line and P its intersection with any n line; PQ is horizontal.

$$PQ=x_3=\frac{s_{\overline{2n}|}}{100},$$

therefore

$$KP=PQ\sec\alpha=\frac{s_{\overline{2n}|}}{100}\sec\alpha.$$

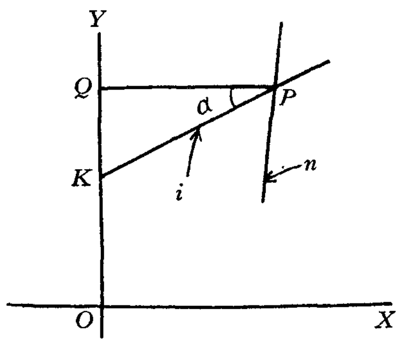

Fig. 4.

For a particular value of n the distances along different i lines are determined and the corresponding points marked. By joining these points we get an n line.

It was found advantageous to draw two charts to include a range of I from $2\frac{1}{2}$ to 6%. The yield rates marked on the charts are equal to twice the half-yearly rates.

Some reproductions from the originals have been made by means of copper-plate engravings. There was some difficulty in obtaining a true representation of all the fine lines, and also in printing on to a hard-surfaced card.

The Nomogram No. 3 (between pp. 314 and 315) shows a reproduction of one of the finished charts, such as would be used in the case mentioned in Example (2), given below.

The methods of using the nomogram are best illustrated by worked examples.

Example (1). Consider a 4% bond paying interest half-yearly and maturing at par in $12\frac{1}{2}$ years. Required to find the price which will yield £3. 7*s*. 6*d*.% per annum.

Set one end of the cursor on point marked 4 on the I line and set the other end of the cursor on the point where the radial yield line marked £3. 7*s*. 6*d*. cuts the $12\frac{1}{2}$ years term line. Where the cursor cuts the price line is the required price. It cuts at £106. 6*s*. 6*d*.

Example (2). Consider a $3\frac{7}{8}$% bond paying interest half-yearly and maturing at par in 15 years $7\frac{1}{2}$ months. Required to find the yield given by a price of £106. 10*s*. 0*d*. Deduct from the price interest accruing since the last interest day. Accrued interest is $3\frac{7}{8}$% on £100 for $4\frac{1}{2}$ months and equals £1. 9*s*. 0*d*. Net price = £106. 10*s*. 0*d*. less £1. 9*s*. 0*d*. = £105. 1*s*. 0*d*.

Set one end of cursor on point marked $3\frac{7}{8}$ on the I line and middle of cursor on the net price £105. 1*s*. 0*d*. Follow the cursor line through the year lines until the term ($15\frac{5}{8}$ years) is reached. By interpolating between the yield lines at this point the yield (about £3. 9*s*. 2*d*.%) is obtained.

The accuracy in the price is to about the nearest shilling or possibly sixpence with a good operator. The accuracy of the yield is variable, but over most of the range readings can be made to within a penny or twopence. For very short tenures the accuracy is not so good. The author is at present investigating a nomogram that will overcome this to some extent.

There is described in Vol. LI of the *Journal* a type of graphical device using a special graduated ruler. The position of the ruler is fixed by fitting it on three points on the diagram. The simple straight line cursor previously described is fixed by two points only. This ruler is therefore much more difficult to set than the cursor. A line on a piece of glass or celluloid allows interpolation, whereas with a ruler it is necessary to extrapolate. For reading by eye interpolation is an easier and more accurate process than extrapolation.

In conclusion it is pointed out that the nomogram is essentially of practical use both to the actuary and the layman. The device is not merely a mathematical curiosity. It was constructed to solve the equation more quickly and effectively than other methods.

Editorial Note. While the nomogram, for which our thanks to the author are tendered, is new and enables yields to be found readily by anybody whether or not he is familiar with the underlying mathematical principles, it is right to refer to a note by Mr J. F. L. Bray in *J.S.S.* Vol. II, p. 322 where the same methods are employed, so that in that respect the discovery independently by the author of this note is not an original one.