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POLYTOPES OF ROOTS OF TYPE A_N

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Polytopes of roots of type A_{n-1} are investigated, which we call P_n. The polytopes, P_n^+, of positive roots and the origin have been considered in relation to other branches of mathematics [4]. We show that exactly n copies of P_n^+ forms a disjoint cover of P_n. Moreover, those n copies of P_n^+ can be obtained by letting the elements of a subgroup of the symmetric group S_n generated by an n-cycle act on P_n^+. We also characterise the faces of P_n and some facets of P_n^+, which we believe to be useful in some optimisation problems. As by-products, we obtain an interesting identity on the number of lattice paths and a triangulation of the product of two simplices.

1. INTRODUCTION

The polytope P_n^+ of positive roots of type A_{n-1} and the origin has been considered by Gelfand, Graev and Postnikov in relation to hypergeometric functions [4]. Many combinatorial problems have been considered: for example, the volume is calculated and some facets are characterised. In this article, we consider the polytope P_n of all roots of type A_{n-1} in relation to the polytope P_n^+. P_n itself is an object of interest since it is related to many combinatorial objects. Notice that P_n is a *Young orbit polytope* corresponding to the partition $(n-1,1)$, which was introduced as a framework for many combinatorial optimisation problems [8]. Moreover, the set of roots of type A_{n-1} is the set of minimal null designs of a certain type [2]. Hence, it is worth characterising the faces of P_n. It is also interesting to observe how the polytope P_n^+ sits inside P_n and calculate the volume of P_n.

In this paper, we characterise all faces of P_n and give a proof for the characterisation of certain facets of P_n^+. Then we use these results to show that exactly n copies of P_n^+ form a disjoint (in the sense that the intersection has volume zero in $\mathbb{R}^{n-1}$) cover of P_n, and there, the cyclic group generated by a Coxeter element (n-cycle) plays a role. While proving the main theorem, we also obtain an interesting identity on the number of lattice paths and a triangulation of the product of two simplices.

We refer to [1] and [10] for detailed information on convex polytopes, while we give some basic definitions. A *(convex) polytope* is the convex hull $\mathrm{Conv}(K) = \left\{\sum_{i=1}^{l} \lambda_i \mathrm{u}_i :\right.$

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$\sum_i \lambda_i = 1,\ \lambda_i \geqslant 0\Big\}$ of a finite set $K = \{\mathbf{u}_1, \dots, \mathbf{u}_l\}$ in $\mathbb{R}^d$ for some d. The *dimension* of a polytope Conv(K) is the dimension of its *affine hull* $\Big\{\sum_{i=1}^{l} \lambda_i \mathbf{u}_i \ :\ \sum_i \lambda_i = 1\Big\}$, that is, the size of the largest affinely independent subset of K subtracted by 1. For vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^d$, let $(\mathbf{u}, \mathbf{v})$ denote the usual inner product in $\mathbb{R}^d$. A *face* of a polytope $P \in \mathbb{R}^d$ is any set of the form $F = P \cap \{\mathbf{x} \in \mathbb{R}^d : (\mathbf{c}, \mathbf{x}) = c_0\}$, where $\mathbf{c} \in \mathbb{R}^d$, $c_0 \in \mathbb{R}$ and $(\mathbf{c}, \mathbf{x}) \leqslant c_0$ for all $\mathbf{x} \in P$. The *dimension* of a face is the dimension of its affine hull. P itself is a face with $(\mathbf{0}, \mathbf{x}) \leqslant 0$, and $\emptyset$ is a face given by $(\mathbf{0}, \mathbf{x}) \leqslant -1$. We call these faces *trivial.* A face F of a polytope P is called a *facet* if the dimension of F is one less than the dimension of P. Observe that faces are characterised as the subsets of P, whose elements maximise a given linear functional. In the definition, we can replace $\mathbf{c}$, c_0 by $-\mathbf{c}$ and $-c_0$, respectively. Hence, the faces of a polytope are also characterised as subsets of P whose elements *minimise* a linear functional.

We denote the elementary vectors of $\mathbb{R}^n$ by ε_i, $i = 1, \dots, n$. Then P_n^+ is the convex hull of $\{\varepsilon_i - \varepsilon_j : i < j\}$ and the origin, and P_n is the convex hull of $\{\varepsilon_i - \varepsilon_j : i \neq j\}$. Obviously, the dimension of P_n and P_n^+ is $n-1$.

In Section 2, the polytope P_n is considered and all faces are characterised. In Section 3, we first summarise some combinatorial results of Gelfand, Graev and Postnikov. Then, in Section 4, P_n and P_n^+ are considered together and the main theorem is proved that shows how the two polytopes are related.

2. Polytopes P_n

In this section, we consider the polytope $P_n = \mathrm{Conv}\{\varepsilon_i - \varepsilon_j : i \neq j,\ i, j \in [n]\}$ of all roots of type A_{n-1}. Note that P_n is a special case of the polytopes called *generalised permutahedrons* since it is a convex hull of all the vectors given by all permutations of the vector $(1, -1, 0, \dots, 0) \in \mathbb{R}^n$. (*Permutahedron* Π_{n-1} is a classical object defined as the convex hull of all vectors of permutations of the vector $(1, 2, \dots, n)$, see [**10**].) We give an explicit description of every face of P_n. P_n is obtained from Π_{n-1} by identifying many vertices, hence the faces of P_n are the ones collapsed down from the faces of Π_{n-1}. Remember that the k-faces of Π_{n-1} are in one to one correspondence with the ordered partitions of the set $[n]$ into $n-k$ non-empty parts. Hence Theorem 1 does not surprise us.

Each face of a (finite) convex hull can be described as a subset of the given polytope, which maximises (or minimises) a linear functional. So, to investigate the faces of P_n, we consider all possible linear functionals defined on $\mathbb{R}^n$. Observe that every linear functional f can be written as

$$f(\mathbf{y}) = \sum_{i \in I} \lambda_i y_i - \sum_{j \in J} \lambda_j y_j \quad \text{for } \lambda_i \geqslant 0,\ \lambda_j > 0 \text{ and } I \cap J = \emptyset,\ I \cup J = [n],$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)$.

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Theorem 1. *Every m-dimensional ($m = 0, \dots, n-2$) face of P_n is given by the convex hull of the vectors in $\{\varepsilon_i - \varepsilon_j : i \in I,\ j \in J\}$ where I, J are disjoint non-empty subsets of $[n]$ such that $|I| + |J| = m + 2$. Hence, there is a one to one correspondence between the set of non-trivial faces of P_n and the set of ordered partitions of subsets of $[n]$ with two blocks, where the dimension of the face corresponding to (I, J) is $|I| + |J| - 2$.*

Proof: For a given non-zero linear functional $f(\mathbf{y}) = \sum_{i \in I} \lambda_i y_i - \sum_{j \in J} \lambda_j y_j$, $\lambda_i \geqslant 0$, $\lambda_j > 0$, determining a non-trivial face F, where (I, J) is a partition of $[n]$, we define (I', J') as follows:

1. If $I \neq \emptyset$ and $J \neq \emptyset$, then $I' = \{i : \lambda_i = \max(\lambda_l : l \in I)\}$, $J' = \{j : \lambda_j = \max(\lambda_l : l \in J)\}$,
2. If $I = \emptyset$ then $I' = \{i : \lambda_i = \min(\lambda_l : l \in J)\}$, $J' = \{j : \lambda_j = \max(\lambda_l : l \in J)\}$,
3. If $J = \emptyset$ then $I' = \{i : \lambda_i = \max(\lambda_l : l \in I)\}$, $J' = \{j : \lambda_j = \min(\lambda_l : l \in I)\}$.

If $I = \emptyset$ (hence $J = [n]$) and λ_j is a constant for all $j \in J$, then $F = P_n$. If $J = \emptyset$ and λ_i is a constant for all $i \in I$, then $F = P_n$ also. Hence, $I' \neq \emptyset$, $J' \neq \emptyset$ and $I' \cap J' = \emptyset$. Note that F is the convex hull of the vectors in $\{\varepsilon_i - \varepsilon_j : i \in I', j \in J'\}$, hence $I' \neq \emptyset$ and $J' \neq \emptyset$ are determined uniquely and independently of the choice of a linear functional f which characterises the given face. Conversely, for a pair of disjoint non-empty subsets I, J of $[n]$, if we define $f_{IJ}(\mathbf{y}) = \sum_{i \in I} y_i - \sum_{j \in J} y_j$, then the convex hull of vectors in $\{\varepsilon_i - \varepsilon_j : i \in I,\ j \in J\}$ is the face which maximises f_{IJ} in P_n.

Now, we show that the dimension of the face of P_n, determined by disjoint non-empty subsets I, J of $[n]$ is $|I| + |J| - 2$. Let $a = |I|$, $b = |J|$ and $I = \{i_1, \dots, i_a\}$, $J = \{j_1, \dots, j_b\}$. Then

$$X = \{\varepsilon_{i_1} - \varepsilon_{j_l} : l = 1, \dots, b\} \cup \{\varepsilon_{i_l} - \varepsilon_{j_1} : l = 2, \dots, a\}$$

is a linearly independent set of minimal vectors, hence is an affinely independent set. In addition, for any $i \in I$, $j \in J$, $\varepsilon_i - \varepsilon_j$ is in X or

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{j_1}) - (\varepsilon_{i_1} - \varepsilon_{j_1}) + (\varepsilon_{i_1} - \varepsilon_j),$$

an affine combination of vectors in X. Hence, X is an affine basis of the face we are considering, and the dimension of the face is $|X| - 1 = b + a - 1 - 1 = a + b - 2$. ∎

Corollary 2. *For $m = 0, 1, \dots, n-2$, the number of m-dimensional faces of P_n is*

$$\binom{n}{m+2}(2^{m+2} - 2).$$

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PROOF: By Theorem 1, the number of m-dimensional faces is the number of ordered partitions of $(m+2)$-subsets of $[n]$ with two blocks. The result is immediate since $\sum_{l=1}^{m+1} \binom{m+2}{l} = 2^{m+2} - 2$. ∎

Remember that a d-dimensional polytope is *simple* if every vertex is in d facets.

COROLLARY 3. *P_n is not a simple polytope if $n > 3$, whereas the permutahedron Π_{n-1} is always simple.*

PROOF: When we fix a vertex $\varepsilon_i - \varepsilon_j$ in P_n, the number of facets containing $\varepsilon_i - \varepsilon_j$ is $\sum_{l=0}^{n-2} \binom{n-2}{l} = 2^{n-2}$ which is strictly bigger than the dimension $n-1$ of P_n, if $n > 3$. ∎

As a reminder, a *simplex* is the convex hull of vectors in $U = \{\mathbf{u}_1, \dots, \mathbf{u}_l\}$ with the property that U is an affinely independent set. Also, an *m-simplex* Δ_m is a simplex of dimension m. Given two polytopes $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$, the *product* of two polytopes is also a convex polytope $P \times Q = \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in P, \mathbf{v} \in Q\} \subseteq \mathbb{R}^{p+q}$.

COROLLARY 4. *Every nontrivial face of P_n is a product of two simplices. Moreover, if the face F corresponds to a disjoint pair of non-empty subsets I, J, then the face is the product of a $(|I|-1)$-simplex and a $(|J|-1)$-simplex.*

PROOF: Let $P = \mathrm{Conv}(\varepsilon_i : i \in I) \simeq \Delta_{|I|-1}$ and $Q = \mathrm{Conv}(\varepsilon_i : j \in J) \simeq \Delta_{|J|-1}$, then by Theorem 1, F is affinely isomorphic to $P \times Q$. ∎

3. Combinatorics of P_n^+

In this section, we summarise some combinatorial results from [**4**] about the polytope P_n^+. These will be needed in Section 4.

Let H_n^+ be the sublattice in $\mathbb{Z}^n$ generated by $\varepsilon_i - \varepsilon_j$, $1 \leqslant i < j \leqslant n$ and $\mathrm{Vol}_{H_n^+}$ be the form of volume on the space $H_n^+ \otimes_{\mathbb{Z}} \mathbb{R}$ such that volume of the identity cube is equal to 1.

DEFINITION 5: Let $\Gamma = \{(i,j) : 1 \leqslant i < j \leqslant n\}$ be a tree on the set $[n]$. Γ is *admissible* if there are no $1 \leqslant i < j < k \leqslant n$ such that both (i,j) and (j,k) are edges of Γ. We say that Γ has *intersections* if there are $1 \leqslant i < k < j < l \leqslant n$ such that (i,j) and (k,l) are edges of Γ. Γ is defined to be *standard* if it is admissible and there is no intersection. For a given standard tree Γ, let $\mathcal{I}_\Gamma = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i < j \leqslant n, (i,j) \text{ is an edge of } \Gamma\}$. Let $\Theta = \{\mathcal{I}_\Gamma : \Gamma \text{ is a standard tree on } [n]\}$. It is well known that $\mathcal{I}_\Gamma$, where Γ is a standard tree, forms a basis of the linear space $H_n^+ \otimes_{\mathbb{Z}} \mathbb{R}$. Hence $\mathrm{Conv}(\mathcal{I}_\Gamma \cup \{\mathbf{0}\})$ is an $(n-1)$-dimensional simplex and we let $\Delta_{\mathcal{I}_\Gamma}$ be this simplex.

THEOREM 6. *Θ is a local triangulation of P_n^+, in other words,*

$$\bigcup_{\mathcal{I}_\Gamma \in \Theta} \Delta_{\mathcal{I}_\Gamma} = P_n^+$$

and $\Delta_{\mathcal{I}_{\Gamma_1}} \cap \Delta_{\mathcal{I}_{\Gamma_2}}$ is the common face of $\Delta_{\mathcal{I}_{\Gamma_1}}$ and $\Delta_{\mathcal{I}_{\Gamma_2}}$ for all $\mathcal{I}_{\Gamma_1}, \mathcal{I}_{\Gamma_2} \in \Theta$.

Lemma 7. *$(n-1)!\,\mathrm{Vol}_{H_n^+}\Delta_{\mathcal{I}_\Gamma} = 1$ for any $\mathcal{I}_\Gamma \in \Theta$.*

Theorem 8. *The number of standard trees on $[n]$ is equal to the Catalan number*

$$C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

Hence, by Theorem 6 and Lemma 7,

$$(n-1)!\,\mathrm{Vol}_{H_n^+}(P_n^+) = C_{n-1}.$$

For a disjoint pair (I, J) of subsets of $[n]$, let

$$S_{IJ} = \{\varepsilon_i - \varepsilon_j : i \in I,\ j \in J,\ i < j\}.$$

Definition 9: Let I, J be disjoint subsets of $[n]$ such that $I \cup J = [n]$ and $1 \in I$, $n \in J$. We let Γ be a tree on $[n]$.

1. Γ is *of type* (I, J) if for every edge (i, j), $i < j$, in Γ, $i \in I$ and $j \in J$.
2. Let $\Theta_{IJ} = \{\mathcal{I}_\Gamma : \Gamma \text{ is standard of type } (I,J)\}$, and $P_{IJ}^+ = \mathrm{Conv}(S_{IJ} \cup \{0\})$.
3. A *word w of type* (p, q) is the sequence $w = (w_1, w_2, \dots, w_{p+q})$, $w_r \in \{0, 1\}$ such that $|\{r : w_r = 0\}| = p$ and $|\{r : w_r = 1\}| = q$. Let $w = (w_1, w_2, \dots, w_{p+q})$ and $w' = (w_1', w_2', \dots, w_{p+q}')$ be two words of type (p, q). We say that w' *exceeds* w if $w_1' + \dots + w_r' \geqslant w_1 + \dots + w_r$ for all $r = 1, 2, \dots, p+q$. If we present a word w of type (p, q) as the path P_w from $(0,0)$ to (p, q) by the correspondence $1 \leftrightarrow N$, $0 \leftrightarrow E$, where N, E mean north and east respectively, then w' exceed w if and only if $P_{w'}$ is above the path P_w.
4. Let $I = \{1\} \cup I'$, $J = \{n\} \cup J'$. Let $|I'| = p$, $|J'| = q$ and $I' \cup J' = \{t_1 < t_2 < \dots < t_{p+q}\}$. Associate with the pair (I, J) the word $w_{IJ} = (w_1, w_2, \dots, w_{p+q})$ of type (p, q) such that $w_r = 0$ if $t_r \in I$ and $w_r = 1$ if $t_r \in J$ for all $r = 1, 2, \dots, p+q$.

Lemma 10. *$(n-1)!\,\mathrm{Vol}_{H_n^+}\Delta_{\mathcal{I}_\Gamma} = 1$ for each $\mathcal{I}_\Gamma \in \Theta_{IJ}$, where (I, J) is a pair of disjoint subsets of $[n]$ such that $I \cup J = [n]$ and $1 \in I$, $n \in J$.*

Theorem 11. *Let (I, J) be a pair of disjoint subsets of $[n]$ such that $I \cup J = [n]$ and $1 \in I$, $n \in J$. Then, Θ_{IJ} forms a local triangulation of P_{IJ}^+. Moreover, the number of standard trees of type (I, J) is equal to the number of words w' of type $(|I|-1, |J|-1)$, which exceeds the word $w = w_{IJ}$.*

Corollary 12. *If $I = \{1, 2, \dots, i\}$ and $J = \{i+1, i+2, \dots, n\}$ then $w_{IJ} = (0, \dots, 0, 1, \dots, 1)$, hence $(n-1)!$ times the volume of P_{IJ}^+ is the number of paths from $(0,0)$ to $(i-1, n-i-1)$, which is $\binom{n-2}{i-1}$.*

4. P_n AND P_n^+

In this section, we look at the polytope P_n in relation to P_n^+. We first characterise the facets (which do not contain the origin) of P_n^+. (In [4], there is a statement about the facets of P_n^+, but it is slightly incorrect and there is no proof given, so we give a proof of the characterisation of the facets of P_n^+.) Then, we prove the main theorem which shows how P_n^+ sits inside P_n. From this observation, we obtain an interesting identity on the number of paths, and find a triangulation of the product of two simplices. Remember that $S_{IJ} = \{\varepsilon_i - \varepsilon_j : i \in I, j \in J, i < j\}$, for a pair of disjoint subsets I, J of $[n]$.

PROPOSITION 13. *Let $\mathcal{A}$ be the set of facets of P_n^+ which do not contain the origin, and $\mathcal{B} = \{(I,J) : I \cup J = [n], I \cap J = \emptyset$ and $1 \in I, n \in J\}$. Then there is a one to one correspondence between $\mathcal{A}$ and $\mathcal{B}$, such that the corresponding facet of $(I,J) \in \mathcal{B}$ is* $\mathrm{Conv}(S_{IJ})$.

PROOF: Note that when $n = 3$, the Proposition is clear. Let F be a facet of P_n^+ not containing the origin, and $S = \{\varepsilon_i - \varepsilon_j \in F\}$. We also let $f(\mathrm{y}) = \sum\limits_{i\in I} \lambda_i y_i - \sum\limits_{j\in J} \lambda_j y_j$, where $I \cap J = \emptyset, I \cup J = [n]$, $\lambda_i \geqslant 0$, $\lambda_j > 0$, be a corresponding linear functional such that F maximises f on P_n^+. Moreover, let M be the maximum value of f on P_n^+. We let (I', J') be the disjoint pair of non-empty subsets of $[n]$ given in the proof of Theorem 1. If $S_{I'J'} \neq \emptyset$ then $S = S_{I'J'}$. Moreover, if $I' \cup J' \neq [n]$, then we can ignore the number missed in the union of I' and J' and the case goes down to the case $n-1$. Hence F can not be a facet, by induction. Hence $I' \cup J' = [n]$. If $1 \notin I'$ or $n \notin J'$, then 1 or n is completely ignored in $S_{I'J'}$ hence, by induction again, $1 \in I'$, $n \in J'$.

Suppose that $S_{I'J'} = \emptyset$. We first state two basic facts.

1. $\{l : \varepsilon_l - \varepsilon_j \in S$ or $\varepsilon_i - \varepsilon_l \in S\} = [n]$, since with an $(n-1)$-set, the maximum dimension of a face is $n-3$.
2. $M > 0$, since the origin is not contained in F.

There are two cases to be considered, either I and J are non-empty, or one of I, J is empty.

Suppose that $I \neq \emptyset$ and $J \neq \emptyset$. Note that, by considering the elements in I' in the context of fact 1, we have $S_{I'\,J-J'} \cap S \neq \emptyset$ or $S_{I'\,I-I'} \cap S \neq \emptyset$, and those two cases are exclusive because of the difference of the possible values of M. (If $S_{I'I'} \cap S \neq \emptyset$ then the maximum of f on P_n^+ is 0, contrary to fact 2.)

We assume that $S_{I'\,J-J'} \cap S \neq \emptyset$. Then

$$S \subset S_{I'\,J-J'} \cup S_{I-I'\,J'} \cup S_{I-I'\,J-J'} \cup S_{J-J'\,J'} \cup S_{J-J'\,J-J'} \ .$$

Let

$$\begin{aligned} I_1 &= \{i \in I - I' : \varepsilon_i - \varepsilon_j \in S \text{ for some } j \in J'\} \\ I_2 &= \{i \in I - I' : \varepsilon_i - \varepsilon_j \in S \text{ for some } j \in J - J'\} \\ J_1 &= \{j \in J - J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } i \in I'\} \\ J_2 &= \{j \in J - J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } i \in I - I'\} \\ J_3 &= \{i \in J - J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } j \in J'\} \\ J_4 &= \{i \in J - J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } j \in J - J'\} \\ J_5 &= \{j \in J - J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } i \in J - J'\} . \end{aligned}$$

Then $I_1 \cap I_2 = \emptyset$ because of the possible values of M. (Note that λ_i is constant on I'.) Moreover, $J_1, J_2, \ldots, J_5$ are mutually disjoint sets: It is easy to show that J_1, J_2, J_5 are mutually disjoint and J_3, J_4 are disjoint. To show that $J_2 \cap J_3 = \emptyset$, assume that there is $\varepsilon_j \in J_2 \cap J_3$; then $\varepsilon_i - \varepsilon_j \in S$ and $\varepsilon_j - \varepsilon_{j'} \in S$ for some $i \in I - I'$, $j' \in J'$. Since $i < j < j'$, $\varepsilon_i - \varepsilon_{j'} \in P_n^+$ and $f(\varepsilon_i - \varepsilon_{j'}) = f(\varepsilon_i - \varepsilon_j) + f(\varepsilon_j - \varepsilon_{j'}) = 2M > M$, we have a contradiction. Other cases can be proved in the same way.
We also define two subsets J_1' and J_2' of J' by

$$\begin{aligned} J_1' &= \{j \in J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } i \in I - I'\} \\ J_2' &= \{j \in J' : \varepsilon_i - \varepsilon_j \in S \text{ for some } i \in J - J'\} . \end{aligned}$$

Then $J_1' \cap J_2' = \emptyset$. Now, if we count the possible number of affinely independent vectors in S, by the proof of Theorem 1, it is at most

$$\begin{aligned} &(|I'| + |J_1| - 1) + (|I_1| + |J_1'| - 1) + (|I_2| + |J_2| - 1) \\ &\qquad + (|J_3| + |J_2'| - 1) + (|J_4| + |J_5| - 1) \leqslant |I| + |J| - 5 = n - 5 . \end{aligned}$$

Hence, S can not make an $(n-2)$-dimensional face.

If we assume that $S_{I'\,I-I'} \cap S \neq \emptyset$, then

$$S \subset S_{I'\,I-I'} \cup S_{J-J'\,J'} \cup S_{I-I'\,I-I'} \cup S_{J-J'\,J-J'} \cup S_{I-I'\,J-J'} \cup S_{I-I'\,J'} .$$

As we did for the previous case, we define five mutually disjoint subsets $I_1, \ldots, I_5$ of $I - I'$, four mutually disjoint subsets $J_1, \ldots, J_4$ of $J - J'$ and two disjoint subsets J_1', J_2' of J'. Then, the number of possible affinely independent vectors is at most

$$\begin{aligned} &(|I'| + |I_1| - 1) + (|J_1| + |J_1'| - 1) + (|I_2| + |I_3| - 1) + (|J_2| + |J_3| - 1) \\ &\qquad + (|I_4| + |J_4| - 1) + (|I_5| + |J_2'| - 1) \leqslant |I| + |J| - 6 = n - 6 . \end{aligned}$$

Hence S can not make a facet.

As for second case, we assume that $J = \emptyset$. Applying fact 1 to the elements of I', we have $S \subset S_{I'\,I-I'-J'}$ and the number of affinely independent vectors of S is at most $|I'| + |I - I' - J'| - 1 = |I - J'| - 1 \leqslant n - 2$. Therefore S can not form a facet. The proof for the case $I = \emptyset$ goes just the same.

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For a given facet F, we produced $(I', J') \in \mathcal{B}$ so that $F = \text{Conv}(S_{I'J'})$ and the choice is unique as we proved in Theorem 1.

Conversely, if we have $(I, J) \in \mathcal{B}$ then $F = \text{Conv}(S_{IJ})$ is a facet, since $1 \in I, n \in J$. Moreover, since $f_{IJ}(\mathbf{y}) = \sum_{i \in I} y_i + \sum_{j \in J} y_j$ is a linear functional producing F, this is the inverse process of what we did above. ∎

Observe that S_n (the symmetric group on n letters) acts on P_n as a linear transformation in the obvious way, by $\sigma \in S_n$ sending the vertex $\varepsilon_i - \varepsilon_j$ to another vertex $\varepsilon_{\sigma(i)} - \varepsilon_{\sigma(j)}$ (geometric representation of S_n). Let G be the cyclic subgroup of S_n generated by the n-cycle $(12 \ldots n)$. Let F_{IJ} be the corresponding facet of P_n and F_{IJ}^+ be the corresponding facet of P_n^+ of the given pair of disjoint subsets I, J such that $I \cup J = [n]$. (For F_{IJ}^+, $1 \in I$ and $n \in J$ should be satisfied also.) We say that a convex polytope F' is a *sub-face* of a face F of a polytope P if F' and F have the same dimension and $F' \subseteq F$. Two sub-faces of a given face are said to be *disjoint* if the dimension of the intersection is strictly less than the dimension of the given face.

Proposition 14. *Let $(I, J), (I', J')$ be two pairs of disjoint subsets of $[n]$ such that $I \cup J = I' \cup J' = [n]$ and $1 \in I'$, $n \in J'$. Let $g \in G$. Then $g(F_{I'J'}^+)$ is a sub-face of F_{IJ} if and only if $g(I') = I$ and $g(J') = J$.*

Proof: The 'if' part is trivial. Let us assume that $g(F_{I'J'}^+)$ is a sub-face of F_{IJ}. Then $g(S_{I'J'}) \subseteq \{\varepsilon_i - \varepsilon_j : i \in I, j \in J\}$. Note that for each $i \in I'$ (or $j \in J'$), $\varepsilon_i - \varepsilon_n$ ($\varepsilon_1 - \varepsilon_j$ respectively) is in $S_{I'J'}$, hence $g(i) \in I$ and $g(j) \in J$. The proof is completed since $I \cup J = I' \cup J' = [n]$. ∎

Proposition 15. *Let I, J be a pair of disjoint subsets of $[n]$ such that $I \cup J = [n]$ and $\mathcal{S} = \{(I', J', g_{I'J'}) : g_{I'J'}(I') = I,\ g_{I'J'}(J') = J,\ \text{for } g_{I'J'} \in G \text{ and } 1 \in I',\ n \in J'\}$. Then $\{g_{I'J'}(F_{I'J'}^+) : (I', J', g_{I'J'}) \in \mathcal{S}\}$ forms a set of disjoint sub-faces of F_{IJ}.*

Proof: Note that for $(I', J') \in \mathcal{S}$, since $g_{I'J'}$ is a power of the n-cycle $(1\,2 \ldots n)$ and $1 \in I'$, $n \in J'$, there must be $i \in J$ such that $i + 1 \in I$. (If $g_{I'J'} = id$, then $i = n$, $i + 1 = 1$.)

Let $(I_1, J_1, g_{I_1J_1})$, $(I_2, J_2, g_{I_2J_2}) \in \mathcal{S}$ be distinct and $g_{I_1J_1}(F_{I_1J_1}^+)$, $g_{I_2J_2}(F_{I_2J_2}^+)$ be sub-faces of F_{IJ}. Then there are two numbers i_1, i_2 such that $i_1, i_2 \in J$, $i_1 + 1$, $i_2 + 1 \in I$ and $g_{I_1J_1}(1) = i_1 + 1$, $g_{I_2J_2}(1) = i_2 + 1$ (hence $g_{I_1J_1}(n) = i_1$, $g_{I_2J_2}(n) = i_2$). Without loss of generality, we assume that $i_1 < i_2$. (If $i_1 = i_2$ then $g_{I_1J_1}(n) = g_{I_2J_2}(n)$ so $g_{I_1J_1} = g_{I_2J_2} \in G$. Hence $I_1 = g_{I_1J_1}^{-1}(I) = g_{I_2J_2}^{-1}(I) = I_2$ and $J_1 = J_2$.) We let $A = \{i_2 + 1, \ldots, n, 1, \ldots, i_1\}$, $|A| = a \neq 0$ (if $i_2 = n$ then $A = \{1, \ldots, i_1\}$) and $B = [n] - A$, $|B| = b \neq 0$. If there is $\varepsilon_k - \varepsilon_l \in g_{I_1J_1}(F_{I_1J_1}^+) \cap g_{I_2J_2}(F_{I_2J_2}^+)$ such that $k \in A$, $l \in B$, then $g_{I_1J_1}^{-1}(k) < g_{I_1J_1}^{-1}(l)$ but $A = g_{I_1J_1}(\{i_2 - i_1 + 1, \ldots, n\})$ and $B = g_{I_1J_1}(\{1, \ldots, i_2 - i_1\})$, hence we have a contradiction. The same argument excludes the case $k \in B$, $l \in A$ also. Therefore, the vertices of $g_{I_1J_1}(F_{I_1J_1}^+) \cap g_{I_2J_2}(F_{I_2J_2}^+)$ are $\varepsilon_k - \varepsilon_l$ where ($k \in A \cap I$ and $l \in A \cap J$) or ($k \in B \cap I$ and $l \in B \cap J$). The biggest possible number of affinely independent

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vertices of $g_{I_1J_1}(F^+_{I_1J_1}) \cap g_{I_2J_2}(F^+_{I_2J_2})$ is $(a-1)+(b-1)=n-2$. Hence the dimension of $g_{I_1J_1}(F^+_{I_1J_1}) \cap g_{I_2J_2}(F^+_{I_2J_2})$ is strictly less than $n-2$. ∎

For a given disjoint pair (I,J) of subsets of $[n]$, we let P_{IJ} be the convex hull generated by the vectors in $\{\varepsilon_i - \varepsilon_j : i \in I, j \in J\} \cup \{\mathbf{0}\}$.

The main theorem is the following.

THEOREM 16. [Main Theorem]

$$\bigcup_{g \in G} g(P_n^+) = P_n .$$

Furthermore, if $g_1(P_n^+) \cap g_2(P_n^+) \neq \emptyset$ for $g_1 \neq g_2 \in G$, then the volume of the intersection is 0.

PROOF: The disjointness of $g(P_n^+)$ follows from Proposition 15. Therefore, we are only left to show that $G(P_n^+)$ is not only a part of P_n but also P_n itself. It is sufficient to show that

$$|G| \ \mathrm{Vol}_{H_n^+}(P_n^+) = n\,\mathrm{Vol}_{H_n^+}(P_n^+) = \sum_{i=1}^{n-1} \binom{n}{i} t_{i,n-i}, \tag{1}$$

where $t_{i,n-i}$ is the volume of P_{IJ}, $|I| = i$ and $|J| = n-i$, since

$$P_n = \bigcup_{\substack{(I,J)\\ I \cup J = [n]}} P_{IJ}.$$

Note that $t_{i,n-i} = \left(\binom{n-2}{i-1}/(n-1)!\right)$ by Corollary 12, since P_{IJ}, $|I| = i$, $|J| = n-j$, is exactly the same polytope as $P^+_{\{1,\dots,i\}\{i+1,\dots,n\}}$.

By Theorem 8, the left hand side of Eqation (1) is

$$\frac{nC_{n-1}}{(n-1)!} = \frac{\binom{2(n-1)}{n-1}}{(n-1)!},$$

and the right hand side of Eqation (1) is

$$\frac{\sum_{i=1}^{n-1} \binom{n}{i}\binom{n-2}{i-1}}{(n-1)!}.$$

Equation (1) is verified because of the following well known equation: for fixed integers k, l, m,

$$\sum_r \binom{k}{r}\binom{l}{m-r} = \binom{k+l}{m}.$$

∎

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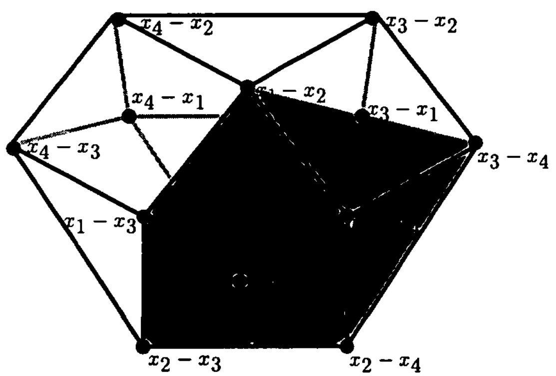
REMARK 1. The n-cycle $(12\dots n)$ is a Coxeter element of S_n. Hence, Theorem 16 explains how the Coxeter elements of type A_{n-1} play a role in one way. However, Theorem 16 does not hold for the other types of root systems (at least for B_2 and G_2). To understand Theorem 16 in the wider context of finite reflection groups (or Coxeter groups), n-cycles of S_n might have to be interpreted differently (other than Coxeter elements), or a more general rule would be needed which covers the A_n case. Although we could not find a general version (in the context of Coxeter groups) of Theorem 16, we believe that it is a very interesting property in itself.

EXAMPLE 1. The following picture is P_4, which is a 3-dimensional polytope. The shaded region is the intersection of P_4^+ with the boundary of P_4. It is easy to check that exactly 4 copies of the shaded region form the boundary of P_4.



As a corollary of Theorem 8 and Theorem 16, we have the following.

COROLLARY 17.

$$\mathrm{Vol}_{H_n^+}(P_n) = n\ \mathrm{Vol}_{H_n^+}(P_n^+) = \frac{1}{(n-1)!}\binom{2(n-1)}{n-1}.$$

Some facets of P_n are a union of images of facets of P_n^+. If we use the identity on the volume then we obtain an interesting result.

COROLLARY 18. *For given n, i, j such that $i + j = n$, $i \neq 0$, $j \neq 0$ let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i)$ be a sequence of non-zero integers such that $\sum_{l=1}^{i} \lambda_l = n$. Then the number of paths from $(0,0)$ to $(i-1, j-1)$, equally $\binom{n-2}{i-1}$, is equal to the sum*

$$\sum_{\substack{(\lambda_{h(1)},\dots,\lambda_{h(i)}),\ \lambda_{h(i)}\neq 1\\ h\in\langle(1,2,\dots,i)\rangle}} \begin{array}{l}\text{number of paths from } (0,0) \text{ to } (i-1,j-1)\\ \text{which exceed } P_{(\lambda_{h(1)},\dots,\lambda_{h(i)})},\end{array}$$

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where $P_{(m_1,\dots,m_l)} = \phi(m_1)\dots\phi(m_{l-1})\psi(m_l)$ *is the path from* $(0,0)$ *to* $(i-1, j-1)$ *obtained from* $(m_1,\dots,m_l)$ *by the following correspondence*

$$\phi(m) = \underbrace{N\dots N}_{m-1 \text{ times}} \quad E,$$

and

$$\psi(m) = \underbrace{N\dots N}_{m-2 \text{ times}} \quad .$$

PROOF: Define two subsets of $[n]$ by

$$I = \{1, 1+\lambda_1, 1+\lambda_1+\lambda_2, \dots, 1+\lambda_1+\dots+\lambda_{i-1}\}$$

and $J = [n] - I$. Then $|I| = i$, $|J| = j$ and $(n-1)!\,\mathrm{Vol}_{H_n^+}(P_{IJ})$ is the number of paths from $(0,0)$ to $(i-1, j-1)$ by Corollary 12. On the other hand, by Proposition 15 and Theorem 16, $\mathrm{Vol}_{H_n^+}(P_{IJ}) = \sum_{(I',J',g_{I'J'})\in\mathcal{S}} \mathrm{Vol}_{H_n^+}(P^+_{I'J'})$. Remember that $\mathcal{S}$ was defined by

$$\mathcal{S} = \big\{(I', J', g_{I'J'}) : g_{I'J'}(I') = I, g_{I'J'}(J') = J, \text{ for } g_{I'J'} \in G \text{ and } 1 \in I', n \in J'\big\}.$$

Moreover, by Theorem 7, $(n-1)!\,\mathrm{Vol}_{H_n^+}(P^+_{I'J'})$ is the number of paths from $(0,0)$ to $(i-1, j-1)$ which exceed the word $w_{I'J'}$.

For a given subset $A = \{a_1, a_2, \dots, a_i\}$ of $[n]$, such that $1 = a_1 < a_2 < \dots < a_i$, we define the *type* of A as $type(A) = (a_2-a_1, a_3-a_2, \dots, a_i-a_{i-1}, n+1-a_i)$. Then $type(I) = (\lambda_1, \lambda_2, \dots, \lambda_i)$. Note that $(I', J', g_{I'J'}) \in \mathcal{S}$ if and only if $type(I') = (\lambda_{h(1)}, \lambda_{h(2)}, \dots, \lambda_{h(i)})$ for some $h \in \langle(1, 2, \dots, i)\rangle$ and $\lambda_{h(i)} \neq 1$ since if $\lambda_{h(i)} = 1$ then $1, n \in I'$. Hence, we are only left to show that the path $P_{w_{I'J'}}$ is exactly the same as $P_{type(I')}$, and this is immediate from Definition 9, (4). ∎

EXAMPLE 2. Let $n = 6$, $i = 3$, $j = 3$, $\lambda = (1, 3, 2)$, and let $N_{P_{(m_1,m_2,m_3)}}$ be the number of paths from $(0,0)$ to $(2,2)$ which exceed $P_{(m_1,m_2,m_3)}$. Then

$$\text{the number of paths from } (0,0) \text{ to } (2,2) = N(P_{(1,3,2)}) + N(P_{(2,1,3)}) = 3 + 3 = \binom{4}{2},$$

since $P_{(1,3,2)} = ENNE$ and $P_{(2,1,3)} = NEEN$.

COROLLARY 19. *There is a triangulation with no new vertices of* $\Delta_p \times \Delta_q$, $p, q \geqslant 0$, *with* $\binom{p+q}{p}$ *simplices, where* Δ_i *is the* i*-dimensional simplex.*

PROOF: Let $I = \{1, 2, \dots, p+1\}$, $J = \{p+2, p+3, \dots, p+q+2\}$ then $F_{IJ} \cong \Delta_p \times \Delta_q$ by Corollary 4. Now, Corollary 12 finishes the proof. ∎

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