SOLUTIONS OF THE DIFFUSION EQUATION FOR A MEDIUM GENERATING HEAT

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1. *Introduction.* The problem of solving the equation of thermal conduction for cases in which heat is generated in the interior of the medium under consideration arises frequently in physics and engineering. It occurs, for instance, when we consider the diffusion of heat in a solid undergoing radioactive decay (**1**) or which is absorbing radiation (**2**). Complications of a similar nature arise when there is a generation or absorption of heat in the solid as a result of a chemical change—for example, the hydration of cement (**3**). The particular case in which the rate of generation of heat is independent of the temperature arises in the theory of the ripening of apples and has been discussed by Awberry (**4**).

In many cases of practical interest the rate of generation of heat may be taken to be a linear function of the temperature. Explicit solutions of a very general kind have been stated by Paterson (**5**) for this case with the additional assumption that the solid has constant thermal conductivity. No proofs are given by Paterson but he indicates that the results may be obtained by the method of sources, an account of which has resently been given by Carslaw and Jaeger (**6**). The object of this note is to show how the solutions may be derived by the use of the theory of Fourier transforms.

2. *The Differential Equation.* The equation governing the variation with time of the temperature, θ, in a homogeneous isotropic solid within which heat is absorbed or generated is

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + \Theta(\mathbf{r}, \theta, t), \quad \text{.........(1)}$$

where κ denotes the thermal conductivity, here assumed constant, and Θ is a known function of position, of the temperature θ, and the time t. We assume that the function Θ is of the form

$$\Theta(\mathbf{r}, \theta, t) = \phi(\mathbf{r}, t) + \theta\psi(t), \quad \text{.........(2)}$$

where $\phi(\mathbf{r}, t)$ is a function of the coordinates and the time and $\psi(t)$ is a function of the time only. If we now substitute from equation (2) into equation (1) and make the transformations

$$u = \theta \exp\left\{-\int_0^t \psi(\tau)\,d\tau\right\}, \qquad \chi(\mathbf{r}, t) = \phi(\mathbf{r}, t) \exp\left\{-\int_0^t \psi(\tau)\,d\tau\right\}, \quad \text{.........(3)}$$

we find that equation (1) assumes the form

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \chi(\mathbf{r}, t). \quad \text{.........(4)}$$

We suppose that the initial distribution of temperature is known that is, that $\theta = \theta_0(\mathbf{r})$ when $t = 0$; the initial condition on the function u is therefore $u = \theta_0(\mathbf{r})$.

3. *Infinite solid.* To solve equation (4) subject to the initial condition $u = \theta_0(\mathrm{r})$, we reduce it to an ordinary differential equation by the introduction of the Fourier transform of the function $u(r, t)$ defined by the equation

$$U(\xi, \eta, \zeta, t) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z, t) e^{i(x\xi + y\eta + z\zeta)}\, dx\, dy\, dz. \quad \text{.........(5)}$$

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Multiplying both sides of equation (4) by $\exp\{i(\xi x+\eta y+\zeta z)\}$ and integrating with respect to x, y and z over the entire range $-\infty \leqslant x, y, z \leqslant \infty$, we have after a little reduction

$$\frac{dU}{dt}+\kappa(\xi^2+\eta^2+\zeta^2)U=X(\xi,\eta,\zeta,t), \quad\ldots\ldots(6)$$

where $X(\xi,\eta,\zeta;t)$ is the Fourier transform of $\chi(x,y,z,t)$.

Since the initial condition on u is known, it follows that at $t=0$,

$$U=U_0(\xi,\eta,\zeta)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_0(x,y,z)e^{i(\xi x+\eta y+\zeta z)}\,dx\,dy\,dz,$$

so that the solution of equation (6) is

$$U=U_0(\xi,\eta,\zeta)e^{-\kappa t^2(\xi+\eta^2+\zeta^2)}+\int_0^t X(\xi,\eta,\zeta;\tau)e^{-\kappa(t-\tau)(\xi^2+\eta^2+\zeta^2)}\,dt.$$

Inverting this result by means of the formula (7)

$$u(x,y,z)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(\xi,\eta,\zeta;t)e^{-i(\xi x+\eta y+\zeta z)}\,d\xi\,d\eta\,d\zeta, \quad\ldots\ldots(7)$$

we have

$$u(x,y,z;t)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U_0(\xi,\eta,\zeta)e^{-\kappa t(\xi^2+\eta^2+\zeta^2)-i(\xi x+\eta y+\zeta z)}\,d\xi\,d\eta\,d\zeta$$
$$+\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^t X(\xi,\eta,\zeta;\tau)e^{-\kappa(t-\tau)(\xi^2+\eta^2+\zeta^2)-i(\xi x+\eta y+\zeta z)}\,d\xi\,d\eta\,d\zeta\,d\tau. \ldots(8)$$

Now the function

$$G(\xi,\eta,\zeta)=e^{-\kappa t(\xi^2+\eta^2+\zeta^2)}$$

is the Fourier transform of the function

$$g(x,y,z)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi x+\eta y+\zeta z)-\kappa t(\xi^2+\eta^2+\zeta^2)}\,d\xi\,d\eta\,d\zeta$$
$$=\left(\frac{1}{2\kappa t}\right)^{\frac{3}{2}}e^{-(x^2+y^2+z^2)/4\kappa t}.$$

Substituting this result in the resultant theorem (7)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(\xi,\eta,\zeta)G(\xi,\eta,\zeta)e^{-i(\xi x+\eta y+\zeta z)}\,d\xi\,d\eta\,d\zeta$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\alpha,\beta\,\gamma)g(x+\alpha,y-\beta,z-\gamma)\,d\alpha\,d\beta\,d\gamma, \quad\ldots\ldots(9)$$

which relates the functions $u(x,y,z)$, $g(x,y,z)$ to their Fourier transforms $U(\xi,\eta,\zeta)$ and $G(\xi,\eta,\zeta)$, we find that the first integral on the right-hand side of equation (8) has the value

$$\left(\frac{1}{4\pi\kappa t}\right)^{\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_0(\alpha,\beta,\gamma)\exp\left\{-\frac{\Sigma(\alpha-x)^2}{4\kappa t}\right\}d\alpha\,d\beta\,d\gamma.$$

The second integral can be evaluated in an exactly similar way. Performing this integration and reverting to the original variable defined by the equations (3) we have finally

$$\theta(x,y,z;t)=\left(\frac{1}{4\pi\kappa t}\right)^{\frac{3}{2}}\exp\left(\int_0^t\psi(\tau)\,d\tau\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\theta_0(\alpha,\beta,\gamma)\exp\left\{-\frac{\Sigma(\alpha-x)^2}{4\kappa t}\right\}d\alpha\,d\beta\,d\gamma$$
$$+\left(\frac{1}{4\pi\kappa}\right)^{\frac{3}{2}}\exp\left(\int_0^t\psi(\tau)\,d\tau\right)\int_0^t\frac{e^{-\int_0^t\psi(\tau)\,d\tau}}{(t-\tau)^{\frac{3}{2}}}\,d\tau\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\alpha,\beta,\gamma,\tau)\exp\left\{-\frac{\Sigma(\alpha-x)^2}{4\kappa\tau}\right\}d\alpha\,d\beta\,d\gamma. \quad\ldots\ldots(10)$$

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If we take the functions ϕ and ψ to be identically zero we get the solution corresponding to the case in which there is no generation of heat in the medium. Substituting $\phi \equiv \psi \equiv 0$,

$$(\xi, \eta, \zeta) = (4\kappa t)^{-\frac{1}{2}} (\alpha - x, \beta - y, \gamma - z)$$

in equation (10) we get Fourier's well-known solution (**8**),

$$\theta = \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_0\{x + 2\xi\surd(\kappa t),\, y + 2\eta\surd(\kappa t),\, z + 2\zeta\surd(\kappa t)\} e^{-(\xi^2+\eta^2+\zeta^2)} d\xi\, d\eta\, d\zeta. \quad \text{...(11)}$$

The general formula (10) was obtained by Paterson (**5**) by the method of sources.

The solution of the partial differential equation

$$\frac{\partial \theta}{\partial t} = \kappa \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) + \phi(x, y\,;\, t) + \psi(t)\theta,$$

subject to the initial condition $u = \theta_0(x, y)$ is similarly found to be

$$\theta = \frac{\exp\left\{\int_0^t \psi(\tau)\, d\tau\right\}}{4\pi\kappa t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_0(\alpha, \beta) \exp\left\{\frac{-(\alpha - x)^2 - (\beta - y)^2}{4\kappa t}\right\} d\alpha\, d\beta$$
$$+ \frac{\exp\left\{\int_0^t \psi(\tau)\, d\tau\right\}}{4\pi\kappa} \int_0^t \frac{e^{-\int_0^\tau \psi(\lambda) d\lambda}}{t - \tau} d\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\alpha, \beta\,;\, \tau) \exp\left\{\frac{-(\alpha - x)^2 - (\beta - y)^2}{4\kappa(t - \tau)}\right\} d\alpha\, d\beta. \quad \text{..(12)}$$

The solution for the case in which there is axial symmetry about the z-axis so that the functions θ, θ_0, and ϕ are functions of ρ and t only ($\rho^2 = x^2 + y^2$) can be deduced readily from this last equation. If the initial value of θ is

$$\theta_0(x, y) \equiv \theta_0(\rho),$$

we obtain the expression for the temperature at a subsequent time t by substituting the values

$$x = \rho, \quad y = 0, \quad \alpha = \varpi \cos\phi', \quad \beta = \varpi \sin\phi', \quad \theta_0(\alpha, \beta) = \theta_0(\varpi), \quad \phi(\alpha, \beta, \tau) = \phi(\varpi, \tau)$$

into equation (12). Making use of the result (**9**)

$$\int_0^{2\pi} e^{\mu \cos\phi'} d\phi' = 2\pi I_0(\mu),$$

we then have

$$\theta = \frac{\exp\left\{\int_0^t \psi(\tau)\, d\tau - \frac{\rho^2}{4\kappa t}\right\}}{4\pi\kappa t} \int_0^\infty \varpi\theta_0(\varpi) e^{-\varpi^2/4\kappa t} I_0(\varpi\rho/2\kappa t)\, d\varpi$$
$$+ \frac{\exp\left\{\int_0^t \psi(\tau)\, d\tau\right\}}{4\pi\kappa} \int_0^t \frac{\exp\left\{-\frac{\rho^2}{4\kappa(t-\tau)} - \int_0^\tau \psi(\lambda)\, d\lambda\right\}}{t - \tau} d\tau \int_0^\infty \varpi\phi(\varpi, \tau) e^{-\varpi^2/4\kappa(t-\tau)} I_0(\varpi\rho/2\kappa(t - \tau)\, d\varpi. \quad \text{(13)}$$

It is of interest to note that this result can also be established directly from the equation of axially symmetrical radial flow

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} \right) + \chi(\rho, t), \quad \text{.....(14)}$$

by means of the Hankel transform. If we denote by $\bar{u}(\xi, t)$ the Hankel transform of $u(\rho, t)$ so that

$$\bar{u}(\xi, t) = \int_0^\infty \rho u(\rho, t) J_0(\xi\rho)\, d\rho, \quad \text{.....(15)}$$

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then, as a result of a pair of integrations by parts,

$$\int_0^\infty \rho\left(\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}\right)J_0(\xi\rho)\,d\rho=-\xi^2\bar{u}(\xi,t). \quad\ldots\ldots(16)$$

Multiplying equation (14) throughout by $\rho J_0(\xi\rho)$ and integrating with respect to ρ over the range $(0, \infty)$ we find that $\bar{u}(\xi, t)$ is determined by the solution of the ordinary differential equation

$$\frac{d\bar{u}}{dt}+\kappa\xi^2\bar{u}=\bar{\chi}(\xi,t), \quad\ldots\ldots(17)$$

where $\bar{\chi}(\xi, t)$ denotes the Hankel transform of the function $\chi(\rho, t)$. The solution of equation (17) is similar to that of equation (6), namely,

$$\bar{u}=\bar{\theta}_0(\xi)\,e^{-\kappa\xi^2 t}+\int_0^t \bar{\chi}(\xi,\tau)\,e^{-\kappa\xi^2(t-\tau)}\,d\tau. \quad\ldots\ldots(18)$$

The expression for $u(\rho, t)$ is now found from this equation by using the Hankel inversion theorem (**10**)

$$u(\rho,t)=\int_0^\infty \xi\bar{u}(\xi,t)J_0(\xi\rho)\,d\xi. \quad\ldots\ldots(19)$$

In this way we get two terms for $u(\rho, t)$ the first of which is

$$\int_0^\infty \xi J_0(\xi\rho)\,d\xi\int_0^\infty \eta\theta_0(\eta)J_0(\eta\xi)\,e^{-\kappa\xi^2 t}\,d\eta,$$

which on interchange of the order of the integrations becomes

$$\int_0^\infty \eta\theta_0(\eta)\,d\eta\int_0^\infty \xi J_0(\xi\rho)J_0(\xi\eta)\,e^{-\kappa t\xi^2}\,d\xi.$$

The inner integral may be evaluated by Weber's formula (**11**)

$$\int_0^\infty \xi J_0(\rho\xi)J_0(\eta\xi)\,e^{-\kappa t\xi^2}\,d\xi=\frac{1}{2\kappa t}\exp\left(-\frac{\rho^2+\eta^2}{4\kappa t}\right)I_0\left(\frac{\rho\eta}{2\kappa t}\right)$$

to give

$$\frac{1}{2\kappa t}\,e^{-\rho^2/4\kappa t}\int_0^\infty \eta\theta_0(\eta)\,e^{-\eta^2/4\kappa t}\,I_0(\rho\eta/2\kappa t)\,d\eta$$

for the value of the first term. In a similar fashion we find

$$\int_0^t \frac{\exp\{-\rho^2/4\kappa(t-\tau)\}}{2\kappa(t-\tau)}\,d\tau\int_0^\infty \eta\chi(\eta,\tau)\exp\{-\eta^2/4\kappa(t-\tau)\}I_0(\rho\eta/2\kappa\overline{t-\tau})\,d\eta$$

for the second term. Substituting for the original variables from equation (3) we recover the formula (13). The difficulties involved in the changes of the order of the integrations can be overcome by using the analogue for Hankel transforms (**12**) of the resultant theorem.

4. *Conduction in a semi-infinite solid.* We now consider the conduction of heat in a medium which is bounded by the plane $x=0$ but is otherwise of unlimited extent. The medium which is taken to be isotropic and homogeneous will be supposed to occupy the half-space $x\geqslant 0$. Since the range of variation of x is now restricted to $(0, \infty)$ it is no longer permissible to employ the exponential form of the Fourier transform.

(i) *No surface loss.* The problem here is to solve the partial differential equation (1)—with $\Theta(\mathbf{r}, \theta, t)$ given by equation (2)—subject to the initial condition $\theta=\theta_0(x, y, z)$ at $t=0$ and the boundary condition $\partial\theta/\partial x=0$ at $x=0$. This is then reduced to the solution of (4) with

$$u=\theta_0(x,y,z) \quad \text{at } t=0$$

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and

$$\frac{\partial u}{\partial x} = 0, \qquad x = 0.$$

To solve this boundary value problem let

$$U(x, \eta, \zeta, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z, t) e^{i(\eta y + \zeta z)} dy\, dz\,; \quad \text{.......................(20)}$$

then equation (4) reduces to

$$\frac{\partial U}{\partial t} = \kappa \left(\frac{\partial^2}{\partial x^2} - \eta^2 - \zeta^2 \right) U + X(x, \eta, \zeta, t) \quad \text{.............................(21)}$$

subject to the conditions

$$U = \Theta_0(x, \eta, \zeta),\ t = 0\,; \quad \frac{\partial U}{\partial x} = 0, \quad x = 0. \quad \text{...........................(22)}$$

If we now introduce the Fourier cosine transform defined by

$$\bar{U}(\xi, \eta, \zeta, t) = \int_0^{\infty} U(x, \eta, \zeta, t) \cos x\xi\, dx,$$

then

$$\int_0^{\infty} \frac{\partial^2 U}{\partial x^2} \cos \xi x\, dx = \left[\frac{\partial U}{\partial x} \cos x\xi + \xi U \sin x\xi \right]_0^{\infty} - \xi^2 \bar{U}.$$

The expression in the square bracket vanishes because of the boundary conditions. We then obtain as before

$$\bar{U} = \bar{U}_0(\xi, \eta, \zeta, t) e^{-\kappa t(\xi^2+\eta^2+\zeta^2)} + \int_0^t \bar{X}(\xi, \eta, \zeta, \tau) e^{-\kappa(t-\tau)(\xi^2+\eta^2+\zeta^2)}\, d\tau,$$

so that

$$U(x, \eta, \zeta, t) = \frac{2}{\pi} e^{-(\eta^2+\zeta^2)\kappa t} \int_0^{\infty} \Theta_0(\alpha, \eta, \zeta, t) d\alpha \int_0^{\infty} e^{-\kappa t \xi^2} \cos \alpha\xi \cos x\xi\, d\xi$$

$$+ \frac{2}{\pi} \int_0^t e^{-\kappa(t-\tau)(\eta^2+\zeta^2)} d\tau \int_0^{\infty} X(\alpha, \eta, \zeta, \tau) d\alpha \int_0^{\infty} e^{-\kappa(t-\tau)\xi^2} \cos \alpha\xi \cos x\xi\, d\xi. \quad \text{......(23)}$$

Using the result (**13**),

$$\frac{1}{\pi} \int_0^{\infty} e^{-\kappa t \xi^2} \cos \lambda\xi\, d\xi = \frac{1}{2\sqrt{(\pi\kappa t)}} e^{-\lambda^2/4\kappa t}, \quad \text{..............................(24)}$$

inverting by the rule for two-dimensional exponential transforms and making use of the corresponding resultant theorem we have on returning to the original variables

$$\theta = \frac{e^{\int_0^t \psi(\tau) d\tau}}{(4\pi\kappa)^{\frac{3}{2}}} \left\{ \int_0^t \frac{e^{-\int_0^t \psi(\tau) d\tau}}{(t-\tau)^{\frac{3}{2}}} d\tau \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\alpha, \beta, \gamma, \tau) e^{-\frac{\Sigma(x-\alpha)^2}{4\kappa(t-\tau)}} [1 + e^{-\alpha x/\kappa(-\tau)}] d\alpha\, d\beta\, d\gamma \right.$$

$$\left. + \frac{1}{t^{\frac{3}{2}}} \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta_0(\alpha, \beta, \gamma) e^{-\frac{\Sigma(x-\alpha)^2}{4\kappa t}} [1 + e^{-\alpha x/\kappa t}] d\alpha\, d\beta\, d\gamma \right\} \quad \text{.........(25)}$$

in agreement with Paterson's result.

If we substitute $\psi \equiv \phi \equiv 0$ in equation (25) and take for θ_0 the function

$$\theta_0(x, y, z) = \delta(x-a)\, \delta(y)\, \delta(z),$$

where, as before, the δ's are Dirac delta-functions, we obtain the elementary solution

$$\theta = \frac{e^{-(y^2+z^2)/4\kappa t}}{(4\pi\kappa t)^{\frac{3}{2}}} [e^{-(a-x)^2/4\kappa t} + e^{-(a+x)^2/4\kappa t}].$$

(ii) *Prescribed surface temperature.* When the temperature is prescribed over the surface $x=0$, the problem is that of solving equation (1) subject to the initial condition $\theta=\theta_0(x, y, z)$ at $t=0$ and the boundary condition $\theta=\theta_0(y, z, t)$ when $x=0$. The corresponding conditions on u are

$$u=\theta_0(x, y, z), \quad t=0\,; \; u=\theta_2(y, z, t)\equiv\theta_1(y, z, t)e^{-\int_0^t \psi(\tau)\,d\tau}, \quad x=0.$$

Proceeding as in the case of no surface loss we have to solve the equation (21) subject now to the conditions

$$U=\Theta_0(x, \eta, \zeta), \quad t=0\,; \; \Theta=\Theta_2(\eta, \zeta, t), \quad x=0.$$

If we now introduce the Fourier sine transform

$$\bar{U}(\xi, \eta, \zeta, t)=\int_0^\infty U(x, \eta, \zeta, t)\sin x\xi\,dx$$

and make use of the result

$$\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin x\xi\,dx=\xi\Theta_2(\eta, \zeta, t)-\xi^2\bar{U}(\xi, \eta, \zeta, t),$$

we find that equation (23) is replaced by

$$\begin{aligned}U(x, \eta, \zeta, t)=&\frac{2}{\pi}e^{-(\eta^2+\zeta^2)\kappa t}\int_0^\infty \Theta_0(\alpha, \eta, \zeta, t)\,d\alpha\int_0^\infty e^{-\kappa t\xi^2}\sin\alpha\xi\sin x\xi\,d\xi\\ &+\frac{2}{\pi}\int_0^t \Theta_2(\eta, \zeta, \tau)e^{-\kappa(t-\tau)(\eta^2+\zeta^2)}\,d\tau\int_0^\infty \xi e^{-\kappa(t-\tau)\xi^2}\sin x\xi\,d\xi\\ &+\frac{2}{\pi}\int_0^t e^{-\kappa(t-\tau)(\eta^2+\zeta^2)}\,d\tau\int_0^\infty X(\alpha, \eta, \zeta, \tau)\,d\tau\int_0^\infty e^{-\kappa(t-\tau)\xi^2}\cos\alpha\xi\cos x\xi\,d\xi.\end{aligned}$$

Using the result (24) and

$$\frac{1}{\pi}\int_0^\infty \xi e^{-\kappa t\xi^2}\sin x\xi\,d\xi=\frac{x}{4\pi^{\frac{1}{2}}(\kappa t)^{\frac{3}{2}}}e^{-x^2/4\kappa t},$$

inverting by means of the exponential form of the Fourier transform theorem and proceeding as before, we obtain finally

$$\begin{aligned}\theta=\frac{e^{\int_0^t g(\tau)\,d\tau}}{(4\pi\kappa)^{\frac{3}{2}}}\Bigg\{&\frac{1}{t^{\frac{3}{2}}}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \theta_0(\alpha, \beta, \gamma)e^{-\frac{\Sigma(x-\alpha)^2}{4\kappa t}}[1-e^{-\alpha x/\kappa t}]\,d\alpha\,d\beta\,d\gamma\\ &+x\int_0^t \frac{e^{-\int_0^\tau g(\lambda)\,d\lambda}}{(t-\tau)}\,d\tau\int_{-\infty}^\infty\int_{-\infty}^\infty \theta_1(\beta, \gamma, \tau)e^{-\{x^2+(y-\beta)^2+(z-\gamma)^2\}/4\kappa(t-\tau)}\,d\beta\,d\gamma\\ &+\int_0^t \frac{e^{-\int_0^\tau g(\lambda)\,d\lambda}}{(t-\tau)^{\frac{3}{2}}}\,d\tau\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \phi(\alpha, \beta, \gamma, \tau)e^{-\frac{\Sigma(x-\alpha)^2}{4\kappa(t-\tau)}}[1-e^{-\alpha x/\kappa(t-\tau)}]\,d\alpha\,d\beta\,d\gamma\Bigg\}.\end{aligned}$$

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