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Revisiting G-Dedekind domains

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Dedicated to fairness

Abstract. Let R be an integral domain with $qf(R) = K$, and let $F(R)$ be the set of nonzero fractional ideals of R. Call R a dually compact domain (DCD) if, for each $I \in F(R)$, the ideal $I_v = (I^{-1})^{-1}$ is a finite intersection of principal fractional ideals. We characterize DCDs and show that the class of DCDs properly contains various classes of integral domains, such as Noetherian, Mori, and Krull domains. In addition, we show that a Schreier DCD is a greatest common divisor (GCD) (Greatest Common Divisor) domain with the property that, for each $A \in F(R)$, the ideal A_v is principal. We show that a domain R is G-Dedekind (i.e., has the property that A_v is invertible for each $A \in F(R)$) if and only if R is a DCD satisfying the property $*$: For all pairs of subsets $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\} \subseteq K \backslash \{0\}$, $(\cap_{i=1}^m (a_i)(\cap_{j=1}^n (b_j)) = \cap_{i,j=1}^{m,n} (a_i b_j)$. We discuss what the appropriate names for G-Dedekind domains and related notions should be. We also make some observations about how the DCDs behave under localizations and polynomial ring extensions.

1 Introduction

Let R be an integral domain with quotient field K, and let $F(R)$ be the set of nonzero fractional ideals of R. Call R a *dually compact domain* (*DCD*) if, for each set, $\{a_\alpha\}_{\alpha \in I} \subseteq K\backslash\{0\}$ with $\cap a_\alpha R \neq (0)$, there is a finite set of elements $\{x_1, \ldots, x_r\} \subseteq K\backslash\{0\}$ such that $\cap a_\alpha R = \cap_{i=1}^r x_i R$, or equivalently, for each $I \in F(R)$, the ideal $I_v = (I^{-1})^{-1}$ is a finite intersection of principal fractional ideals of R. We characterize DCDs (in Section 3) and show that the class of DCDs properly contains various classes of integral domains of import, such as Noetherian, Mori, and Krull domains, in Section 4. In addition, we show that a *pre-Schreier* DCD is a GCD domain with the property that, for each $A \in F(R)$, the ideal A_v is principal. (Here, R is pre-Schreier if, for all $x, y, z \in R\backslash\{0\}$, $x|yz \Rightarrow x = rs$, with $r, s \in R$ such that $r|y$ and $s|z$.) We show that a domain R is a *G-Dedekind domain* (i.e., has the property that A_v is invertible for each $A \in F(R)$) if and only if R is a DCD satisfying the property $*$: for all pairs of subsets $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\} \subseteq K\backslash\{0\}$, $(\cap_{i=1}^m (a_i)(\cap_{j=1}^n (b_j)) = \cap_{i,j=1}^{m,n} (a_i b_j)$ from [26]. We discuss in Section 2 what names for G-Dedekind domains and related notions should be appropriate. We also make some observations about how the DCDs behave under localizations and polynomial ring extensions.

In [25], this author studied integral domains R with the property that A_v is invertible for every nonzero ideal A, and called these domains "generalized Dedekind

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domains" or G-Dedekind domains. Later, using a new form of the abovementioned $\star$-property, Anderson and Kang [6] published a much improved version of [25]. Calling the G-Dedekind domains "Pseudo-Dedekind" domains, they showed that R is a Pseudo-Dedekind domain if and only if, for all sets $\{a_\alpha\}_{\alpha\in I}, \{b_\beta\}_{\beta\in J} \subseteq K\backslash\{0\}$, we have $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$, where α ranges over I and β over J. Call a set of elements $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$ *allowable* if $\cap(a_\alpha) \neq (0)$. In this article, we show, among things already indicated, that, for a given star operation $\star$, $A^\star$ is invertible for any ideal A of R if and only if, for any allowable set of elements $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$ and for every nonzero ideal A, we have $A^\star(\cap(a_\alpha)) = \cap A^\star a_\alpha$. As a consequence, we show that R is a Pseudo-Dedekind/G-Dedekind domain if and only if, for each nonzero ideal A of R and for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A_v(\cap(a_\alpha)) = \cap A_v(a_\alpha)$. We show that R is a DCD if and only if, for any allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, there is a set $\{b_1, b_2, \ldots, b_n\} \subseteq K\backslash\{0\}$ such that $\cap_{\alpha\in I}(a_\alpha) = (b_1, b_2, \ldots, b_n)_v$. We also show that R is a G-Dedekind/Pseudo-Dedekind domain if and only if R is a DCD with the abovementioned property $\star$. That is, the DCDs were essentially at work behind the scenes in the results in [25] and subsequently in [6].

We use star operations, in order to approach the subject from a more general standpoint. Included below is a brief introduction to star operations. The reader may consult [14], [10], or [15] for more information on star operations. For our purposes, we include below some information that may be helpful in reading this article.

Let R be an integral domain with quotient field K, and let $F(R)$ be the set of nonzero fractional ideals of R. A star operation is a function $A \mapsto A^\star$ on $F(R)$ with the following properties:

If $A, B \in F(R)$ and $a \in K\backslash\{0\}$, then

(i) $(a)^\star = (a)$ and $(aA)^\star = aA^\star$.
(ii) $A \subseteq A^\star$, and if $A \subseteq B$, then $A^\star \subseteq B^\star$.
(iii) $(A^\star)^\star = A^\star$.

We may call $A^\star$ the *$\star$-image* (or *$\star$-envelope*) of A. An ideal A is said to be a *$\star$-ideal* if $A^\star = A$. Thus, $A^\star$ is a $\star$-ideal (by (iii)). Moreover, (by (i)), every principal fractional ideal, including $R = (1)$, is a $\star$- ideal for any star operation $\star$.

For all $A, B \in F(R)$ and for each star operation $\star$, we can show that $(AB)^\star = (A^\star B)^\star = (A^\star B^\star)^\star$. These equations define what is called *$\star$-multiplication (or $\star$-product)*. Associated with each star operation $\star$ is a star operation $\star_f$ defined by $A^{\star_f} = \bigcup\{J^\star | 0 \neq J$ is a finitely generated subideal of $A\}$, for each $A \in F(D)$. We say that a star operation $\star$ is of finite type or of finite character if $\star = \star_f$, i.e., $A^\star = A^{\star_f}$ for each $A \in F(R)$.

Define $A^{-1} = \{x \in K | xA \subseteq R\}$, for $A \in F(R)$. Thus, $A^{-1} = \cap_{a\in A\backslash\{0\}}(\frac{1}{a})$. Furthermore, define $A_v = (A^{-1})^{-1}$ and $A_t = A_{v_f} = \bigcup\{J_v | 0 \neq J$ is a finitely generated subideal of $A\}$. By the definition, $A_t = A_v$ for each finitely generated nonzero ideal of R. The functions $A \mapsto A_v$ and $A \mapsto A_t$ on $F(R)$ are more familiar examples of star operations defined on an integral domain. A fractional ideal $A \in F(R)$ is *$\star$-invertible* if $(AA^{-1})^\star = R$. An invertible ideal is a $\star$-invertible $\star$-ideal for each $\star$-operation $\star$, and so is a v-ideal. A v-ideal is better known as a divisorial ideal, and using the definition, it can be shown that $A_v = \cap_{\substack{x\in K\backslash\{0\} \\ A\subseteq xR}} xR$. The identity function d on $F(R)$, defined by $A \mapsto A$, is another example of a star operation. Indeed, a "d-invertible" ideal is the usual

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invertible ideal. There are of course many more star operations that can be defined on an integral domain R. But for any star operation $\star$ and for any $A \in F(R), A^\star \subseteq A_v$. Some other useful relations are: For any $A \in F(R), (A^{-1})^\star = A^{-1} = (A^\star)^{-1}$, and so $(A_v)^\star = A_v = (A^\star)_v$. Using the definition of the t-operation, one can show that an ideal that is maximal w.r.t. being a proper integral t-ideal is a prime ideal of R, each nonzero ideal A of R with $A_t \neq R$ is contained in a maximal t-ideal of R, and $R = \cap R_M$, where M ranges over maximal t-ideals of R. The set of maximal t-ideals of R is denoted by $t\text{-}Max(R)$. For more on v- and t-operations, the reader may consult Sections 32 and 34 of Gilmer [14], or the other two books cited above. This is the barest minimum of description to get us started, and we shall expand on it when need arises. Our terminology comes from [14]. Of course, we have called a subset $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$ allowable $\cap(a_\alpha) \neq (0)$ (a) to save on space and (b) because the characteristic property of a G-Dedekind/Pseudo-Dedekind domain is that, for all $A, B \in F(D)$, we have $(AB)^{-1} = A^{-1}B^{-1}$, and this does not require any set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$ with $\cap(a_\alpha) = (0)$. The reader may expect such purpose-oriented terminology in the sequel as well. In Section 2, we fix a name for G-/Pseudo-Dedekind domains, indicating reasons why we should, and in Section 3, we show that R is a G-/Pseudo-Dedekind domain if and only if R is a dually compact (DC) $*$-domain. Finally, in Section 4, we touch on some related questions and indicate how DCDs behave under some extensions such as quotient ring formation or, in case of integrally closed DCDs, polynomial ring formation.

2 What is in a name?

Popescu [20] introduced the notion of a generalized Dedekind domain via localizing systems. Nowadays, the following equivalent definition is usually given: An integral domain is a generalized Dedekind domain if it is a strongly discrete Prüfer domain (i.e., $P \neq P^2$ for every prime ideal P) and every (prime) ideal I has $\sqrt{I} = \sqrt{(a_1, \ldots, a_n)}$ for some $a_1, \ldots, a_n \in I$ (or equivalently, every principal ideal has only finitely many minimal prime ideals). Unbeknownst to this author, [20] was already out. I personally do not think there is anything pseudo about the G-Dedekind domains. On the other hand, some serious studies related to G-Dedekind prime rings, introduced by Akalan [1], are being carried out. This indicates that there is need for a name close to G-Dedekind domains. The aim of this section is to fix a suitable name for the domains that are given two different names, one not quite appropriate and the other overshadowed by a previously adopted name. Of course, as a result, we end up with one more notion a more suitable name for a so-calledπ-domain.

The following lemma essentially comes from [2], yet we use it to give a more general view of what started as G-Dedekind domains or Pseudo-Dedekind domains. Of course, our statement is different and more streamlined.

Lemma 2.1 *Let $\star$ be a star operation defined on an integral domain R, and let $A \in F(R)$. Then $A^\star$ is invertible if and only if, for any allowable set of elements $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A^\star(\cap(a_\alpha)) = \cap A^\star a_\alpha$.*

The proof of the above lemma has been used as part of the proof of the following theorem.

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Theorem 2.2 *Let $\star$ be a star operation defined on an integral domain R. Then $A^\star$ is invertible for every nonzero fractional ideal A of R if and only if, for any allowable set of elements $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$ and for every nonzero ideal A, we have $A^\star(\cap(a_\alpha)) = \cap A^\star a_\alpha$.*

Proof Let A be any nonzero ideal of R, and suppose that, for every allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A^\star(\cap(a_\alpha)) = \cap A^\star a_\alpha$. Then $R \supseteq A^\star A^{-1} = A^\star(\cap_{a_\beta\in A\backslash\{0\}}(\frac{1}{a_\beta})) = \cap_{a_\beta\in A\backslash\{0\}} A^\star(\frac{1}{a_\beta})$ by the condition. Since for each of $a_\beta \in A\backslash\{0\}$ we have $A^\star(\frac{1}{a_\beta}) \supseteq R$ and so $R \supseteq (A^\star A^{-1} = A^\star(\cap_{a_\beta\in A\backslash\{0\}}(\frac{1}{a_\beta})) = \cap_{a_\beta\in A\backslash\{0\}} A^\star(\frac{1}{a_\beta}) \supseteq R$, showing that $R = A^\star(\cap_{a_\beta\in A\backslash\{0\}}(\frac{1}{a_\beta})) = A^\star A^{-1}$. Thus, as A was chosen arbitrarily, the condition implies that, for every nonzero ideal A, we have that $A^\star$ is invertible. Conversely, suppose that A is a nonzero ideal such that $A^\star$ is invertible. Then, by an exercise on page 80 of [14], we have, for any allowable set, $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, $A^\star(\cap(a_\alpha)) = \cap A^\star(a_\alpha)$. Thus, for every nonzero ideal A, the ideal $A^\star$ being invertible implies that, for every nonzero ideal A and for every allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A^\star(\cap(a_\alpha)) = \cap A^\star a_\alpha$. ■

Corollary 1 *For $\star = d$, R is a Dedekind domain if and only if, for each nonzero ideal A of R and for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A(\cap(a_\alpha)) = \cap A(a_\alpha)$.*

Proof Indeed, it is well known that R is a Dedekind domain if and only if every nonzero ideal of R is invertible and Theorem 2.2 provides the, general, necessary and sufficient conditions for every nonzero ideal to be $\star$-invertible, when $\star = d$. ■

Let us recall that an ideal I is t-invertible if I is v-invertible and I^{-1} is a v-ideal of finite type, that an ideal I that is invertible is $\star$-invertible for every star operation $\star$ [28], and that an integral domain R is a Krull domain if and only if every nonzero ideal of R is t-invertible [19]. Now, if I_t is invertible, then I_t and hence I are t-invertible. Thus, if, for each ideal I, I_t is invertible in a domain R, then R is at least a Krull domain. According to Theorem 1.10 of [25], if I_t is invertible for every nonzero ideal of R, then R is a locally factorial Krull domain. Such domains are often called π-domains for some reason. Now, Theorem 2.2 characterizes π-domains for $\star = t$ as follows.

Corollary 2 *A domain R is a π-domain if and only if, for each nonzero ideal A of R and for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A_t(\cap(a_\alpha)) = \cap A_t(a_\alpha)$.*

Proof Indeed, R is a π-domain if and only if A_t is invertible for every nonzero ideal A of R (Theorem 1.10 of [25]) and Theorem 2.2 provides the, general, necessary and sufficient conditions for every nonzero ideal to be $\star$-invertible, when $\star = t$. ■

Of course, by saying that an ideal A of R is v-invertible, we mean that $(AA^{-1})_v = R$. Similar to earlier comments, we note that A_v being invertible entails A being v-invertible. We also note that domains R with A_v invertible for each nonzero ideal A are the G-Dedekind/Pseudo-Dedekind domains. So, for $\star = v$, Theorem 2.2 provides the following characterization of G-Dedekind/Pseudo-Dedekind domains.

Corollary 3 *A domain R is a G-Dedekind/Pseudo-Dedekind domain if and only if, for each nonzero ideal A of R and for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have $A_v(\cap(a_\alpha)) = \cap A_v(a_\alpha)$.*

The proof is the same as the one provided by Theorem 2.2 replacing $\star$ by v.

Remark 2.3 Looking at Theorem 2.2 and Corollaries 1–3, we may call R a $\star$-Dedekind domain if $A^\star$ is invertible for each $A \in F(R)$ and note that in a $\star$-Dedekind domain, we have $A^\star = A_v$ for all $A \in F(R)$. This is because an invertible ideal is divisorial. Thus, Corollary 2 gives for $\star = t$ a t-Dedekind domain, and Corollary 3 gives the name of a v-Dedekind domain to the G-Dedekind domain of [25] and the Pseudo-Dedekind domain of [6]. But there is a slight problem with this naming system; Elliott in [10] calls a Krull domain a t-Dedekind domain. So, perhaps, $\star$ -G-Dedekind may be the general name with the note that a d-G-Dedekind domain is the usual Dedekind domain and a t-G-Dedekind domain is a locally factorial Krull domain, while the v-G-Dedekind domain is the usual Pseudo-Dedekind domain, or the old G-Dedekind domain. Of course, as $\star = v$ in a $\star$-G-Dedekind domain, each $\star$-G-Dedekind domain has the properties listed in [25] for G-Dedekind domains and are shared by $\star$-G-Dedekind domains, or in [6] for Pseudo-Dedekind domains. Thus, if $\star$ is of finite character, then $\star = \star_f = v_f = t$. That is, if R is a $\star$-G-Dedekind domain and $\star$ is of finite character, then R is a t-G-Dedekind domain. Recall that an integral domain R with quotient field K is completely integrally closed if whenever $rx^n \in R$ for $x \in K$, $0 \neq r \in R$, and every integer $n \geq 1$, then $x \in R$. Equivalently, R is completely integrally closed if and only if $(AA^{-1})_v = R$ for every $A \in F(R)$ [14]. If, for a star operation $\star$ defined on R, A is $\star$ invertible for each $A \in F(R)$ following [4], we may call R a $\star$-CICD. Thus, as noted in [25], a $\star$-G-Dedekind domain is a completely integrally closed domain (CICD), for each star operation $\star$ that it is defined for. There is a star operation called the w-operation, defined in terms of the t-operation as $A \mapsto A_w = \cap_{M\in t\text{-}Max(R)} AR_M$ (see [5] and the references therein). As indicated in [5], this operation is of finite character. Thus, in view of earlier comments in this remark, a w-G-Dedekind domain is a t-G-Dedekind domain. (While the w-operation has been around for some time, Wang and McCasland adopted it in [23].)

3 Dually compact domains

Cohn [9] called an element $x \in R$ primal if, for all $y, z \in R$, $x|yz$ implies that $x = rs$ where $r|y$ and $s|z$. A domain all of whose nonzero elements are primal was called a pre-Schreier domain in [26]; this was a break from Cohn who called R a Schreier domain if R was integrally closed with all elements primal. Based on a study of the group of divisibility of a pre-Schreier domain, this author extracted, in [26], what he called the $*$ property, saying: R is a $*$ domain if, for all pairs of subsets $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\} \subseteq K\backslash\{0\}$, $(\cap_{i=1}^m (a_i))(\cap_{j=1}^n (b_j)) = \cap_{i,j=1}^{m,n} (a_i b_j)$.

We now look at the facts working behind the Anderson–Kang/Zafrullah results. For this, let us call a domain R DC if, for any allowable subset $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, there is a set $\{x_1, x_2, \ldots, x_n\} \subseteq K\backslash\{0\}$ such that $\cap_{\alpha\in I}(a_\alpha) = (x_1) \cap (x_2) \cap \cdots \cap (x_n)$. Let us also note that a fractional ideal A being divisorial ideal (or v -ideal) of finite type means that there are elements $s_1, \ldots, s_r \in K\backslash\{0\}$ such that $A = (s_1, \ldots, s_r)_v$.

Theorem 3.1 *The following are equivalent for an integral domain R that is different from its quotient field K.*

(1) R is DC.
(2) A_v is a v-ideal of finite type for every $A \in F(R)$.
(3) A^{-1} is of finite type for each $A \in F(R)$.
(4) A^{-1} is of finite type for each nonzero integral ideal A of R.
(5) A_v is of finite type for each nonzero integral ideal A of R.
(6) Every divisorial ideal of R is expressible as a finite intersection of principal fractional ideals.
(7) For a star operation $\star$ and for each $A \in F(R)$, the ideal $A^\star$ is a v-ideal of finite type.
(8) For a star operation $\star$ and for each $A \in F(R)$, the ideal $A^\star$ is a finite intersection of principal ideals from $F(R)$.

Proof (1) $\Rightarrow$ (2). Suppose that R is DC, and suppose that A is a nonzero fractional ideal generated by $\{c_\gamma\}_{\gamma \in J}$. Then $A^{-1} = \cap_\gamma(\frac{1}{c_\gamma}) \neq (0)$. So, by the DC property, there is a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq K\backslash\{0\}$ such that $\cap_\gamma(\frac{1}{c_\gamma}) = \cap_{i=1}^n (x_i) = (x_1^{-1}, \ldots, x_n^{-1})^{-1}$. Now, $A^{-1} = (x_1^{-1}, \ldots, x_n^{-1})^{-1}$ implies that $A_v = (x_1^{-1}, \ldots, x_n^{-1})_v$.

(2) $\Rightarrow$ (1). Suppose that, for each $A \in F(R)$, there are $x_1, x_2, \ldots, x_n \in K\backslash\{0\}$ such that $A_v = (x_1, x_2, \ldots, x_n)_v$. Then, since A^{-1} is a divisorial ideal, as $(A^{-1})_v = A^{-1}$, we have $A^{-1} = (y_1, \ldots, y_r)_v$. Or, assuming that all the y_i are nonzero, $A_v = \cap(\frac{1}{y_i})$ for each $A \in F(R)$. Thus, if, for each $A \in F(R)$, A_v is of finite type, then, for each $A \in F(R)$, we can find some $b_i \in K\backslash\{0\}$ such that $A_v = \cap_{i=1}^r (b_i)$. Now, let $\{a_\alpha\}_{\alpha \in I} \subseteq K\backslash\{0\}$ be allowable, and let $A = \cap(a_\alpha)$. Since $\cap(a_\alpha) \neq (0)$, A is a divisorial ideal by [14] and hence of finite type, and so, by (2), for some $x_1, \ldots, x_n \in K\backslash\{0\}$, we have $A = \cap(a_\alpha) = \cap_{i=1}^n (x_i)$.

Next, (2) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (4) because every fractional ideal A of R is of the form $\frac{B}{d}$ where B is an integral ideal. (2) $\Rightarrow$ (3) because $A^{-1} = (A^{-1})_v$ and (3) $\Rightarrow$ (2) because $A_v = (A^{-1})^{-1}$.

(1) $\Rightarrow$ (6). A nonzero ideal A is divisorial if and only if $A = \cap_{\substack{A \subseteq x_\alpha R \\ x_\alpha \in K\backslash\{0\}}} x_\alpha R$. By the DC condition, there are $x_1, \ldots, x_n$ in $K\backslash\{0\}$ such that $A = \cap_{\substack{A \subseteq x_\alpha R \\ x_\alpha \in K\backslash\{0\}}} x_\alpha R = \cap_{i=1}^n x_i R$,

(6) $\Rightarrow$ (1). Let for $\{a_\alpha\}_{\alpha \in I} \subseteq K\backslash\{0\}$, $\cap(a_\alpha) \neq (0)$. Note that $\cap(a_\alpha)$ is divisorial. So, by (6), there are elements $x_1, \ldots, x_r \in K\backslash\{0\}$ such that $\cap(a_\alpha) = \cap_{i=1}^r (x_i)$. Finally, each of (7) and (8) holds if and only if $A^\star = A_v$ for all $A \in F(R)$ because in both cases $A^\star = B$ where B is divisorial and so $A_v = B = A^\star$. Observing that, we have (7) $\Leftrightarrow$ (5) and (8) $\Leftrightarrow$ (6). ∎

Remark 3.2

(1) There are a number of integral domains that fit the description of DCDs. Noetherian domains do fit nicely, as do the so-called Mori domains. Recall that R is a Mori domain if R satisfies ascending chain conditions on divisorial ideals. It is well known that R is a Mori domain if and only if, for each $A \in F(R)$, there is a finitely generated fractional ideal $B \subseteq A$ such that $A_v = B_v$. Indeed, a Krull

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domain is a DCD, being a Mori domain as indicated in Fossum [13]. On the other hand, there are DCDs, such as the ring of entire functions in which A_v is principal for each $A \in F(R)$ and the ring of entire functions is neither a Mori domain, nor a Krull domain.

(2) If the star operation $\star$ defined on R is such that $A^\star$ is a v-ideal of finite type for each $A \in F(R)$, we can call R a $\star$ -DCD.

(3) It may be somewhat hard to see, for some, that any t-invertible t -ideal, and hence any invertible ideal, is a finite intersection of principal fractional ideals. Let me note for the record that if A is a t -invertible t-ideal, then A^{-1} is of finite type. Say, for some $B = \{b_1, \ldots, b_n\}$, we have $A^{-1} = B_v$. But then, $A = A_v = (B_v)^{-1} = \cap_{i=1}^{n}(\frac{1}{b_i})$.

Indeed, as in a DCD, the inverse of every nonzero fractional ideal is of finite type, and all we need for the v-G-Dedekind property to hold is the $*$-property.

Theorem 3.3 *Let R be a DCD. Then R is a v-G-Dedekind domain if and only if R is a $*$-domain.*

Proof It is easy to see, from the treatment of it in [6], that a v -G-Dedekind domain is a $*$-domain. (This fact was also mentioned in [25].) But, of course, we need to show that a v-G-Dedekind domain is DC. This follows from the fact that, in a v-G-Dedekind domain R, A_v is invertible for each $A \in F(R)$ and, as shown in Remark 3.2, an intersection of finitely many principal fractional ideals. For the converse, we show that DC plus the $*$-property imply the v -G-Dedekind property. For this, consider for allowable sets $\{a_\alpha\}_{\alpha \in I} \subseteq K\backslash\{0\}, \{b_\beta\}_{\beta \in J} \subseteq K\backslash\{0\}$ the product $P = (\cap(a_\alpha))(\cap(b_\beta))$. By the DC property of R, we can find $\{x_1, \ldots, x_m\}, \{y_1, \ldots, y_n\} \subseteq K\backslash\{0\}$ such that $\cap(a_\alpha) = \cap_{i=1}^{m}(x_i)$ and $\cap(b_\beta) = (\cap_{j=1}^{n}(y_j)$. Thus, using DC plus $*$, $P = (\cap(a_\alpha))(\cap(b_\beta)) = (\cap(x_i))(\cap(y_j)) = \cap_{i,j=1}^{m,n}(x_i y_j) = \cap_i x_i(\cap(y_j)) = \cap x_i(\cap(b_\beta))$ (using $\cap(y_j) = \cap(b_\beta)$). This gives $P = \cap_i x_i(\cap(b_\beta)) = \cap_{i,\beta}(x_i b_\beta) = \cap_\beta b_\beta(\cap(x_i)) = \cap_\beta b_\beta(\cap(a_\alpha)) = \cap_{\alpha,\beta}(a_\alpha b_\beta)$. ■

Recall that an integral domain R is called v-coherent if every finite intersection of v-ideals of finite type is a v-ideal of finite type, equivalently, if A^{-1} is a v-ideal of finite for all finitely generated $A \in F(R)$ [11]. Also that R is a generalized greatest common divisor (GGCD) domain if $aR \cap bR$ is invertible for all $a, b \in R\backslash\{0\}$ [3]. According to Corollary 1.7 of [25], a v-coherent domain R is a GGCD domain if and only if R is a $*$-domain. Now, a DCD R is slightly more than a v-coherent domain.

Corollary 4 *Suppose that for a star operation $\star$, $A^\star$ is a v-ideal of finite type for each $A \in F(R)$. Then R is a $\star$ -G-Dedekind domain if and only if R is a $*$-domain. Consequently, a Mori (and hence a Krull) domain R is a locally factorial Krull domain if and only if R is a $*$-domain.*

The proof can be easily constructed from the preceding comments, and so is left to the reader.

Considered in [6] was also the notion of a pseudo principal ideal domain (pseudo PID) or, in our terminology, v-G-PID by requiring that A_v is principal, for each nonzero ideal A of R. This notion appeared in Bourbaki too, as pointed out in [6]. Using the DC approach, one can prove the following result.

Theorem 3.4 *A domain R is a v-G-PID if and only if R is a DC Schreier domain.*

Proof Suppose that R is a DC Schreier domain. Now, a Schreier domain is a $*$ -domain too [26, Corollary 1.7]. So, a DCD that is a Schreier domain is at least a v-G-Dedekind domain, by Theorem 3.3. But then, A_v is invertible for each nonzero ideal A of R, and in a Schreier domain, every invertible ideal is principal [26, Theorem 3.6]. Thus, A_v is principal for each nonzero ideal A of R, and R is a v -G-PID. Conversely, note that a v-G-PID is DC and is at least a greatest common divisor (GCD) domain, and a GCD domain is Schreier [9]. Thus, a v-G-PID R is a DC Schreier domain. ∎

Theorem 3.5 *Let $\star$ be a star operation defined on R such that for each $A \in F(R)$, $A^\star$ is a v-ideal of finite type. Then the following are equivalent.*

(1) R is a $\star$-G-PID.
(2) R is a Schreier domain.
(3) R is a $\star$-G-Dedekind domain with $Pic(R) = (0)$.

Proof (1) ⇔ (2). A $\star$-G-PID, by definition, is a GCD domain and so a Schreier domain. Conversely, "for each $A \in F(R)$, $A^\star$ is a v-ideal of finite type" makes each $A^\star \in F(R)$ a v-ideal of finite type. But then, for each $A \in F(D)$, $A^\star$ is a finite intersection of principal fractional ideals and of finite type and by Theorem 3.6 of [26] principal because R is Schreier.

(1) ⇔ (3). $\star$-GPID is $\star$-G-Dedekind with every invertible ideal principal which is exactly (3).

For the converse, note that (3) implies that R is at least a GCD domain and a GCD domain is Schreier. That is, (2) holds, and (2) is equivalent to (1). ∎

4 Related stuff

We end this article with some interesting characterizations of the $\star$ -GPIDs and a discussion of related material.

Recall that a Riesz group is a directed group that satisfies the Riesz interpolation property: Given that $x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n \in G$ such that $x_i \leq y_j$ for all $i \in [1, m], j \in [1, n]$, there is $z \in G$ such that $x_i \leq z \leq y_j$ for all $(i, j) \in [1, m] \times [1, n]$). It was shown in [26] that the Riesz interpolation property translates in the commutative ring theory set up to: For all $x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n \in K\backslash\{0\}$ with $x_1, x_2, \ldots, x_m \in \cap y_i R$, there is a $z \in \cap y_i R$ such that $(x_1, \ldots, x_m) \subseteq zR \subseteq \cap y_i R$. So, a pre-Schreier domain is actually a pre-Riesz domain. We have no interest in changing existing names; we only want to add a new name. Call R a super Riesz domain if, for any divisorial ideal A of R and for any set $\{x_\alpha\}$ of elements contained in A, with $\cap(x_\alpha) \neq (0)$, there is a $d \in A$ such that $(x_\alpha) \subseteq (d)$. Because a divisorial ideal

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is expressible as an intersection of principal fractional ideals, a super Riesz domain can be easily seen to be pre-Schreier. Furthermore, let us call a product AB of ideals A, B subtle if, for each $x \in AB$, we have $x = ab$ where $a \in A$ and $b \in B$. The authors of [6] also touched on the following question: Let R be an integral domain that satisfies $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$ for all subsets $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$. Is R Pseudo-Dedekind?

We try to give a partial answer below.

Proposition 1 *Let R be an integral domain. Then the following are equivalent.*

(1) For all allowable $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq K\backslash\{0\}$, we have $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$.
(2) For all allowable $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$, we have $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$, and for all allowable sets $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we have a $d \in R\backslash\{0\}$ such that $d(\cap(a_\alpha))$ is expressible as an intersection of principal integral ideals.
(3) R is a DCD that satisfies: For all allowable $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$, we have $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$.

Proof (1) ⇒ (2). Obvious because the first part follows directly and (1) means R is DC and so for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we can find $x_1, \ldots, x_n$ such that $\cap(a_\alpha) = \cap_{i=1}^n(x_i)$. But, for the right-hand side, we can find a nonzero d such that dx_i are all integral.

(2) ⇒ (1). Consider $(\cap(a_\alpha))(\cap(b_\beta))$ for all allowable $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq K\backslash\{0\}$. Since $\cap(a_\alpha), \cap(b_\beta)$ are nonzero fractional ideals by the given property of R, we have, for some $r, s \in R$, $r(\cap(a_\alpha)) = \cap(x_\gamma)$ and $s(\cap(b_\beta)) = \cap(y_\delta)$, where x_γ and y_δ are in R, by (2). But then, $(\cap(x_\gamma))(\cap(y_\delta)) = \cap(x_\gamma y_\delta) = P$. Now, $P = \cap(x_\gamma y_\delta) = \cap_\gamma x_\gamma(\cap(y_\delta))$ (substituting for $\cap(y_\delta)) = \cap_\gamma x_\gamma(\cap(sb_\beta)) = \cap_{\gamma,\beta}(x_\gamma sb_\beta) = \cap_\beta sb_\beta(\cap(x_\gamma))$ and substituting for $\cap(x_\gamma))$ we get $P = \cap_\beta sb_\beta(\cap(ra_\alpha)) = \cap(ra_\alpha sb_\beta) = rs(\cap(a_\alpha b_\beta))$. But, on the other hand, $P = (\cap(x_\gamma))(\cap(y_\delta)) = (r(\cap(a_\alpha)))(s(\cap(b_\beta)))$. Thus, $rs(\cap(a_\alpha))(\cap(b_\beta)) = rs(\cap(a_\alpha b_\beta))$. Canceling rs from both sides, we get the desired equality.

(1) ⇒ (3). Obvious, in light of (1) ⇒ (2).

(3) ⇒ (2). Note that because R is DC for each allowable set $\{a_\alpha\}_{\alpha\in I} \subseteq K\backslash\{0\}$, we can find $x_1, \ldots, x_n$ such that $(\cap(a_\alpha)) = (\cap_{i=1}^n(x_i))$ and so a $d \in R\backslash\{0\}$ such that $dx_i \in R$. But then, $d(\cap(a_\alpha)) = \cap_{i-1}^n dx_i$ an intersection of principal fractional ideals. ■

Recall from [26], again, that R is a pre-Schreier domain if and only if, for all $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\} \subseteq R\backslash\{0\}$, $a_i b_j | x$ implies $x = rs$ where $a_i | r$ and $b_j | s$. Since this property sprang in the context of pre-Schreier domains, we can call a domain R super pre-Schreier if $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$ such that $a_\alpha b_\beta | x$, then $x = rs$ such that $a_\alpha | r$ and $b_\beta | s$, and ask if a super pre-Schreier domain must be a v-GPID. The reason for this question is provided by the following proposition.

Proposition 2 *An integral domain R is super pre-Schreier if and only if, for all allowable* $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$, *we have* $(\cap(a_\alpha))(\cap(b_\beta)) = \cap(a_\alpha b_\beta)$ *such that for each* $x \in (\cap(a_\alpha))(\cap(b_\beta))$, $x = rs$ *where* $r \in \cap(a_\alpha)$ *and* $s \in \cap(b_\beta)$.

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Proof Suppose that R is super pre-Schreier. Then $(\cap(a_\alpha))(\cap(b_\beta)) \subseteq (\cap(a_\alpha b_\beta))$ holds, always. So let $x \in \cap(a_\alpha b_\beta)$. This means $a_\alpha b_\beta | x$. But super pre-Schreier property requires that $x = rs$ where $r \in \cap(a_\alpha)$ and $s \in \cap(b_\beta)$ and that puts $x \in (\cap(a_\alpha))(\cap(b_\beta))$ and the other requirement is met. The converse is a similar translation. ■

Call the product IJ of two nonzero integral ideals I, J of R subtle if each $d \in IJ\backslash\{0\}$ can be written as $d = rs$ where $r \in I$ and $s \in J$. It was shown in [26] that R is pre-Schreier if and only if R has the property $*$ and for each pair of subsets $\{a_1, \ldots, a_m\}, \{b_1, \ldots, b_n\} \subseteq R\backslash\{0\}$ the product $(\cap_{i=1}^{n}(a_i))\,(\cap_{i=1}^{n}(a_i))$ is subtle. (Following an earlier version of [26], Anderson and Dobbs [8] studied domains products of whose ideals were all subtle.)

Corollary 5 *The following are equivalent for an integral domain R. (1) For all allowable $\{a_\alpha\}_{\alpha\in I} \subseteq R\backslash\{0\}, \{b_\beta\}_{\beta\in J} \subseteq R\backslash\{0\}$, we have $(\cap(a_\alpha))(\cap(b_\beta)) = (\cap(a_\alpha b_\beta))$ such that for each $x \in (\cap(a_\alpha))(\cap(b_\beta))$, $x = rs$ where $r \in \cap(a_\alpha)$ and $s \in \cap(b_\beta)$, (2) for every pair of nonzero ideals A, B, we have $(AB)_v = A_v B_v$, and for every nonzero integral ideal A of R, there is a $d \in R\backslash\{0\}$ with dA_v an intersection of principal integral ideals and the product is subtle, and (3) R is a super pre-Schreier domain.*

Finally, a word about DCDs. Indeed, a DCD can be characterized by: A_v is a v-ideal of finite type for each $A \in F(R)$. The first thing that comes to mind as a general property is that A^{-1} is of finite type for each $A \in F(R)$. This leads to the following result. But we need to recall some terminology. R is $\star$-Prüfer if, for each finitely generated $A \in F(R)$, A is $\star$-invertible [4]. Since, for $A \in F(R)$, A being $\star$-invertible implies $A^\star = A_v$. So, for each finitely generated nonzero ideal J in a $\star$-Prüfer domain R we have $J^\star = J_v$. Furthermore, as $(JJ^{-1})^\star = R$ implies $((JJ^{-1})^\star)_v = (JJ^{-1})_v = R$, a $\star$-Prüfer domain is a v-domain and $\star_f = t$. If $\star$ is of finite character, then A^{-1} is a v-ideal of finite type for each finitely generated $A \in F(R)$ and so, if $\star$ is of finite type, a $\star$-Prüfer domain is a Prüfer $\star$-multiplication domain and, $\star = t$. $\star$-Prüfer domains, for a finite character $\star$-operation $\star$, were studied in [17] where they were called $\star$-multiplication domains. They were later called Prüfer $\star$-multiplication domains (P$\star$MDs). An in-depth study of these domains can be found in [12], along with an introduction to semistar operations. These days this concept is defined by: R is a P$\star$MDs, for a star operation $\star$, if, for each finitely generated $A \in F(R)$, A is $\star_f$-invertible. If R is a P$\star$ MD for a finite character star operation, then $\star = t$ over R. On the other hand, if R is a P$\star$MD for a "general" star-operation $\star$, then $\star_f = t$.

Proposition 3 *Let R be a DCD, and let $\star$ be a star operation defined on R. If A is $\star$-invertible, for $A \in F(R)$, then $A^\star$ is $\star_f$-invertible. Consequently a DCD R is a P$\star$ MD if and only if R is a $\star$-Prüfer domain.*

Proof Note that A being $\star_f$-invertible means that A^{-1} is a v-ideal of finite type. But A^{-1} is a v-ideal of finite type in a DCD. Thus, a DC $\star$-Prüfer domain is a P$\star$MD. The converse is obvious in the case of a DCD. ■

Remark 4.1 It may be noted that there is a marked difference between "A is $\star_f$-invertible" and "$A^\star$ is $\star_f$ -invertible." That is if you require that every $A \in F(R)$

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is $\star_f$ -invertible, you will end up with a Krull domain, for $\star_f$ -invertible is t-invertible and every $A \in F(R)$ being t-invertible implies that R is Krull [19, p. 82]. On the other hand, if you require that, for each $A \in F(R)$, the ideal $A^\star$ is $\star_f$ -invertible, you get a PVMD that acts and behaves very much like $\star$ -G-Dedekind domains. These domains were studied under the name of pre-Krull domains in [27], just for $\star = v$ and later, under the name of (t, v)-Dedekind domains, in [4], essentially based on the line taken in [27], describing the (t, v)-Dedekind domains as domains in which $(AB)^{-1} = (A^{-1}B^{-1})^t$, or as domains in which A_v is t -invertible for each $A \in F(R)$. The closest to that [4] came to was in its Theorem 1.14. These and similar concepts were also studied by Halter-Koch under "mixed invertibility." Hopefully, with the introduction of DC property, the situation will become clearer.

For now, we have the following corollary.

Corollary 6 *The following are equivalent for an integral domain R, with a star operation $\star$ defined on it.*

(1) $A^\star$ is $\star_f$-invertible for each $A \in F(R)$.
(2) In R, we have A is $\star$-invertible and $(AB)^\star = (A^\star B^\star)^{\star_f}$, for all $A, B \in F(R)$.
(3) R is a $\star$-DC $\star$-Prüfer domain.
(4) R is a $\star$-CICD and $(AB)^\star = (A^\star B^\star)^{\star_f}$, for all $A, B \in F(R)$.

Proof (1) ⇒ (2). Let $A^\star$ be $\star_f$-invertible, for all $A \in F(R)$. Then $A^\star$ and hence A are $\star$-invertible, forcing $A^\star = A_v$ [28], for all $A \in F(R)$. Thus, $\star = v$ over R. Moreover, as $\star_f$ is of finite character, $A^\star$ is a $\star$-ideal of finite type, because $A^\star$ is $\star_f$-invertible . Now, consider for $A, B \in R$ the product $(AB)^\star$. Indeed, $(AB)^\star = (A^\star B^\star)^\star$. Because each of $A^\star, B^\star$ is of finite type, $(AB)^\star = (A^\star B^\star)^{\star_f}$.

(2) ⇒ (1). $R = (AA^{-1})^\star = (A^\star A^{-1})^{\star_f}$, by (2).

(2) ⇒ (3). By (2), A is $\star$-invertible for every $A \in F(R)$ we conclude that R is a $\star$-CICD and hence $\star$-Prüfer. Since by (2) we have $(AB)^\star = (A^\star B^\star)^{\star_f}$, we conclude that $R = (AA^{-1})^\star = (A^\star A^{-1})^{\star_f}$ which established that $A^\star$ is a $\star$-ideal of finite type. But, as $\star = v$, we conclude that R is $\star$-DC.

(3) ⇒ (1). Follows from Proposition 3.

Finally, (4) and (2) are restatements of each other. ∎

The domain characterized by Corollary 6 is a completely integrally closed PVMD, that is a Dedekind domain for $\star = d$, a Krull domain for $\star$ of finite character, or for $\star = t$, and a pre-Krull domain of [27] for a star operation $\star$ that is not of finite type, or $\star = v$. The following proposition is a reason why I tend to call these domains $(\star, v)$-G-Dedekind domains. For this, let us recall that R is $\star$-DC if, for each $A \in F(R)$, we have that $A^\star$ is a v-ideal of finite type.

Proposition 4 *A $(\star, v)$-G-Dedekind domain R is a $\star$ -G-Dedekind domain if and only if R is a $\star$-domain.*

Proof A $(\star, v)$-G-Dedekind domain R is $\star$-DC, with $\star = v$, and so DC, whereas a DC $\star$-domain is a indeed a $\star$-G-Dedekind domain, by Theorem 3.3. Conversely, a

$\star$-G-Dedekind domain is a $\star$ -domain and DC. This means that it is $\star$-DC, and as every $\star$ -ideal in a $\star$-G-Dedekind domain is invertible, we conclude that a $\star$-G-Dedekind domain is a $\star$-domain that is a $(\star, v)$ -G-Dedekind domain for every finite type star operation $\star$. ■

It may be noted that although $(\star, v)$-G-Dedekind domains have been studied before, Corollary 6 and Proposition 4 provide a better view and highlight the connection of v-G-Dedekind domains with their generalization, the $(\star, v)$-G-Dedekind domains. Now let's take a look at some general properties of DCDs.

It was indicated in [25], using the example of the ring of entire functions, a v-G-Dedekind domain does not behave well under localizations. Using the same example, we can conclude that a DCD does not behave well under localization. But let us be clear about that example. Let $\mathcal{E}$ denote the ring of entire functions. Using the fact that $\mathcal{E}$ is a Bezout domain and that an element s of $\mathcal{E}$ that is not divisible by any principal (height one) prime (i.e., one that does not have a zero) is a unit, one can show that $\mathcal{E} = \cap \mathcal{E}_{(p)}$ where p ranges over principal height one primes. In fact, we have the following lemma.

Lemma 4.2 *Let R be a GCD domain such that an element $s \in R\backslash\{0\}$ is a nonunit if and only if s is divisible by at least one principal prime of R. Then $R = \cap R_{(p)}$ where p ranges over principal height one primes of R. Such a ring is completely integrally closed.*

Proof Obviously, if R has no principal height, one primes then R is a field and R is trivially an intersection of localizations at its height one primes. So let us assume that R does have a set $\wp$ of principal height one primes and set $S = \cap_{p \in \wp} R_{(p)}$. We already have $R \subseteq \cap R_{(p)}$. If there is $x \in S\backslash R$, then $x = r/s$, and we can assume that $GCD(r, s) = 1$. Now, as $pR_{(p)}$ is of height one and R is a GCD domain, $R_{(p)}$ is a discrete rank one valuation domain. Next, $r/s \in R_{(p)}$ for $p \in \wp$ forces s to be a unit in $R_{(p)}$ meaning s is not divisible by p. But then, $x = r/s$ where s is not divisible by any of the $p \in \wp$. This forces s to be a unit, by the rule. But then, x is an associate of $r \in R$. Thus, $R = \cap R_{(p)}$. Finally, each $R_{(p)}$ for each $p \in \wp$ is a discrete rank one valuation domain and hence completely integrally closed, and an intersection of CICDs is completely integrally closed. ■

Thus, we see the reason why one concludes that $\mathcal{E}$ is completely integrally closed. Now, $\mathcal{E}$ is Bezout and hence a PVMD such that every nonzero ideal of $\mathcal{E}$ is a t-ideal. Following [16], let P be a nonzero nonmaximal prime t-ideal of a PVMD R and set $S = \cap R_M$ where M ranges over all the maximal t-ideals which do not contain P. Then $P^{-1} = R_P \cap S$ [16, Proposition 1.1]. (Houston had this to say about mentioning [16, Proposition 1.1]: You are applying it to a Prüfer domain, so Theorem 3.2 of the Huckaba–Papick paper [18], on which my paper is based, probably should take precedence. I would go even further. You can give a very simple proof as follows: Let P be a prime ideal of height > 1 [maximal or not]. Let $u \in P^{-1}$. Then $P \subseteq (\mathcal{E} : u)$. If $(\mathcal{E} : u)$ is not all of $\mathcal{E}$, then it is a proper principal ideal [since $\mathcal{E}$ is Bezout] and must be contained in some principal prime. But this puts P inside a principal prime, a contradiction.) The reader may note the reference and the simple fact. As far as I am concerned, given any prime ideal P of $\mathcal{E}$ of height greater than one, we must have P_v

principal, as noted on page 292 of [25]. Now, if $P_v = d\mathcal{E} \neq \mathcal{E}$, P must be contained in a principal prime. But, in $\mathcal{E}$, every principal prime is of height one. Thus, whatever the reference and whatever the reason, the following result stands.

Proposition 5 *Let P be a nonzero prime of $\mathcal{E}$ of height greater than one. Then $P_v = \mathcal{E}$.*

Proof Note that P is contained in no height one prime of $\mathcal{E}$. But then, principal primes are maximal (t-) ideals in $\mathcal{E}$ and so by Lemma 4.2, $S = \mathcal{E}$, forcing $P^{-1} = \mathcal{E}_P \cap \mathcal{E} = \mathcal{E}$. Indeed, then $P_v = \mathcal{E}$. Finally, if M is a maximal ideal of $\mathcal{E}$ of height greater than one, then M must contain a nonzero nonmaximal prime $\wp$ of height greater than one. But then, $\mathcal{E} = (\wp)_v \subseteq M_v$. ■

Corollary 7 *In $\mathcal{E}$ it is possible to have a multiplicative set S and an ideal A such that $(A\mathcal{E}_S)_{v'} \subsetneq (A_{v_\mathcal{E}}\mathcal{E}_S)_{v'}$ where v' and $v_\mathcal{E}$ are the v-operations in $\mathcal{E}_S$ and $\mathcal{E}$, respectively.*

Proof Let P be a nonzero nonmaximal prime ideal of $\mathcal{E}$ of height greater than one contained in a maximal ideal M of $\mathcal{E}$. Then, by Proposition 5, $P_{v_\mathcal{E}} = \mathcal{E}$, where $v_\mathcal{E}$ denotes the v-operation on $\mathcal{E}$. On the other hand, in $\mathcal{E}_M$, which is a valuation ring, we have $P\mathcal{E}_M$ a nonmaximal prime of $\mathcal{E}_M$ and so must be divisorial being the intersection of all the principal ideals containing $P\mathcal{E}_M$. Thus, if v' is the v-operation on $\mathcal{E}_M$, then $(P\mathcal{E}_M)_{v'} = P\mathcal{E}_M \subsetneq \mathcal{E}_M = (P_{v_\mathcal{E}}E_M)_{v'}$. ■

We now establish that it is the DC property going missing in localization that is responsible for Corollary 7. For this, note that a DCD that is also a $\star$-domain is a v-G-Dedekind domain. Now, the $\star$-property, as indicated in [26], fares nicely under localization, and so if the v-G-Dedekind property goes missing in localizing, it is the DC property that goes missing in localizing. However, the situation can be brought under control, once we introduce some restriction.

Proposition 6 *If R is DC such that for each ideal $A \in F(R)$ there is a finitely generated $B \subseteq A$ with $A_v = B_v$, then for each multiplicative set S, the ring R_S is DC.*

Proof Note that a domain R with the given property is DC because for every ideal $A \in F(R)$ we have a finitely generated ideal $B \subseteq A$ with $A_v = B_v$, thus making A_v of finite type, for each $A \in F(R)$. On the other hand, a domain with the given property is known to be a Mori domain ([27] and the references therein) and a Mori domain stays a Mori domain under localization. We include the proof below.

Let S be a multiplicative set in R such that for each $A \in F(R)$ there is a finitely generated $B \subseteq A$ with $A_v = B_v$, and let α be an ideal of R_S. Then $\alpha = AR_S$ where $A = \alpha \cap R$. Let $B \subseteq A$ where B is finitely generated such that $A_v = B_v$. Note that $A \subseteq B_v$ and so $\alpha = AR_S \subseteq B_vR_S$. Thus, $\alpha_v = (AR_S)_v \subseteq (B_vR_S)_v = (BR_S)_v$, since B is finitely generated [24]. Since $B \subseteq A$, we have $BR_S \subseteq AR_S$ and so $(BR_S)_v \subseteq (AR_S)_v$. Thus, $(BR_S)_v = (AR_S)_v$ with BR_S finitely generated and contained in AR_S. ■

Because, in a DCD R, A_v is a finite intersection of principal fractional ideals, we conclude that A_vR_S is divisorial. Thus, $(AR_S)_v \subseteq A_vR_S$. However, as Corollary 7 indicates, the inclusion may be strict on occasion. Now, generally, if $A \in F(R)$ is

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finitely generated and S is a multiplicative set of R, then $(AR_S)_{v'} = (A_{v_R}R_S)_{v'}$, where v' and v_R are v-operations on R_S and R, respectively [24]. This is a general formula, and so it works for DCDs too, but in a slightly modified form.

Proposition 7 *If $A \in F(R)$ is nonzero finitely generated, S is a multiplicative set of R, and R is DC, then $(AR_S)_{v'} = A_{v_R}R_S$, where v' and v_R are v-operations on R_S and R, respectively.*

Proof The proof follows from the fact that A_{v_R} is a finite intersection of principal fractional ideals, and so $A_{v_R}R_S$ is a divisorial ideal of R_S and that makes $A_{v_R}R_S = (A_{v_R}R_S)_{v'}$. ∎

A prime example of a DCD is a Mori domain, and Roitman [22] has produced an example of a Mori domain R such that $R[X]$ is not Mori. On the other hand, for integrally closed integral domains, Querre [21] proved the following result.

Theorem 4.3 *An integral domain R is integrally closed if and only if, for every integral ideal B of $R[X]$ with $B \cap R \neq 0$, we have $B_v = (A_B[X])_v = (A_B)_v[X]$ where A_B is the ideal of R generated by the coefficients of elements of B.*

For a clearer treatment of Theorem 4.3, see Section 3 of [7]. For now, we use this theorem to prove the following result.

Theorem 4.4 *Let R be an integrally closed integral domain. The polynomial ring $R[X]$ is DC if and only if R is.*

Proof Suppose that R is DC. Then, for every ideal $I \in F(R)$ of R, we have $I_v = J_v$ where J is finitely generated. Now, let $H \in F(R[X])$. Because R is integrally closed, according to Theorem 2.1 of [7], $H = \frac{f(X)}{g(X)}B$ where $f(X), g(X) \in R[X]$ and B is an ideal of $R[X]$ with $B \cap R \neq (0)$. But then, $H_v = \frac{f(X)}{g(X)}B_v = \frac{f(X)}{g(X)}((A_B)_v[X])$ is a v-ideal of finite type because, as A_B is an integral ideal of R, $(A_B)_v$ is a v-ideal of finite type. Conversely, let I be a nonzero integral ideal of R, and suppose that $R[X]$ is DC. Then $(I[X])_v = I_v[X]$ is a v-ideal of finite type, and this forces I_v to be a v-ideal of finite type. ∎

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