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TILTED ALGEBRAS AND CROSSED PRODUCTS*

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Abstract. We consider an artin algebra A and its crossed product algebra $A_\alpha\#_\sigma G$, where G is a finite group with its order invertible in A. Then, we prove that A is a tilted algebra if and only if so is $A_\alpha\#_\sigma G$.

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1. Introduction. Let K be a commutative artin ring. A K-algebra A is a ring A together with a ring homomorphism $K \longrightarrow A$ whose image is contained in the centre $Z(A)$ of A. We say that A is an *artin K-algebra*, or *artin algebra* for short, if A is finitely generated as a K-module.

Let A be an artin algebra, and G a finite group. By an *action* of G on A, we mean a group homomorphism $\sigma : G \longrightarrow \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of all automorphisms of A. If a finite group G acts on an artin algebra A such that the order $|G|$ is invertible in A, and $\alpha : G \times G \longrightarrow U(A) \bigcap Z(A)$ is a 2-cocycle map in the sense of Section 2, where $U(A)$ is the group of the units of A, we can form the crossed product algebra $A_\alpha\#_\sigma G$ with respect to A and G (see Section 2). A result in [**12**] has attracted our attention, which stated that: let A be an artin algebra and G a finite group acting on A with the order $|G|$ invertible in A, and $\alpha : G \times G \longrightarrow U(A) \bigcap Z(A)$ a 2-cocycle map. Then, A is a representation-finite tilted algebra if and only if so is $A_\alpha\#_\sigma G$ [**12**, Theorem 4.6].

Here, an artin algebra A is said to be *tilted* provided that there exists a hereditary artin algebra R and a tilting R-module T such that $A = \mathrm{End}_R(T)$ (see [**7**] and [**13**]); and A is said to be *representation-finite* if the number of the isomorphism classes of indecomposable modules in mod A is finite.

The aim of this paper is to generalize the original result of Reiten and Riedtmann [**12**, Theorem 4.6] without the restriction on the representation type. The main result is the following theorem.

THEOREM 1.1. *Let A be an artin algebra, G a finite group whose order $|G|$ is invertible in A, $\sigma : G \longrightarrow \mathrm{Aut}(A)$ a group homomorphism, and $\alpha : G \times G \longrightarrow U(A) \bigcap Z(A)$ a 2-cocycle map. Then, A is a tilted algebra if and only if so is $A_\alpha\#_\sigma G$.*

We mention that the problem would be different from representation-finite tilted algebras when ones consider representation-infinite titled algebras. Since a

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representation-finite artin algebra has only one component, while a representation-infinite artin algebra is not the case.

The main idea of the proof of Theorem 1.1 is applying the criterion of tilted algebras (see [**8**, Theorem 1.6] and [**15**, Theorem 3]) to find a generalized standard component with a faithful section for $A * G$ (or A) when A (or $A * G$) is supposed to be tilted. While the proof of [**12**, Theorem 4.6] is based on finding a stable section under the action of G on mod A by using all the projective modules.

Let us fix the notations and conventions of this paper. For an artin algebra A, we always assume that A is connected. By a module, we always mean a finitely generated right module. The category of all finitely generated right A-modules is denoted by mod A. τ_A is the Auslander–Reiten translation of mod A, and $\Gamma(\text{mod}\, A)$ denotes the Auslander–Reiten quiver of A. When no possible confusion will occur, we do not distinguish between an indecomposable module M in mod A and the corresponding vertex $[M]$ in $\Gamma(\text{mod}\, A)$. $\text{Aut}(A)$ denotes the group of all automorphisms of A, $U(A)$ denotes the group of the units of A, and $Z(A)$ is the centre of A. We denote by $\text{add}(M)$ the full subcategory of mod A consisting of all summands of a direct sum of copies of a module M. For all unexplained notions and notations, see [**1, 2, 6, 11**] and [**14**]. The reader is also referred to the recent papers [**3,4**] and [**17**] for a discussion of representation theory problems over crossed product algebras and twisted group algebras.

2. Preliminaries. In this paper, we follow the construction of crossed product algebras in [**12**]. The classical definition of a crossed product is introduced in [**11**, Section 14.1] and [**6**, Chapter 3, Section 28]. A more generalized definition of crossed product algebras can be found in [**10**, Chapter 1, 1.4].

Let A be an artin algebra, G a finite group acting on A, that is, there is a group homomorphism $\sigma : G \longrightarrow \text{Aut}(A)$. Following [**11**, Section 14.1], a map $\alpha : G \times G \longrightarrow U(A) \bigcap Z(A)$ is defined to be a 2-cocycle if the following two conditions are satisfied:

(1) $\alpha(x, y)\alpha(xy, z) = {}^{x}\alpha(y, z)\alpha(x, yz)$ for all $x, y, z \in G$;
(2) $\alpha(x, 1_G) = 1_A = \alpha(1_G, x)$ for $x \in G$, and 1_G the identity of G,
where we have denoted the action $\sigma(x)(a)$ by ${}^{x}a$ for $x \in G$, $a \in A$. The *crossed product algebra* $A_\alpha\#_\sigma G$ is defined to be the free left A-module $\bigoplus_{x\in G} Ax$ with the basis G, and the multiplication is defined by

$$(ax)(by) = a\, {}^{x}b\, \alpha(x, y)\, xy$$

for $a, b \in A$ and $x, y \in G$.

The crossed product algebra $A_\alpha\#_\sigma G$ is still an artin algebra. Usually, we denote the identity elements 1_A of A and 1_G of G by 1 if there is no confusion. If $\alpha : G \times G \longrightarrow U(A)$ is the trivial map, that is, $\alpha(x, y) = 1_A$ for all $x, y \in G$, then we have a special kind of crossed product algebra construction, which is called *skew group algebra*, and denote by AG instead of $A_\alpha\#_\sigma G$.

For the convenience of the reader, we collect some basic facts about mod A and mod $A_\alpha\#_\sigma G$. We mention here that there is a slight difference between the results we recall in this section and the ones in [**12**], since we deal with the right modules, while the results in [**12**] are stated in the left module version.

Let A be an artin algebra, G a finite group acting on A with the order $|G|$ invertible in A. Then, the action σ induces a right action of G on mod A, that is, there is a group homomorphism from G^{op} to the group of all autofunctors of mod A (compare [**12**, Section 1, 1.5]). We give an explicit description for the action of an element x on A-modules and A-module homomorphisms.

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For an element $x \in G$ and a right A-module M, define the action of x on M to be the right A-module ${}^xM$ such that ${}^xM = M$ as a K-module, and the right A-multiplication is given by $m \cdot a = m\,{}^xa$ for $m \in M$ and $a \in A$. Let $f : M \longrightarrow N$ be an A-module homomorphism, ${}^xf : {}^xM \longrightarrow {}^xN$ is defined by ${}^xf(m) = f(m)$ for $m \in {}^xM$.

From now on, we fix a group homomorphism $\sigma : G \longrightarrow \mathrm{Aut}(A)$ and a 2-cocycle map $\alpha : G \times G \longrightarrow U(A) \bigcap Z(A)$, and we set

$$B = A_\alpha \#_\sigma G.$$

There is a natural algebra monomorphism $i : A \longrightarrow B$ by assigning that $i(a) = a\,1_G$ with 1_G the identity of G. Then, we have two induced exact functors, the tensor functor $F = -\bigotimes_A B : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,B$ and the restriction functor $H = \mathrm{Hom}_B(B, -) : \mathrm{mod}\,B \longrightarrow \mathrm{mod}\,A$, we list some properties related to these two functors for later use.

Lemma 2.1. *Keep the notations as above. Then, we have the following.*

(1) *(F, H) and (H, F) are two adjoint pairs of exact functors.*
(2) *For the adjoint pair (F, H), the unit $\eta : 1_{\mathrm{mod}\,A} \longrightarrow HF$ is a split monomorphism and the counit $\varepsilon : FH \longrightarrow 1_{\mathrm{mod}\,B}$ is a split epimorphism.*
(3) *For the adjoint pair (H, F), the unit $\eta' : 1_{\mathrm{mod}\,B} \longrightarrow FH$ is a split monomorphism and the counit $\varepsilon' : HF \longrightarrow 1_{\mathrm{mod}\,A}$ is a split epimorphism.*
(4) *Let M be an indecomposable right A-module, then $HF(M) \simeq \bigoplus_{x\in G} {}^xM$.*
(5) *Let M and N be two indecomposable right A-modules, then $FM \simeq FN$ if and only if $M \simeq {}^xN$ for some $x \in G$.*

Proof. We refer to [**12**, Section 1, 1.1 and 1.8]. □

Lemma 2.2.

(1) *Let M and N be two A-modules. Then, $f : M \longrightarrow N$ is monic (or epic) if and only if so is $Ff : FM \longrightarrow FN$.*
(2) *Let V and W be two B-modules. Then, $f : V \longrightarrow W$ is monic (or epic) if and only if so is $Hf : HV \longrightarrow HW$.*

Proof. We only prove (1), the proof of (2) is similar. Suppose that $f : M \longrightarrow N$ is monic (or epic), then so is Ff since F is an exact functor. Now assume that $Ff : FM \longrightarrow FN$ is monic. It follows that $HF(f)$ is also monic from the exactness of H. We show that $f : M \longrightarrow N$ is monic. Notice that $\eta : 1_{\mathrm{mod}\,A} \longrightarrow HF$ is a split monomorphism by Lemma 2.1(2), then by the naturality of η we have that $\eta_N f = HF(f)\,\eta_M$ is monic, and hence $f : M \longrightarrow N$ is monic. The proof of the epimorphism case can be proved similarly by using the split epimorphism $\varepsilon' : HF \longrightarrow 1_{\mathrm{mod}\,A}$ from Lemma 2.1(3). We have completed the proof. □

Let us recall the notions of almost split morphisms and almost split sequences. Let A be an artin algebra, and let M, N, L be modules in $\mathrm{mod}\,A$. An A-module homomorphism $g : M \longrightarrow N$ is called *right almost split* if g is not a retraction, and if every A-module homomorphism $L \longrightarrow N$, which is not a retraction, can factor through g. An A-module homomorphism $g : M \longrightarrow N$ is called *right minimal* if every endomorphism $u : M \longrightarrow M$ such that $gu = g$ is an automorphism. An A-module homomorphism $f : M \longrightarrow N$ is called *right minimal almost split* if it is both right almost split and right minimal. Left almost split morphisms, left minimal morphisms

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and left minimal almost split morphisms are defined dually. An exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is called an *almost split sequence* provided f is left minimal almost split and g is right minimal almost split.

We have the following result about the relationship of almost split morphisms and sequences between A and $A_\alpha \#_\sigma G$, whose proof in the version of a dualizing K-variety and its skew category can be found in [**12**, Section 3, Theorem 3.8].

LEMMA 2.3. *Let $B = A_\alpha \#_\sigma G$ be the crossed product algebra, $F : \operatorname{mod} A \longrightarrow \operatorname{mod} B$ and $H : \operatorname{mod} B \longrightarrow \operatorname{mod} A$ the two exact functors as before. Then, we have the following.*

(1) *If $g : M \longrightarrow N$ is a right (or left) minimal almost split morphism in* $\operatorname{mod} A$*, then $Fg : FM \longrightarrow FN$ is a direct sum of right (or left) minimal almost split morphisms in* $\operatorname{mod} B$*. Conversely, if $g : V \longrightarrow W$ is a right (or left) minimal almost split morphism in* $\operatorname{mod} B$*, then $Hg : HV \longrightarrow HW$ is a direct sum of right (or left) minimal almost split morphisms in* $\operatorname{mod} A$.
(2) *If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an almost split sequence in* $\operatorname{mod} A$*, then $0 \longrightarrow FL \longrightarrow FM \longrightarrow FN \longrightarrow 0$ is a direct sum of almost split sequences in* $\operatorname{mod} B$*. Conversely, if $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is an almost split sequence in* $\operatorname{mod} B$*, then $0 \longrightarrow HU \longrightarrow HV \longrightarrow HW \longrightarrow 0$ is a direct sum of almost split sequences in* $\operatorname{mod} A$.

Throughout, we denote by $\tau_A := \operatorname{Tr}_A D$, $\tau_A^{-1} := D\operatorname{Tr}_A$, $\tau_B := \operatorname{Tr}_B D$, $\tau_B^{-1} := D\operatorname{Tr}_B$ the Auslander–Reiten translation operators, see [**1**, Chapter IV, Section 2] and [**2**, Chapter IV, Section 1]. The following result is an immediate consequence of the above lemma.

COROLLARY 2.4. *The functors F and H commute with τ and τ^{-1}.*

Let V be an indecomposable B-module, then HV is an A-module, which can be decomposed into a direct sum of indecomposable A-modules. Therefore, we can select an indecomposable summand M of HV, such that V is a summand of FM by using Lemma 2.1(2). In this case, we call M an *indecomposable A-module related to V*. Notice that, let M' be an indecomposable A-modules such that V is a summand of FM', then by applying the functor H and Lemma 2.1(4), there exists an element $x \in G$ such that ${}^xM'$ is an indecomposable A-module related to V.

Since irreducible morphisms can be viewed as components of minimal almost split morphisms (see [**1**, Chapter IV, 1.10]), then we also get a connection between irreducible morphisms in $\operatorname{mod} A$ and the ones in $\operatorname{mod} B$. The following is a restatement of [**12**, Section 4, Lemma 4.1].

COROLLARY 2.5.

(1) *Let M and N be two indecomposable A-modules. If $g : {}^xM \longrightarrow {}^yN$ is an irreducible morphism in* $\operatorname{mod} A$*, where $x, y \in G$. Then for every indecomposable summand V of FM, there exists an irreducible morphism $V \longrightarrow W$ in* $\operatorname{mod} B$ *for some indecomposable summand W of FN.*

(2) *Let L and M be two indecomposable A-modules. If $f : {}^xL \longrightarrow {}^yM$ is an irreducible morphism in* mod A*, where $x, y \in G$. Then for every indecomposable summand V of FM, there exists an irreducible morphism $U \longrightarrow V$ in* mod B *for some indecomposable summand U of FL.*
(3) *Let V and W be two indecomposable B-modules. If $g : V \longrightarrow W$ is an irreducible morphism in* mod B*. Then for every indecomposable A-module M related to V, there exists an irreducible morphism $M \longrightarrow N$ in* mod A *for some indecomposable A-module N related to W.*
(4) *Let U and V be two indecomposable B-modules. If $f : U \longrightarrow V$ is an irreducible morphism in* mod B*. Then for every indecomposable A-module M related to V, there exists an irreducible morphism $L \longrightarrow M$ in* mod A *for some indecomposable A-module L related to U.*

In the sequel of this section, we recall some notions related to the proof of Theorem 1.1 and a result about the Jacobson radical of mod A.

We denote by rad_A the Jacobson radical of mod A (see [**1**, A. Appendix, A.3] for definition), and denote by rad_A^i the ith power of rad_A. The infinite radical $\bigcap_{i=1}^{\infty} \mathrm{rad}_A^i$ of mod A is denoted by rad_A^{∞}. Let $\mathcal{C}$ be a component of $\Gamma(\mathrm{mod}\, A)$, if $\mathrm{rad}_A^{\infty}(M, N) = 0$ for all modules $M, N \in \mathcal{C}$, then $\mathcal{C}$ is called a *generalized standard component* of $\Gamma(\mathrm{mod}\, A)$ (see [**15**] and [**16**]).

For a component $\mathcal{C}$ of $\Gamma(\mathrm{mod}\, A)$, we denote by $\mathrm{ann}_A(\mathcal{C})$ the *annihilator* of $\mathcal{C}$ in A, that is, the intersection of the annihilators $\mathrm{ann}_A(M)$ of all modules M in $\mathcal{C}$. If $\mathrm{ann}_A(\mathcal{C}) = 0$, then we call $\mathcal{C}$ a *faithful component*. Likewise, for a subset $\mathcal{D}$ of $\mathcal{C}$, the annihilator $\mathrm{ann}_A(\mathcal{D})$ in A is the intersection of the annihilators of all modules in $\mathcal{D}$. And $\mathcal{D}$ is *faithful* if $\mathrm{ann}_A(\mathcal{D}) = 0$.

Let $\mathcal{C}$ be a component of $\Gamma(\mathrm{mod}\, A)$. A connected full subquiver Σ in $\mathcal{C}$ is a *section*, if it is subject to the three conditions: first, Σ is acyclic; second, Σ meets each τ_A-orbit in $\mathcal{C}$ exactly once; third, Σ is *convex* in $\mathcal{C}$, that is, for a path $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t$ in $\mathcal{C}$, if M_0 and M_t belong to Σ, then M_i belong to Σ for $i = 0, \ldots, t$.

Let M and N be two indecomposable A-modules. A *walk* in mod A from M to N is a sequence of A-module homomorphisms

$$M = M_0 \xrightarrow{f_1^*} M_2 \xrightarrow{f_2^*} \cdots \longrightarrow \cdots \xrightarrow{f_t^*} M_t = N,$$

where all M_i are indecomposable, and for each i, f_i^* is either a nonzero nonisomorphism $g_i : M_{i-1} \longrightarrow M_i$ or a nonzero nonisomorphism $h_i : M_i \longrightarrow M_{i-1}$ in mod A. And a *path* in mod A from M to N is a sequence of A-module homomorphisms as above such that for each i, f_i^* is an A-module homomorphism $g_i : M_{i-1} \longrightarrow M_i$ in mod A. A path from an indecomposable A-module M to itself is called a *cycle* in mod A.

Especially, if the morphisms involved are irreducible, then we call them a walk of irreducible morphisms, a path of irreducible morphisms and a cycle of irreducible morphisms respectively. Two indecomposable A-modules M and N are in the same component if and only if there is a walk of irreducible morphisms from M to N. A component is *acyclic* provided that there are no cycles of irreducible morphisms.

The following result is well known, whose proof in the version of an additive category can be found in ([**1**, A. Appendix, Lemma 3.4]).

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LEMMA 2.6. *Let*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1u} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{v1} & \phi_{v2} & \cdots & \phi_{vu} \end{pmatrix} : L = \bigoplus_{s=1}^{u} L_s \longrightarrow L' = \bigoplus_{t=1}^{v} L'_t$$

be an A-module homomorphism. The A-module homomorphism ϕ belongs to $\mathrm{rad}_A(L, L')$, *if and only if each of the A-module homomorphisms ϕ_{ts} belongs to* $\mathrm{rad}_A(L_s, L'_t)$, *for* $s = 1, \ldots, u$ *and* $t = 1, \ldots, v$.

3. The Proof of Theorem 1.1. In this section, we give the proof of Theorem 1.1. It is worthy to notice that the crossed product algebra $A_\alpha \#_\sigma G$ is not necessarily a connected artin algebra even if A is. However, we prove Theorem 1.1 under the assumption that $A_\alpha \#_\sigma G$ is connected, based on the following observation.

LEMMA 3.1. *Let A be an artin algebra with a decomposition $A = \prod_{i=1}^{t} A_i$, where each A_i is an artin algebra. Then, A is a tilted algebra if and only if A_i is a titled algebra, for $i = 1, \ldots, t$.*

Here, recall that, a right R-module T is called a *tilting module* if T is subject to the three conditions:

(1) the projective dimension proj.dim. $T \leq 1$;
(2) $\mathrm{Ext}_A(T, T) = 0$;
(3) there is an exact sequence $0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$ with $T_0, T_1 \in \mathrm{add}(T)$.

It is well known that, if T_R is a tilting module with $A = \mathrm{End}_R(T)$, then ${}_AT$ is a tilting module that induces a canonical algebra isomorphism $R \simeq (\mathrm{End}({}_AT))^{op}$(for instance, see [**1**, Chapter VI, Lemma 3.3] and [**5**, Chapter 3, Proposition 3.2.2]).

Proof. We first prove the "only if" part. Suppose that $A = \prod_{i=1}^{t} A_i$ is tilted. Then, we have the isomorphism of left module categories $\mathrm{mod}\, A^{op} \simeq \prod_{i=1}^{t} \mathrm{mod}\, A_i^{op}$. So the tilting module T can be identified as an object $(T_1, \ldots, T_t)$ in $\prod_{i=1}^{t} \mathrm{mod}\, A_i^{op}$, where each T_i is a left module over the artin algebra A_i. Then, we have that each T_i is a tilting left module over A_i by a direct verification that T_i satisfies the tilting condition. Let $\mathrm{End}({}_{A_i}T_i)^{op} = R_i$, then each R_i is hereditary since $R = \prod_{i=1}^{t} R_i$ is hereditary. Moreover, it immediately follows that each $T_{i\,R_i}$ is a tilting module by the left version of the well-known result we quote above. Therefore, each $A_i = \mathrm{End}(T_{i\,R_i})$ is a tilted algebra.

For the "if" part, suppose that each A_i is tilted, that is, there exists a tilting module $T_{i\,R_i}$ over a hereditary algebra R_i such that $A_i = \mathrm{End}(T_{i\,R_i})$. This gives rise to a tilting module T in $\mathrm{mod}\, R$ which corresponds to the object $(T_1, \ldots, T_t)$ in $\prod_{i=1}^{t} \mathrm{mod}\, R_i$, where $R = \prod_{i=1}^{t} R_i$ is hereditary. Hence, A is a tilted algebra with $A = \mathrm{End}(T_R)$. We have completed the proof. □

The following results immediately follow from Corollary 2.5.

LEMMA 3.2. *Let $B = A_\alpha \#_\sigma G$ be the crossed product algebra. Then, the following statements hold.*

(1) *Let M and N be two indecomposable A-modules. If M and N are in the same component, then for every indecomposable summand V of FM, there exists an indecomposable summand W of FN, such that V and W are in the same component.*

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(2) *Let V and W be two indecomposable B-modules. If V and W are in the same component. Then for every indecomposable A-module M related to V, there exists an indecomposable A-module N related to W such that M and N are in the same component.*

LEMMA 3.3. *Let $B = A_\alpha \#_\sigma G$ be the crossed product algebra and M an indecomposable A-module. A component $\mathcal{C}$ containing M is acyclic in $\Gamma(\mathrm{mod}\, A)$ if and only if, for any indecomposable summand W of FM, the component containing W is acyclic in $\Gamma(\mathrm{mod}\, B)$.*

We need the following observation about $A_\alpha \#_\sigma G$-modules.

LEMMA 3.4. *Let $B = A_\alpha \#_\sigma G$ be the crossed product algebra and W a B-module. Then for any $x \in G$, the map $\beta_x(W) : HW \longrightarrow {}^xHW$ defined by $w \longmapsto wx^{-1}$ defines an A-module isomorphism.*

Proof. The bijectivity of the map $\beta_x(W)$ is obvious. So it suffices to show that $\beta_x(W)$ is an A-module homomorphism. It is well known that HW has a G-action structure since W is an $A * G$-module (see [**2**, Section 4 of Chapter III] for more details). Then, we can directly verify that $\beta_x(W)(wa) = wax^{-1} = wx^{-1}\, {}^xa = f_W(w) \cdot a$, where $f_W(w) \cdot a$ is the right A-multiplication of the module ${}^xW$. Consequently, the map f_W is an A-module homomorphism. □

We introduce the following two notions for later use. A component $\mathcal{C}_A$ of $\Gamma(\mathrm{mod}\, A)$ is called *G-stable* if for any $M \in \mathcal{C}_A$, ${}^xM \in \mathcal{C}_A$ for all $x \in G$. A component $\mathcal{C}_B$ of $\Gamma(\mathrm{mod}\, B)$ is called *F-summands closed*, if for any $V \in \mathcal{C}_B$, all indecomposable summands V' of FM are still in $\mathcal{C}_B$, where M is an indecomposable A-module related to V.

LEMMA 3.5. *Let A be an artin algebra, and $G = \{x_1 = 1, x_2, \ldots, x_n\}$ a finite group acting on A with the usual assumptions. Let $B = A_\alpha \#_\sigma G$ be the crossed product algebra. Then, we have the following.*

(1) *Let $\mathcal{C}_A$ be a G-stable component of $\Gamma(\mathrm{mod}\, A)$. If $\mathcal{C}_A$ is a generalized standard component, then any component $\mathcal{C}_B \subseteq F(\mathcal{C}_A)$ is a generalized standard component.*
(2) *Let $\mathcal{C}_B$ be a F-summands closed component of $\Gamma(\mathrm{mod}\, B)$. If $\mathcal{C}_B$ is a generalized standard component, then any component $\mathcal{C}_A \subseteq H(\mathcal{C}_B)$ is a generalized standard component.*

Proof. We only prove (1), because the statement (2) can be proved by carrying a similar approach. Fix an indecomposable A-module $L \in \mathcal{C}_A$. Then, we get a component $\mathcal{C}_B$ in $\Gamma(\mathrm{mod}\, B)$ which contains an indecomposable summand U of FL. We claim that the component $\mathcal{C}_B$ is generalized standard. In fact, if it is not the case, then there exist two indecomposable B-modules V and W such that $\mathrm{rad}_B^\infty(V, W) \neq 0$. Put

$$\mathcal{S} = \{(M, N) \mid M, N \text{ are indecomposable } A\text{-modules related to } V \text{ and } W \text{ respectively}\},$$

which is a finite set. It follows that $\mathcal{S} \subseteq \mathcal{C}_A$ from Lemma 2.1(5) and the assumption that $\mathcal{C}_A$ is G-stable.

Since $\mathrm{rad}_A^\infty(M, N)$ is a finitely generated K-module, then we have $\mathrm{rad}_A^\infty(M, N) = \mathrm{rad}_A^i(M, N)$, for some $i > 0$, and $\mathrm{rad}_A^\infty(M, N) = \mathrm{rad}_A^i(M, N) = 0$, by the assumption that $\mathcal{C}_A$ is a generalized standard component.

Let l be the maximal number of the set

$$\mathcal{I} = \{i \mid \mathrm{rad}_A^{\infty}(M, N) = \mathrm{rad}_A^{i}(M, N) = 0 \text{ for } (M, N) \in \mathcal{S}\}.$$

Since $\mathrm{rad}_B^{\infty}(V, W) \neq 0$, then $\mathrm{rad}_B^{l}(V, W) \neq 0$. Thus, there exists a nonzero B-module homomorphism $f = f_l f_{l-1} \cdots f_2 f_1 \in \mathrm{rad}_B^{l}(V, W)$, where each nonzero B-module homomorphism $f_j : V_{j-1} \longrightarrow V_j$ belongs to $\mathrm{rad}_B(V_{j-1}, V_j)$, and $V_1 = V$, $V_l = W$.

Observe that, for each B-module homomorphism $f_j : V_{j-1} \longrightarrow V_j$, and M_{j-1} and M_j two indecomposable A-modules related to V_{j-1} and V_j respectively, the A-module homomorphism $H(f_j) : H(V_{j-1}) \longrightarrow H(V_j)$ is a summand of the A-module homomorphism

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{1x_2} & \cdots & \lambda_{1x_n} \\ \lambda_{x_2 1} & \lambda_{x_2 x_2} & \cdots & \lambda_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{x_n 1} & \lambda_{x_n x_2} & \cdots & \lambda_{x_n x_n} \end{pmatrix} : \bigoplus_{x \in G} {}^{x}M_{j-1} \longrightarrow \bigoplus_{y \in G} {}^{y}M_j,$$

where $\lambda_{x_p 1} : M_{j-1} \longrightarrow {}^{x_p}M_j$ are A-module homomorphisms for all $x_p \in G$, and $\lambda_{x_p x_q} : {}^{x_q}M_{j-1} \longrightarrow {}^{x_p}M_j$ is $\alpha(x_p x_q^{-1}, x_q)\, {}^{x_q}\lambda_{(x_p x_q^{-1})1}$. Since $H(f_j)$ is nonzero, then there is at least one nonzero A-module homomorphism $\lambda_{x_p 1} : M_{j-1} \longrightarrow {}^{x_p}M_j$. Notice that $f = f_l f_{l-1} \cdots f_2 f_1$ is nonzero, then there must be a composition $\varphi = \varphi_l \varphi_{l-1} \cdots \varphi_2 \varphi_1$ of nonzero A-module homomorphisms, where each φ_j belongs to $\mathrm{Hom}_A(M_{j-1}, {}^{x_p}M_j)$ for some $x_p \in G$.

We claim that φ_j belongs to $\mathrm{rad}_A(M_{j-1}, {}^{x_p}M_j)$, for $j = 1, \ldots, l$. In fact, each $H(f_j)$ is nonisomorphic since f_j is nonisomorphic, by Lemma 2.2. Therefore, each $H(f_j)$ belongs to $\mathrm{rad}_A(H(V_{j-1}), H(V_j))$. Hence, each fixed φ_j belongs to $\mathrm{rad}_A(M_{j-1}, {}^{x_p}M_j)$, by Lemma 2.6. So we have a nonzero A-module homomorphism

$$\varphi = \varphi_l \varphi_{l-1} \cdots \varphi_2 \varphi_1 \in \mathrm{rad}_A^{l}(M', N') = \mathrm{rad}_A^{\infty}(M', N')$$

with some $(M', N') \in \mathcal{S}$. This contracts to the maximal choice of l and completes the proof. □

LEMMA 3.6. *Let $\mathcal{C}_A$ be a preprojective component. If $\mathcal{C}_A$ is faithful, then $\mathcal{C}_A$ contains all the indecomposable projective A-modules.*

Proof. Recall that a component $\mathcal{C}_A$ is called *sincere* if any simple A-module occurs as a simple composition factor of a module in $\mathcal{C}_A$. Since $\mathcal{C}_A$ is faithful, then it is sincere (see [**16**, Preliminaries]). Therefore, for any indecomposable projective A-module P, there is at least one module $M \in \mathcal{C}_A$, such that $\mathrm{Hom}_A(P, M) \neq 0$. This implies that P lies in the projective component $\mathcal{C}_A$ by [**1**, Chapter VIII, Corollary 2.6], which means that $\mathcal{C}_A$ contains all the indecomposable projective A-modules. □

Finally, let us recall the following useful criterion for tilted algebras (see [**8**, Theorem 1.6] and [**15**, Theorem 3]) and a description of the shapes of all components of tilted algebras ([**9**, Theorem 3.7]).

LEMMA 3.7. *A connected artin algebra A is a tilted algebra if and only if the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A admits a generalized standard component $\mathcal{C}_A$ with a faithful section Σ_A.*

LEMMA 3.8. *Let A be a connected artin tilted algebra and $\mathcal{C}_A$ be a component of $\Gamma(\mathrm{mod}\, A)$. Then, $\mathcal{C}_A$ is of one of the following shapes: the connecting component;*

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*the preprojective component; the preinjective component; quasi-serial; the component obtained from a quasi-serial translation quiver by ray insertions or by coray insertions, see [**14**].*

Proof of Theorem 1.1 First, assume that A be a tilted algebra. We prove that the crossed product $B = A_\alpha \#_\sigma G$ is also a tilted algebra.

By Lemma 3.7, $\Gamma(\operatorname{mod} A)$ has a generalized standard component $\mathcal{C}_A$ with a faithful section Σ_A. Put $\Sigma_A = \{L_1, \ldots, L_t\}$, and define Σ_B as the set of all indecomposable B-modules W which is a summand of $F(L_i)$ for some $L_i \in \Sigma_A$. Obviously, Σ_B is a finite set.

Choose an indecomposable B-module V such that L_1 is an indecomposable A-module related to V. Denote by $\mathcal{C}_B$ the component containing V. It follows that $\mathcal{C}_B$ is acyclic from Lemma 3.3 since the component $\mathcal{C}_A$ is acyclic. We now claim that $\Sigma_B \cap \mathcal{C}_B$ meets each τ_B-orbit in $\mathcal{C}_B$. That is, for any given module W in $\mathcal{C}_B$, there exists some module $U \in \Sigma_B \cap \mathcal{C}_B$ such that $W \simeq \tau_B^i U$ for some integer $i \in \mathbb{Z}$. By Lemma 3.2, there exists an indecomposable A-module M related to W lies in $\mathcal{C}_A$. Then, there is some $L \in \Sigma_A$ such that $M \simeq \tau_A^i L$ for some integer $i \in \mathbb{Z}$. This yields that W is a summand of $FM \simeq F(\tau_A^i L) \simeq \tau_B^i F(L)$ by Corollary 2.4. Hence, there exists a module $U \in \Sigma_B \cap \mathcal{C}_B$ such that $W \simeq \tau_B^i U$, by Lemma 2.3(2).

Now, we claim that one can choose a connected full subquiver Σ'_B of $\Gamma(\operatorname{mod} B)$ in the finite set $\Sigma_B \cap \mathcal{C}_B$, which is a section of $\mathcal{C}_B$.

For this purpose, choose an indecomposable B-module $U \in \Sigma_B \cap \mathcal{C}_B$ with an indecomposable A-module L related to U. Consider its neighbours $U^+ \bigcup U^-$. If $W \in U^-$, that is, there is an irreducible morphism $W \longrightarrow U$ in $\mathcal{C}_B$. Then, there exists an indecomposable A-module M related to W, such that there is an irreducible morphism $M \longrightarrow L$ in $\mathcal{C}_A$ by Corollary 2.5(4). Since Σ_A is a section of $\mathcal{C}_A$, then we have that either $M \in \Sigma_A$ or $\tau_A^{-1}(M) \in \Sigma_A$. If $M \in \Sigma_A$, then $W \in \Sigma_B \cap \mathcal{C}_B$. If $\tau_A^{-1}(M) \in \Sigma_A$, then $\tau_B^{-1}(W) \in \Sigma_B \cap \mathcal{C}_B$. Denote this indecomposable B-module belonging to $\Sigma_B \cap \mathcal{C}_B$ by U'. Likewise, if $W \in U^+$, we can find an indecomposable B-module that belongs to $\Sigma_B \cap \mathcal{C}_B$ and denote it by U''. Now, for each τ_B-orbit in the neighbours of U, we just select one indecomposable B-module $U^* \in \Sigma_B \cap \mathcal{C}_B$ (that is, either $U^* = U'$ or $U^* = U''$), and define that U^* and all arrows between U^* and U belong to Σ'_B. Continue this process, we can get a connected full subquiver Σ'_B of $\Gamma(\operatorname{mod} B)$ from $\Sigma_B \cap \mathcal{C}_B$. This subquiver is acyclic since the component $\mathcal{C}_B$ is. And also, it is convex, and meets each τ_B-orbit exactly once by the construction. So, it is a section of $\mathcal{C}_B$.

We show that $\mathcal{C}_B$ is a generalized standard component. If $\mathcal{C}_A$ is a preprojective component (resp. a preinjective component), then $\mathcal{C}_A$ contains all the indecomposable projective modules (resp. indecomposable injective modules) since $\mathcal{C}_A$ is faithful by Lemma 3.6 (resp. by the dual version of Lemma 3.6). Thus, $\mathcal{C}_A$ is the unique preprojective component (resp. preinjective component). Therefore, ${}^x\Sigma_A \subseteq \mathcal{C}_A$ for all $x \in G$, by using the facts that the exact functors F and H preserve projective modules (resp. injective modules) and F and H commute with the Auslander–Reiten translations (see Corollary 2.4). If $\mathcal{C}_A$ is neither a preprojective component nor a preinjective component, then we also have that ${}^x\Sigma_A \subseteq \mathcal{C}_A$ for all $x \in G$, by comparing the shape of components in $\Gamma(\operatorname{mod} A)$ by Lemma 3.8. This implies that $\mathcal{C}_A$ is a G-stable component since Σ_A is a section of $\mathcal{C}_A$ and ${}^x\Sigma_A \subseteq \mathcal{C}_A$ for all $x \in G$. Thus, it follows from Lemma 3.5 that $\mathcal{C}_B$ is a generalized standard component.

Finally, we show that Σ'_B is faithful. Let $\widetilde{U}$ be the direct sum of all modules forming the vertices of Σ'_B with a set of generators $\{u_1, \ldots, u_s\}$. Let $f : B \longrightarrow \widetilde{U}^s$ be

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the B-module homomorphism defined by $f(1) = (u_1, \ldots, u_s)$. Then, Σ'_B is faithful is and only if the homomorphism f is monic (see [**2**, p. 317]). Notice that H preserves epimorphism by Lemma 2.2(2). Hence, H preserves generators of modules. Therefore, we have an A-module homomorphism $H(f) : A^{|G|} \simeq H(B) \longrightarrow H(\widetilde{U}^s) \simeq H(\widetilde{U})^s$ which sends $(1, 0, \ldots, 0)$ to $(m_1, \ldots, m_s)$ with $\{m_1, \ldots, m_s\}$ a set of generators of $H(\widetilde{U})$. Thus, Σ'_B is faithful in mod B if and only if $H(\widetilde{U})$ is faithful in mod A, by Lemma 2.2(2). Denote by Ω the set of indecomposable summands of $H(\widetilde{U})$. Then $\Omega \subseteq \mathcal{C}_A$, by the fact that $\mathcal{C}_A$ is G-stable. This implies $\mathrm{ann}_A(H(\widetilde{U})) = \mathrm{ann}_A(\Omega) \subseteq \mathrm{ann}_A(\Sigma_A) = 0$, by combing Lemmas 2.1(5) and 3.4. Hence, we conclude that Σ'_B is faithful.

We have shown that Σ'_B is a faithful section of the generalized standard component $\mathcal{C}_B$. By Lemma 3.7, B is a tilted algebra.

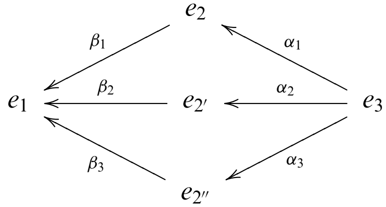
The proof of the converse is analogous, and we just sketch the proof. Let B be a tilted algebra, and let $\Sigma_B = \{W_1, \ldots, W_s\}$ be a faithful section of a generalized standard component $\mathcal{C}_B$ in $\Gamma(\mathrm{mod}\, B)$. Put Σ_A as the set of all indecomposable A-modules M related to some $W \in \Sigma_B$. It is easy to see that Σ_A is finite. Select a component $\mathcal{C}_A$ containing a module $M \in \Sigma_A$. Then, we get a finite set $\Sigma_A \bigcap \mathcal{C}_A$ of $\Gamma(\mathrm{mod}\, A)$, which meets each τ_A-orbit in $\mathcal{C}_A$.

Carrying a similar procedure of the construction of the section Σ'_B as before, one can find a connected full subquiver Σ'_A of $\Gamma(\mathrm{mod}\, A)$ in the finite set $\Sigma_A \cap \mathcal{C}_A$, such that Σ'_A a section of $\mathcal{C}_A$. Moreover, $\mathcal{C}_B$ is closed under taking F-summands by a similar investigating the components in $\Gamma(\mathrm{mod}\, B)$ as the proof of the necessity. Then, it follows that $\mathcal{C}_A$ is a generalized standard component from Lemma 3.5. Thus, we can draw a conclusion that Σ'_A is a section of the generalized standard component $\mathcal{C}_A$.

Finally, we show that Σ'_A is faithful. Let $\widetilde{L}$ be the direct sum of all modules forming the vertices of Σ'_A with a set of generators $\{l_1, \ldots, l_t\}$. Let $f : A \longrightarrow \widetilde{L}^t$ be the A-module homomorphism defined by $f(1) = (l_1, \ldots, l_t)$. Notice that f is monic if and only if $F(f) : F(A) = B \longrightarrow F(\widetilde{L}^t) \simeq F(\widetilde{L})^t$ is monic by Lemma 2.2(1). Then, the faithfulness of Σ'_A follows from the fact that $\mathrm{ann}_B(F(\widetilde{L})) \subseteq \mathrm{ann}_B(\Sigma_B) = 0$. Again by Lemma 3.7, A is a tilted algebra. We have completed all the proof. □

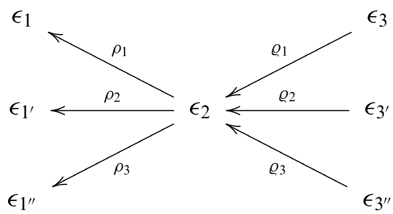
At the end of this paper, we illustrate Theorem 1.1 by the following example, in which the tilted algebras are representation-infinite.

Example 3.9. Let k be an algebraically closed field with char $k \neq 3$, and A the canonical k-algebra $C(2, 2, 2)$ of the Euclidean type $\widetilde{\mathbb{D}}_4$ given by the bound quiver (Q_A, I_A):



with the relation $\beta_1\alpha_1 + \beta_2\alpha_2 + \beta_3\alpha_3 = 0$. Let $G = \{1, x, x^2\}$ be a cyclic group of order 3 with generator x, which acts on A by $x(e_2) = e_{2'}$, $x(e_{2'}) = e_{2''}$, $x(e_{2''}) = e_2$, $x(\beta_1) = \beta_2$, $x(\beta_2) = \beta_3$, $x(\beta_3) = \beta_1$, $x(\alpha_1) = \alpha_2$, $x(\alpha_2) = \alpha_3$, $x(\alpha_3) = \alpha_1$, and by fixing e_1 and e_3. Then, we obtain the skew group algebra $B = AG$, which is Morita equivalent to a basic and connected finite dimensional k-algebra $B' = (B)^{basic}$.

Compute its ordinary quiver $Q_{B'}$ of B' as follows:

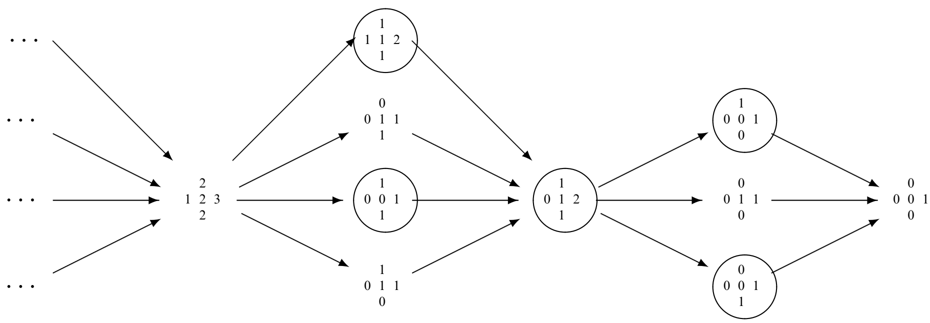


where

$$\epsilon_2 = e_2,$$
$$\epsilon_1 = \tfrac{1}{3}(e_1 + e_1x + e_1x^2),\ \epsilon_{1'} = \tfrac{1}{3}(e_1 - (-1)^{\frac{1}{3}}e_1x + (-1)^{\frac{2}{3}}e_1x^2),$$
$$\epsilon_{1''} = \tfrac{1}{3}(e_1 + (-1)^{\frac{2}{3}}e_1x - (-1)^{\frac{1}{3}}e_1x^2),$$
$$\epsilon_3 = \tfrac{1}{3}(e_3 + e_3x + e_3x^2\ ),\ \epsilon_{3'} = \tfrac{1}{3}(e_3 - (-1)^{\frac{1}{3}}e_3x + (-1)^{\frac{2}{3}}e_3x^2),$$
$$\epsilon_{3''} = \tfrac{1}{3}(e_3 + (-1)^{\frac{2}{3}}e_3x - (-1)^{\frac{1}{3}}e_3x^2),$$
$$\rho_1 = \beta_1 + \beta_2x + \beta_3x^2, \quad \rho_2 = \beta_1 - (-1)^{\frac{1}{3}}\beta_2x + (-1)^{\frac{2}{3}}\beta_3x^2,$$
$$\rho_3 = \beta_1 + (-1)^{\frac{2}{3}}\beta_2x - (-1)^{\frac{1}{3}}\beta_3x^2,$$
$$\varrho_1 = \alpha_1 + \alpha_1x + \alpha_1x^2,\ \varrho_2 = \alpha_1 - (-1)^{\frac{1}{3}}\alpha_1x + (-1)^{\frac{2}{3}}\alpha_1x^2,$$
$$\varrho_3 = \alpha_1 + (-1)^{\frac{2}{3}}\alpha_1x - (-1)^{\frac{1}{3}}\alpha_1x^2.$$

Using the explicit description of the arrows given above, one can directly compute the relations $I_{B'}$ as $\rho_i\varrho_i = 0$ for $i = 1, 2, 3$.

It is well known that $\Gamma(\mathrm{mod}\, A)$ has a preinjective component $\mathcal{C}_A = \mathcal{Q}$ containing a faithful section Σ_A. We depict this component as follows, and denote the modules of Σ_A by circles.



It is not difficult to see that Σ_A is not stable under the action of G. In fact, we have

$${}^{x}\left(\begin{smallmatrix} & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}\right) \simeq \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix},\ {}^{x}\left(\begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right) \simeq \begin{smallmatrix} & 0 & \\ 0 & 0 & 1 \\ & 1 & \end{smallmatrix},\ {}^{x}\left(\begin{smallmatrix} & 1 & \\ 0 & 0 & 1 \\ & 1 & \end{smallmatrix}\right) \simeq \begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix},\ {}^{x}\left(\begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right) \simeq \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{smallmatrix},$$

and the rest modules in Σ_A are fixed points under the action of G. Apply the composition, which is still denoted by F, of the functor $F = -\bigotimes_A B : \mathrm{mod}\, A \longrightarrow$

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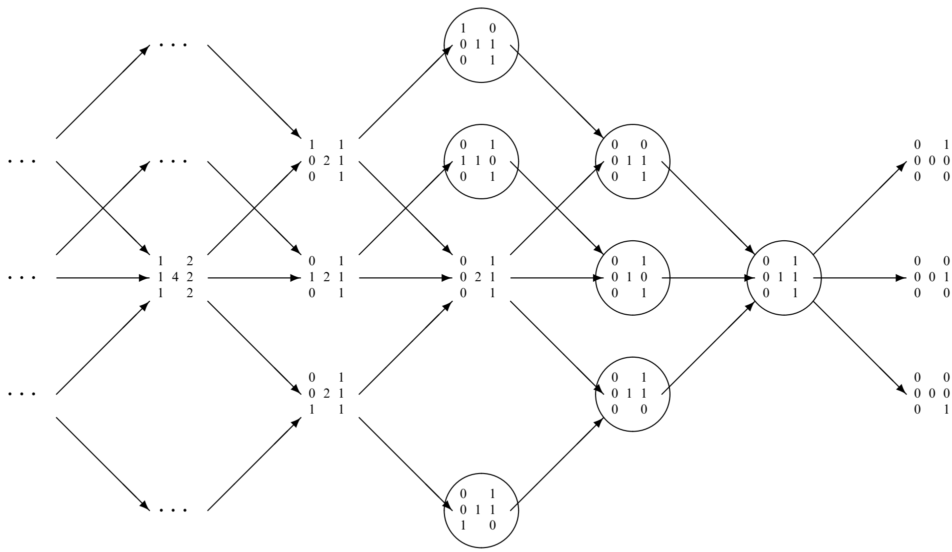
mod B and the Morita equivalent functor $\operatorname{mod} B \xrightarrow{\sim} \operatorname{mod} B'$, on the section Σ_A, we get

$$F\left(\begin{smallmatrix}&1&\\0&0&1\\&0&\end{smallmatrix}\right) \simeq F\left(\begin{smallmatrix}&0&\\0&1&1\\&0&\end{smallmatrix}\right) \simeq F\left(\begin{smallmatrix}&0&\\0&0&1\\&1&\end{smallmatrix}\right) \simeq \begin{smallmatrix}0&&1\\0&1&1\\0&&1\end{smallmatrix} = I(\epsilon_2),\ F\left(\begin{smallmatrix}&1&\\0&0&1\\&1&\end{smallmatrix}\right) \simeq F\left(\begin{smallmatrix}&0&\\0&1&1\\&1&\end{smallmatrix}\right) \simeq F\left(\begin{smallmatrix}&1&\\0&1&1\\&0&\end{smallmatrix}\right) \simeq \begin{smallmatrix}0&&1\\0&2&1\\0&&1\end{smallmatrix},$$

$$F\left(\begin{smallmatrix}&1&\\0&1&2\\&1&\end{smallmatrix}\right) \simeq \begin{smallmatrix}0&&2\\0&3&2\\0&&2\end{smallmatrix} \simeq \begin{smallmatrix}0&&0\\0&1&1\\0&&1\end{smallmatrix} \bigoplus \begin{smallmatrix}0&&1\\0&1&0\\0&&1\end{smallmatrix} \bigoplus \begin{smallmatrix}0&&1\\0&1&1\\0&&0\end{smallmatrix}, \text{ and}$$

$$F\left(\begin{smallmatrix}&1&\\1&1&2\\&1&\end{smallmatrix}\right) \simeq \begin{smallmatrix}1&&2\\1&3&2\\1&&2\end{smallmatrix} \simeq \begin{smallmatrix}1&&0\\0&1&1\\0&&1\end{smallmatrix} \bigoplus \begin{smallmatrix}0&&1\\1&1&0\\0&&1\end{smallmatrix} \bigoplus \begin{smallmatrix}0&&1\\0&1&1\\1&&0\end{smallmatrix} = I(\epsilon_1) \bigoplus I(\epsilon_{1'}) \bigoplus I(\epsilon_{1''}).$$

By Theorem 1.1, $\Gamma(\operatorname{mod} B')$ has a generalized standard component $\mathcal{C}_{B'}$ containing a faithful section $\Sigma_{B'}$ whose elements are the modules denoted by circles as follows.



By [**1**, Chapter VI, Section 6], we know that B' is now tilted by the path algebra C of the quiver

$$\begin{array}{ccccccccc} & & & & 7 & & & & \\ & & & & \uparrow & & & & \\ & & & & 6 & & & & \\ & & & & \uparrow & & & & \\ 1 & \longleftarrow & 2 & \longleftarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 \end{array}$$

of the Euclidean type $\widetilde{\mathbb{E}}_6$. Denote the modules in $\Sigma_{B'}$ by $U_1 = \begin{smallmatrix}1&&0\\0&1&1\\0&&1\end{smallmatrix}$, $U_2 = \begin{smallmatrix}0&&0\\0&1&1\\0&&1\end{smallmatrix}$, $U_3 = \begin{smallmatrix}0&&1\\0&1&1\\0&&1\end{smallmatrix}$, $U_4 = \begin{smallmatrix}0&&1\\0&1&1\\0&&0\end{smallmatrix}$, $U_5 = \begin{smallmatrix}0&&1\\0&1&1\\1&&0\end{smallmatrix}$, $U_6 = \begin{smallmatrix}0&&1\\0&1&0\\0&&1\end{smallmatrix}$, $U_7 = \begin{smallmatrix}0&&1\\1&1&0\\0&&1\end{smallmatrix}$. Then, we get a tilting C-module

$$T = \begin{smallmatrix}&&0&&\\&&0&&\\1&0&0&0&0\end{smallmatrix} \bigoplus \begin{smallmatrix}&&1&&\\&&0&&\\0&0&0&0&0\end{smallmatrix} \bigoplus \begin{smallmatrix}&&0&&\\&&0&&\\0&0&0&0&1\end{smallmatrix} \bigoplus \begin{smallmatrix}&&1&&\\&&1&&\\1&1&1&1&1\end{smallmatrix} \bigoplus \begin{smallmatrix}&&1&&\\&&1&&\\0&0&1&1&1\end{smallmatrix} \bigoplus \begin{smallmatrix}&&0&&\\&&0&&\\1&1&1&1&1\end{smallmatrix} \bigoplus \begin{smallmatrix}&&1&&\\&&1&&\\0&1&1&0&0\end{smallmatrix}$$

such that $\operatorname{Hom}_C(T, T) \simeq B'$.

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