*LMS J. Comput. Math.* 18 (1) (2015) 667–674 © 2015 Author
doi:10.1112/S1461157015000248

# Every $\mathrm{PSL}_2(13)$ in the Monster contains $13A$-elements

Robert A. Wilson

Abstract

We prove the assertion in the title by conducting an exhaustive computational search for subgroups isomorphic to $\mathrm{PSL}_2(13)$ and containing elements in class $13B$.

Supplementary materials are available with this article.

## 1. *Introduction*

The Monster is the largest of the 26 sporadic simple groups, and the only one whose maximal subgroups are not yet completely classified. Much work has however been done on this problem over the years (see for example [**3**, **5**–**8**, **10**]), but a few obstinate cases remain. One of these is the problem of classifying subgroups isomorphic to $\mathrm{PSL}_2(13)$, whose normalizers might be maximal. This case was considered in unpublished work by Holmes, but apparently not completed.

In [**6**] some subgroups isomorphic to $\mathrm{PSL}_2(13)$ are described. Notation here and later generally follows the Atlas [**2**], with one or two modifications following [**9**].

Theorem 1 (Norton). *The following two conjugacy classes of subgroups isomorphic to* $\mathrm{PSL}_2(13)$ *exist in the Monster.*

(1) *One with centralizer* $3^{1+2}.2^2$, *so its normalizer lies in* $(3^{1+2}.2^2 \times G_2(3)).2$. *Such a* $\mathrm{PSL}_2(13)$ *has type* $(2B, 3B, 7A, 13A)$.
(2) *One of type* $(2B, 3B, 7B, 13A)$ *with centralizer of order* 3 *and normalizer contained in* $3{\cdot}\mathrm{F}i_{24}$.

In particular, no subgroup isomorphic to $\mathrm{PSL}_2(13)$ and containing $13B$-elements is known. Our main result is that, in fact, no such subgroup exists.

Theorem 2. *There is no subgroup of the Monster which is isomorphic to* $\mathrm{PSL}_2(13)$ *and contains* $13B$*-elements.*

Theoretical methods do not get us very far, and the main tool is an exhaustive search for all subgroups generated by 13:6 and $D_{12}$ intersecting in a cyclic group of order 6.

Lemma 1. *There is a unique class of subgroups* 13:6 *containing* $13B$*-elements. Every such subgroup contains* $6F$*-elements and has centralizer which is cyclic of order* 4.

*Proof.* Elements in class $13B$ have centralizer $13^{1+2}{:}2{\cdot}A_4$ and normalizer

$$13^{1+2}{:}(3 \times 4S_4).$$

Every 13:6 lies in the subgroup $13^{1+2}{:}(3 \times 4A_4)$ of index 2. Since $4A_4$ contains a unique class of non-central involutions, it follows that there is a unique class of 13:6, which has centralizer

Received 22 January 2015.
*2010 Mathematics Subject Classification* 20D08 (primary).

cyclic of order 4. Note that the 3-elements which cube a $13B$-element are in class $3C$ in the Monster, so the elements of order 6 in this 13:6 are in class $6F$. □

The normalizer of a cyclic group of order 6 containing $6F$-elements is a group of shape $S_3 \times 2^{1+8} \cdot A_9$.

## 2. *Computational techniques*

We use the computer construction described in [4], in which the Monster is generated by a subgroup $\langle a, b\rangle \cong 2^{1+24} \cdot \mathrm{Co}_1$, together with a 'triality element' $T$, which centralizes a subgroup $2^{11} \cdot \mathrm{M}_{24}$ of $\langle a, b\rangle$, all acting on a 196882-space over $\mathbb{F}_3$. In particular, we use the techniques described in [5] and [8]. We repeat some of the most important ones here for convenience.

### 2.1. *Obtaining* $2 \cdot \mathrm{Co}_1$

It is straightforward to obtain elements of $2 \cdot \mathrm{Co}_1$ as $24 \times 24$ matrices over $\mathbb{F}_3$, corresponding in pairs (modulo sign) to elements of the quotient of $2^{1+24} \cdot \mathrm{Co}_1$ by the normal 2-subgroup. To do this, we compute 24 suitable rows of the corresponding $196882 \times 196882$ matrix and extract the $24 \times 24$ matrix from them.

This process can be carried out for any element of the Monster which commutes with the central element $z$ of $2^{1+24}\mathrm{Co}_1$, even if it is only given as a word in the generators of the Monster. We just have to compute the images of a carefully selected set of 24 coordinate vectors and extract another (not necessarily the same) carefully selected set of 24 coordinates from the answer.

### 2.2. *Changing post*

In our computations we are often 'tied to the post', in the sense that we can really only compute elements in the subgroup $2^{1+24} \cdot \mathrm{Co}_1$. However, a method was given in [3] for 'changing post', specifically, finding a word in the generators of the Monster which conjugates any given $2B$-element in $2^{1+24} \cdot \mathrm{Co}_1$ to the central involution. In principle, we pre-compute some representatives for the conjugacy classes of $2B$-elements in this group, as words in the generators. Then we conjugate our arbitrary $2B$-element to one of these by the usual dihedral group method. In practice, however, it turns out not to be too hard to deal with each case as it arises.

The process consists of two steps. In the first step, the involution is conjugated to a standard copy in the quotient $\mathrm{Co}_1$, such that conjugation by $T$ (or $T^{-1}$) takes this standard copy into the normal subgroup $2^{1+24}$. This first stage is achieved by standard dihedral group methods. The second stage is to conjugate this involution in $2^{1+24}$ to a standard one, $z^T$ or $z^{T^{-1}}$. This second stage is achieved by a random search, using the birthday paradox to speed it up, among the roughly 16 million conjugates. Modulo the central involution, this process can be carried out using only the action of $\mathrm{Co}_1$ on the Leech lattice modulo 2. The effect of this central involution is only to swap between the two cases $T$ and $T^{-1}$. (The search could also be done more systematically using base-and-strong-generating-set methods in the permutation representation on 8292375 points.)

## 3. *Finding* 13:6 *and its centralizer*

The group $13{:}6 \times 4$ can be found inside the involution centralizer $2^{1+24} \cdot \mathrm{Co}_1$. Our strategy is to find the centralizing element of order 4 first, and look in its centralizer for the rest of the

https://doi.org/10.1112/S1461157015000248 Published online by Cambridge University Press

group. Therefore, we begin by looking for the centralizer of a $2B$-involution in Co$_1$. Using the generators $a, b$ for $2^{1+24}$Co$_1$ defined in [4], we find that

$$(ab(abab^2)^2)^{21}$$

maps to such an involution and, using standard methods (see [1]), most of its centralizer may be generated by

$$c = (a(ab(abab^2)^2)^{21})^5,$$
$$d = ab(abab^2)^2.$$

In order to be able to use words previously computed for useful subgroups, we first find standard generators for the composition factor $G_2(4)$. These may be taken as

$$e = (cdcdcd^2)^5,$$
$$f = d^8(cd^2)^6 d^{34}.$$

We then use the words stored in [11] for maximal subgroups of $G_2(4)$ to make PSL$_2$(13) (modulo the 2-group) generated by

$$g_1 = (efefef^2)^3((ef)^2(efef^2)^2ef^2)^3(efefef^2)^{18},$$
$$g_2 = (efef^2)^{13}(ef(efef^2)^2)^5(efef^2)^2$$

and various useful elements within PSL$_2$(13) as follows:

$$g_3 = g_2^2(g_1g_2)^2,$$
$$g_4 = g_1g_3g_1g_3^2,$$
$$g_5 = (g_1g_2)^{13}.$$

Modulo the 2-group, $g_1$ and $g_3$ generate 13:6, in which $g_4$ is an element of order 13, and $g_5$ is an element of order 4 commuting with this copy of 13:6. Now we 'apply the formula' a few times to get the elements we really want, that is, elements which normalize $\langle g_4 \rangle$:

$$g_1' = g_1(g_4^{12}g_1^3g_4g_1)^6,$$
$$g_3' = g_3(g_4^3g_3^5g_4g_3)^6,$$
$$g_5' = g_4g_5(g_4g_5^3g_4g_5)^6,$$
$$g_1'' = g_1'g_5',$$
$$g_3'' = g_3'(g_5')^2.$$

Having found the 13:6, generated by $g_1''$ and $g_3''$, and its centralizer, generated by $g_5'$, we now need to find the normalizer of the element of order 6, which is in Monster-class $6F$. As noted above, this normalizer is a subgroup of shape $S_3 \times 2_+^{1+8}{\cdot}A_9$, which lies inside the centralizer of the $2B$-involution $g_1''$.

## 4. *Changing post*

The first step in this process is to 'change post', that is, to conjugate the new involution $g_1''$ to $z$ so that we can work in its centralizer to find the elements we need. (We describe the calculations that we actually did. They could be simplified slightly by using the pre-computation described in § 2.) The first step is to work in the quotient Co$_1$ to conjugate $g_1''$ into the normal $2^{11}$ of the standard copy of $2^{11}$:M$_{24}$. Now this standard copy is generated by $h$ and $i$, where

$$h = (ab)^{34}(abab^2)^3(ab)^6,$$
$$i = (ab^2)^{35}((ababab^2)^2ab)^4(ab^2)^5.$$

https://doi.org/10.1112/S1461157015000248 Published online by Cambridge University Press

We then want to lift to $2^{1+24}2^{11}M_{24}$ and find an element in $2^{1+24}.2^{11}$ outside $2^{1+24}$ which is conjugated by $T$ or $T^{-1}$ into the $2^{1+24}$. To do this, we make an element $k$ which has order 22 in the quotient $2^{11}M_{24}$ as follows.

$$\begin{aligned} k_1 &= hihi^2, \\ k_2 &= hihihi^2, \\ k &= (k_1k_2)^3k_2k_1k_2. \end{aligned}$$

It follows from details in [4] that either $(k^{11})^T$ or $(k^{11})^{T^{-1}}$ lies in the $2^{1+24}$.

We now find that $(g_1'')^{(ab)^4}k^{11}$ has order 15 modulo the central involution and therefore

$$l_1 = (ab)^4(k^{11}(ab)^{36}g_1''(ab)^4)^7$$

conjugates $g_1''$ into the desired place. A simple trial and error then gives us that $T$ conjugates the resulting involution into the normal 2-group.

The second stage of the process of 'changing post' is to conjugate our element $(g_1'')^{l_1T}$ to $z^T$ modulo $\langle z\rangle$. To do this, we translate these elements of $2^{1+24}$ into vectors of the standard module for $Co_1$, as described in [8]. Then we make a few thousand images of each under elements of $Co_1$ and sort the results in order to find coincidences. Any coincidence between the two lists of images gives us an element of $Co_1$ to map one to the other. We find that the element

$$l_2 = (ab)^7(ab^2)^{31}(ab(abab^2)^2)^{25}(ababab^2ab)^{13}(ababab^2)^{10}$$

performs the required conjugation and

$$l = l_1Tl_2T^{-1}$$

conjugates $g_1''$ to $z$.

## 5. *Finding the normalizer of the* 6

We have now reduced the computation of the 6-normalizer to a computation inside the involution centralizer $2^{1+24}\cdot Co_1$. We do this in two stages, first finding the image in the quotient $Co_1$ and then lifting to the involution centralizer.

### 5.1. *Finding $S_3 \times A_9$ in* $Co_1$

Now the element of order 3 which normalizes our 13-element and centralizes $g_1''$ is $g_3''' = g_4^5g_3''g_4^8$. The conjugate of this by $l$ lies in $C(z)$, and our aim is to find its centralizer therein.

First we do some searches in the quotient $Co_1$. Actually, we work in $2{\cdot}Co_1$, where we have the standard generators corresponding to $a$ and $b$, and we make the two elements (differing by a sign) corresponding to $(g_3''')^l$ by the method described in § 2.

We find that the following elements $m_1, m_2, m_3, m_4, m_5$ in $Co_1$-class $3A$ generate the centralizing $A_9$ modulo the 2-group, where

$$\begin{aligned} m_0 &= ((ab)^2(abab^2)^2ab^2)^{22}, \\ m_1 &= (ab)^{13}(ababab^2ab)^{23}(ab^2)^2m_0(ab^2)^{38}(ababab^2ab)^5(ab)^{27}, \\ m_2 &= (ab)^{16}(ababab^2ab)^{17}(ab^2)^2m_0(ab^2)^{38}(ababab^2ab)^{11}(ab)^{24}, \\ m_3 &= (ab)^{28}(ababab^2)(ababab^2ab)^{23}(ab^2)^{19}m_0(ab^2)^{21}(ababab^2ab)^5(ababab^2)^{23}(ab)^{12}, \\ m_4 &= (ab)^{15}(ababab^2)(ababab^2ab)^{16}(ab^2)^{36}m_0(ab^2)^4(ababab^2ab)^{12}(ababab^2)^{23}(ab)^{25}, \\ m_5 &= (ab)^6(ababab^2)^2(ababab^2ab)^{24}(ab^2)^{29}m_0(ab^2)^{11}(ababab^2ab)^4(ababab^2)^{22}(ab)^{34}. \end{aligned}$$

https://doi.org/10.1112/S1461157015000248 Published online by Cambridge University Press

We next find a copy of $J_2$ centralizing $m_3m_4$, generated by

$$\begin{aligned}
m_6 &= (ab)^{16}(ab^2)^{29}m_0(ab^2)^{11}(ab)^{24},\\
m_7 &= (ab)^{27}(ababab^2ab)^{18}(ab^2)^{38}m_0(ab^2)^2(ababab^2ab)^{10}(ab)^{13},\\
m_8 &= (ab)^{14}(ababab^2)(ab^2)^{28}m_0(ab^2)^{12}(ababab^2)^{23}(ab)^{26},\\
m_9 &= (ab)^{19}(ababab^2)(ababab^2ab)^{19}(ab^2)^{15}m_0(ab^2)^{25}(ababab^2ab)^9(ababab^2)^{23}(ab)^{21}.
\end{aligned}$$

In here we look for involutions which centralize the $A_9$. We put $n = m_6m_7$ and $o = m_8m_9$, and $p = ((no)^2(nono^2)^2)^4$, and then

$$\begin{aligned}
m_{10} &= (no)^4(nonono^2no)^2(no^2)^9p(no^2)(nonono^2no)^5(no)^{11},\\
m_{11} &= (no)^{10}(nonono^2no)^5(no^2)^4p(no^2)^6(nonono^2no)^2(no)^5
\end{aligned}$$

to generate the $S_3$ which commutes with the $A_9$ modulo the 2-group.

### 5.2. *Lifting to* $S_3 \times 2^{1+8}A_9$

First we apply the formula to the generators of $S_3$, to make them commute with the element $m_3m_4$ (which has order 10 and fifth power equal to the central involution $z$). That is, we make

$$\begin{aligned}
m'_{10} &= m_3m_4m_{10}(m_3m_4m_{10}^3m_3m_4m_{10})^2,\\
m'_{11} &= m_3m_4m_{11}(m_3m_4m_{11}^3m_3m_4m_{11})^2,\\
m'_{12} &= (m'_{10}m'_{11})^2,
\end{aligned}$$

where $m'_{12}$ is an element of order 3. It is easy to check that $m'_{12}$ is congruent modulo the 2-group to $(g_3''')^l$ (rather than its inverse).

Next we want to find which element of $2^{1+24}$ to conjugate $m'_{12}$ by in order to get $(g_3''')^l$. It is not enough to use the formula, because the latter element is only given as a word involving $T$. We have to test all $2^{16}$ conjugates until we find the right one. We make 16 elements to conjugate by as follows:

$$\begin{aligned}
q_0 &= (a^2m'_{12})^3a^2,\\
q_4 &= (m_1m_2m_5)^{12}q_0(m_1m_2m_5)^2,\\
q_{i+1} &= (m_3m_4)^8q_i(m_3m_4)^2 \quad \text{for } i = 0, 1, 2, 4, 5, 6,\\
q'_i &= (m'_{12})^2q_im'_{12} \quad \text{for all } i.
\end{aligned}$$

Testing such a large number of cases has to be done carefully. We can eliminate all but the correct case very quickly as follows. Pick a random vector $v$ and compute its 256 images under $(g_3''')^{lq}$, where $q$ is any product of the $q_i$. Similarly, compute the 256 images of $v$ under $(m'_{12})^{q'}$, where $q'$ is any product of the $q'_i$. Now comparing these vectors is very quick, and we find that the correct conjugate of $m'_{12}$ is

$$m''_{12} = q_0q_2q_6q'_0q'_1q'_5m'_{12}q'_0q'_1q'_5q_0q_2q_6z.$$

Having found this, we now apply the formula to get the elements we want:

$$\begin{aligned}
r_1 &= m''_{12}m_3m_4m''_{12}(m_3m_4)^9m''_{12}m_3m_4,\\
r_2 &= m''_{12}m_1m_2m_5m''_{12}(m_1m_2m_5)^{13}m''_{12}m_1m_2m_5,\\
r_3 &= m''_{12},\\
r_4 &= m'_{10}(m''_{12})^2m'_{10}m''_{12}m'_{10}.
\end{aligned}$$

We now have that $\langle r_3, r_4\rangle \cong S_3$ and $\langle r_1, r_2\rangle \cong 2_+^{1+8}\cdot A_9$.

## 6. *Enumerating the cases*

From the structure constant for the triple of classes $(2B, 2B, 6F)$ in the Monster we know that the element of order 6 extends to exactly 15255 dihedral groups of order 12 that are generated by $2B$-elements. It is easy to see that 135 of these are generated by the product of an involution in the $S_3$ generated by $r_3$ and $r_4$, with an involution in the normal subgroup $2^{1+8}$. The other 15120 are all generated by involutions mapping to one of cycle type $(2^4, 1)$ in $A_9$. There are 945 such involutions in $A_9$, and each lifts to 16 involutions in $2^{8\cdot}A_9$. (We ignore the central involution, as this is the cube of our element of order 6.)

We first determine a map onto a standard copy of $A_9$, as permutations on nine points. This can be obtained easily using the Meataxe, from the copy of $2^\cdot A_9$ inside $2^\cdot\mathrm{Co}_1$, which we have already used. It turns out that such a map is given by

$$\begin{aligned} r_1 &\mapsto (8, 7, 5, 9, 4), \\ r_2 &\mapsto (1, 2, 3, 4, 5, 6, 7), \\ (g_5')^l &\mapsto (1, 6, 4, 5)(2, 3, 7, 9). \end{aligned}$$

We can then determine (more or less by hand) a set of words in these generators which give representatives of the orbits of $(g_5')^l$ on the involutions of type $(2^4, 1)$ in $A_9$.

Using $(g_5')^l$, we see that we may restrict attention to those involutions whose fixed point is 9, 5 or 8. We make the following elements:

$$\begin{aligned} r_5 = (r_1r_2^2)^5 r_1 (r_1r_2^2)^5 r_1^4 &\mapsto (2, 8, 9), \\ r_2' = r_2^8 &\mapsto (1, 2, 3, 4, 5, 6, 7), \\ r_6 = r_1 r_2' r_5 (r_2')^6 r_1^4 &\mapsto (1, 4, 5), \\ r_7 = (r_2')^6 r_6 r_2' r_6 &\mapsto (1, 4, 5, 6, 2), \\ r_8 = (r_1r_2r_1r_2^2)^3 &\mapsto (1, 5)(2, 7)(3, 8)(4, 6). \end{aligned}$$

It is now easy to see that the 105 involutions in this class which fix the point 9 are given by

$$(r_8)^{r_6^\alpha r_7'^\beta r_2'^\gamma},$$

where $r_7' = r_7^6$ and $0 \leqslant \alpha \leqslant 2$, $0 \leqslant \beta \leqslant 4$, $0 \leqslant \gamma \leqslant 6$.

Similarly, those which fix 5 are given by

$$(r_8)^{r_6^\alpha r_7'^\beta r_2'^\gamma r_1^4}$$

and those which fix 8 are

$$(r_8)^{r_6^\alpha r_7'^\beta r_2'^\gamma r_1^2}.$$

Now we lift to $2^{1+8}A_9$ and see which elements of the $2^{1+8}$ we need to multiply these words by. We make the following generators for $2^{1+8}$:

$$\begin{aligned} s_0 &= r_2^7, \\ s_4 &= (s_0)^{(r_1r_2^8)}, \\ s_{i+1} &= (s_i)^{r_1} \quad \text{for } i = 0, 1, 2, 4, 5, 6. \end{aligned}$$

The result is that $sr_8$ is an involution for the following 16 words $s$:

$$\begin{array}{llll} s_1, & s_0s_1s_2s_3, & s_5, & s_0s_2s_3s_5, \\ s_0s_1s_4s_6, & s_1s_2s_3s_4s_6, & s_0s_4s_5s_6, & s_2s_3s_4s_5s_6, \\ s_7, & s_0s_2s_3s_7, & s_1s_5s_7, & s_0s_1s_2s_3s_5s_7, \\ s_0s_4s_6s_7, & s_2s_3s_4s_6s_7, & s_0s_1s_4s_5s_6s_7, & s_1s_2s_3s_4s_5s_6s_7. \end{array}$$

https://doi.org/10.1112/S1461157015000248 Published online by Cambridge University Press

Slightly more systematically, we may define

$$\begin{aligned} t_0 &= s_1 r_8, \\ t_1 &= s_5 r_8, \\ t_2 &= s_7 r_8, \\ t_3 &= s_1 s_5 t_2, \\ t_{i+4} &= s_0 s_2 s_3 t_i \quad \text{for } 0 \leqslant i \leqslant 3, \\ t_{i+8} &= s_0 s_4 s_6 t_i \quad \text{for } 0 \leqslant i \leqslant 7. \end{aligned}$$

By this stage we have a complete set of 3360 involutions corresponding to fixed points 9 and 5. The first batch of 1680 cases are $t_i$ conjugated by $r_6^{\alpha} r_7'^{\beta} r_2'^{\gamma}$ and the second batch of 1680 cases are $t_i$ conjugated by $r_6^{\alpha} r_7'^{\beta} r_2'^{\gamma} r_1^4$.

The set of 1680 involutions corresponding to fixed point 8 are the $t_i$ conjugated by $r_6^{\alpha} r_7'^{\beta} r_2'^{\gamma} r_1^2$. This set can be further reduced by investigating the action of $(g_5')^l$. In fact, it seems not worth doing this in full generality, but merely in the $A_9$ quotient. By explicit computation we find that this element has 20 orbits of size 4, and 10 orbits of size 2, and five fixed points, making 35 orbits altogether. We may therefore reduce the $16.105 = 1680$ cases to $16.35 = 560$ by taking one from each orbit. We took the following 35 values for $(\gamma, \beta, \alpha)$, which we write as $\gamma\beta\alpha$ to save space:

$$\begin{gathered} 000, 002, 022, 031, 100, 101, 102, 120, 122, 131, \\ 132, 141, 200, 210, 221, 231, 232, 300, 302, 401, \\ 001, 010, 021, 032, 042, 110, 212, 412, 421, 432, \\ 012, 112, 201, 311, 512. \end{gathered}$$

(The first two rows comprise representatives for the orbits of length 4, the third row orbits of length 2 and the last row the fixed points.)

A fourth batch of 255 cases comes from the 255 non-trivial elements of $2^{1+8}/2$. Although only 135 of these cases are involutions, it seemed more trouble than it was worth (and error-prone) to try to select the correct cases. Moreover, the fact that there were exactly 135 involution cases out of the 255 provided an additional sanity check.

## 7. *Results*

For each of the 4175 involutions generated as above, we tested to see whether it extends 13:6 to PSL$_2$(13) as follows. We tested each of the six involutions in the outer half of the $D_{12}$ to see if its product with the element of order 13 has order 3. The element of order 13 is

$$g_4^l = g_4^{l_1 T l_2 T^{-1}} = T l_2^{-1} T^{-1} l_1^{-1} g_4 l_1 T l_2 T^{-1} = T l_2^{21} T^{-1} l_1^{45} g_4 l_1 T l_2 T^{-1}.$$

For the sake of efficiency, we pre-compute the element

$$g_4' = l_1^{-1} g_4 l_1.$$

Since the cyclic part of the $D_{12}$ is generated by $r_3$ of order 3, and $z$ of order 2, we obtain the six involutions by multiplying by the six elements in this cyclic group. More specifically, the involutions we test are the product of one of

$$r_4, r_3 r_4, r_3^2 r_4, z r_4, z r_3 r_4, z r_3^2 r_4$$

with one of the conjugates of one of the $t_i$, or with a product of some $s_i$.

Each of the 4175 cases tested took a little under a minute to check, making a total running time, including making all the required group elements, of around three days, on my geriatric laptop.

We found that none of the cases passed this test. This therefore concludes the proof of Theorem 2. However, a negative result like this is not robust against bugs in the computer programs, and we therefore embarked on extensive checking. The programs used are available as online supplementary material from the publisher's website and can be used as the basis for further checks as necessary.

*References*


1. J. N. Bray, 'An improved method for generating the centralizer of an involution', *Arch. Math. (Basel)* 74 (2000) 241–245.
2. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An atlas of finite groups* (Oxford University Press, 1985).
3. P. E. Holmes and R. A. Wilson, 'A new maximal subgroup of the Monster', *J. Algebra* 251 (2002) 435–447.
4. P. E. Holmes and R. A. Wilson, 'A new computer construction of the Monster using 2-local subgroups', *J. Lond. Math. Soc.* (2) 67 (2003) 349–364.
5. P. E. Holmes and R. A. Wilson, 'On subgroups of the Monster containing $A_5$'s', *J. Algebra* 319 (2008) 2653–2667.
6. S. Norton, 'Anatomy of the Monster: I', *Proceedings of the* Atlas *Ten Years on Conference, Birmingham, 1995* (Cambridge University Press, 1998) 198–214.
7. S. P. Norton and R. A. Wilson, 'Anatomy of the Monster: II', *Proc. Lond. Math. Soc.* (3) 84 (2002) 581–598.
8. S. P. Norton and R. A. Wilson, 'A correction to the 41-structure of the Monster, a construction of a new maximal subgroup $L_2(41)$, and a new Moonshine phenomenon', *J. Lond. Math. Soc.* (2) 87 (2013) 943–962.
9. R. A. Wilson, *The finite simple groups*, Graduate Texts in Mathematics 251 (Springer, 2009).
10. R. A. Wilson, 'Classification of subgroups isomorphic to $PSL_2(27)$ in the Monster', *LMS J. Comput. Math.* 17 (2014) 33–46.
11. R. A. Wilson *et al.*, 'An atlas of group representations', http://brauer.maths.qmul.ac.uk/Atlas/.



*Robert A. Wilson*
*School of Mathematical Sciences*
*Queen Mary University of London*
*Mile End Road*
*London E1 4NS*
*United Kingdom*

r.a.wilson@qmul.ac.uk