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BOREL DIRECTIONS AND ITERATED ORBITS OF MEROMORPHIC FUNCTIONS

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For transcendental meromorphic functions of finite order, we prove that there exist iterated orbits which tend to the Borel directions. This gives a relation between the value distribution theory and the iteration theory of meromorphic functions.

1. INTRODUCTION

Suppose $f : \mathbf{C} \to \overline{\mathbf{C}}$ is a transcendental meromorphic function. If for any $\varepsilon > 0$, f takes every complex value a infinitely many times on the region: $|\arg z - \theta_0| < \varepsilon$, with at most two exceptional values $a \in \overline{\mathbf{C}}$, then the ray $\arg z = \theta_0$ is said to be a Julia direction of $f(z)$. Furthermore, if for any $\varepsilon > 0$,

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, \theta_0, \varepsilon, f = a)}{\log r} \geq \omega > 0,$$

with at most two exceptional values of $a \in \overline{\mathbf{C}}$, where $n(r, \theta_0, \varepsilon, f = a)$ is the number of roots of $f(z) = a$ on the region: $|z| < r$ and $|\arg z - \theta_0| < \varepsilon$, then the ray $\arg z = \theta_0$ is said to be a Borel direction of order at least ω. These are fundamental concepts in value distribution theory [**5**].

In this note, we deal with the problem: Can we choose an iterated orbit such that it approximates to the Borel directions? Define

$$I(f) = \Big\{ z \in \mathbf{C} \;\Big|\; f^n(z) \neq \infty \text{ for all } n \text{ and } f^n(z) \to \infty \text{ as } n \to \infty \Big\},$$

where f^n is the n-th iterate of f, that is, $f^0(z) = z$ and $f^n(z) = f \circ f^{n-1}(z)$ for $n \geq 1$. $f^n(z)$ is defined for all $z \in \mathbf{C}$ except for a countable set which consists of the poles of $f, f^2, \cdots, f^{n-1}$. Obviously, the forward orbit $O^+(a) = \{f^n(a) \mid n \geq 0\}$ is an infinite set if $a \in I(f)$. We want to find a point $a \in \mathbf{C}$ such that $a \in I(f)$ and each limiting direction of $O^+(a)$ (that is, a limit of $\{\arg z \mid z \in O^+(a)\}$) is a Borel direction of f. By $J(f)$ denote the Julia set of f which is the closure of the set of the repelling periodic points; its complement $F(f)$ is the Fatou set (see [**2**]). In this note we shall prove

THEOREM 1. *Let $f(z)$ be a transcendental meromorphic function, then $I(f) \cap J(f) \neq \emptyset$.*

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REMARK. Eremenko [3] has proved this result for transcendental entire functions.

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function of finite order, the lower order $\mu > 0$. Then there exists a point $a \in I(f) \cap J(f)$ such that each limiting direction of $O^+(a)$ is a Borel direction of order at least μ.*

REMARK. It is well known that there exist transcendental meromorphic functions of lower order zero which don't have a Julia direction [5].

Since the backward orbit $O^-(a) = \{z \mid f^n(z) = a \text{ for some } n\}$ is dense on $J(f)$ for every point $a \in J(f)$ with at most one exceptional point [2], we easily have

COROLLARY. *Let $f(z)$ be a transcendental meromorphic function of finite order, the lower order $\mu > 0$. Then there is a dense subset I_B of $J(f)$ such that, for $a \in I_B$, $O^+(a)$ tends to infinity and each limiting direction of $O^+(a)$ is a Borel direction of order at least μ.*

2. THE PROOF OF THEOREM 1

In order to prove Theorem 1, we need the following lemma:

LEMMA 1. [1] *Suppose, in a domain D, the analytic functions f of the family G omit the values $0, 1$, and H is a compact subset of D on which the functions all satisfy $|f(z)| \geq 1$. Then there exist constants k, t, dependent only on H and D, such that for any $z, z' \in H$ and any $f \in G$ we have $|f(z')| < k|f(z)|^t$.*

THE PROOF OF THEOREM 1: We distinguish the following two cases:

A. $f(z)$ has infinitely many poles. Let a_0 be a pole of $f(z)$, then there exists a constant $R > 1$ such that $f(V(a_0)) \supset \{z \mid |z| > R\}$, where $V(\eta) = \{z \mid |z - \eta| < 1\}$. Choose a pole $a_1 \in \{z \mid |z| > R + 2\}$, then $f(V(a_0)) \supset \overline{V(a_1)}$. Since a_1 is also a pole, there exists a constant $l_1 \geq 2$ such that $f(V(a_1)) \supset \{z \mid |z| > R^{l_1}\}$. By repeating this construction, we obtain a sequence of disks $V(a_j)$ (a_j is a pole) such that $V(a_j) \to \infty$ and

$$f(V(a_j)) \supset \overline{V(a_{j+1})} \qquad (j = 0, 1, 2, \ldots).$$

It is obvious that there exists a sequence of domains $B_j \subset V(a_0)$ such that $\overline{B_{j+1}} \subset B_j$ and $f^j(B_j) = V(a_j)$. For a point $a \in \bigcap_{j=1}^{\infty} \overline{B_j}$, we have $a \in I(f)$. Since a_j is a pole, then $V(a_j) \cap J(f) \neq \emptyset$, and thus $B_j \cap J(f) \neq \emptyset$ for all j [2]. So we have $a \in I(f) \cap J(f)$.

B. $f(z)$ has only finitely many poles. By Mittag-Leffler's theorem,

$$f(z) = g(z) + \sum_{j=1}^{m} P_j\left(\frac{1}{z - a_j}\right), \qquad (1)$$

where $g(z)$ is a transcendental entire function, a_j $(j = 1, \cdots, m)$ are m $(< \infty)$ distinct poles of $f(z)$, and P_j is a polynomial with $P_j(0) = 0$. For a transcendental entire function $g(z)$, Eremenko [3] proved: there exist a sequence of positive numbers $r_j \to \infty$, a constant $b > 1$ and a sequence of domains $\sigma_j \subset \{z \mid r_j/b < |z| < br_j\}$ such that

$$g(\sigma_j) \supset \left\{z \,\middle|\, \frac{1}{b_1} r_{j+1} < |z| < b_1 r_{j+1}\right\} \quad (j = 1, 2, \cdots), \tag{2}$$

where $b_1 > b$ is a constant. For a constant $b_2 \in (b,\ b_1)$, by (1) and (2) we deduce that there exists $j_0 > 0$ such that

$$f(\sigma_j) \supset \left\{z \,\middle|\, \frac{1}{b_2} r_{j+1} < |z| < b_2 r_{j+1}\right\} \supset \sigma_{j+1} \tag{3}$$

when $j \geq j_0$. So there exists a sequence of domains $B_p \subset \sigma_{j_0}$ such that

$$f^p(B_p) = \sigma_{j_0+p}, \quad \overline{B_{p+1}} \subset B_p, \quad p = 1, 2, \cdots. \tag{4}$$

It follows that $\bigcap\limits_{p=1}^{\infty} \overline{B_p} \subset I(f)$, thus $I(f) \neq \emptyset$.

If $\left(\bigcap\limits_{p=1}^{\infty} \overline{B_p}\right) \cap J(f) \neq \emptyset$, we have $I(f) \cap J(f) \neq \emptyset$. Below we suppose $\left(\bigcap\limits_{p=1}^{\infty} \overline{B_p}\right) \cap J(f) = \emptyset$, then there exists $p_0 \geq 1$ such that $B_p \subset F(f)$ when $p \geq p_0$. By (3) and (4) we have

$$\left\{z \,\middle|\, \frac{1}{b_2} r_j < |z| < b_2 r_j\right\} \subset F(f) \tag{5}$$

when $j \geq p_0 + j_0 + 1$.

Now, we prove that $F(f)$ has only bounded components: Assume D is an unbounded component of $F(f)$. By (3) and (5) we know that $f(D) \subset D, f^n(z) \to \infty$ for $z \in D$ and $\overline{\sigma_j} \subset D$ when $j \geq p_0 + j_0 + 1$. Put

$$H = \overline{\sigma_{p_0+j_0+1} \cup f(\sigma_{p_0+j_0+1})},$$

then $H \subset D$. Without loss of generality, we may assume $0, 1 \in J(f)$ and $|f^n(z)| \geq 1$ on H for all n. By Lemma 1, for any $z' \in \sigma_{p_0+j_0+1}$ we have

$$|f^{n+1}(z')| < k|f^n(z')|^t, \quad n = 1, 2, \cdots, \tag{6}$$

where k and t are two constants. Put $\Omega = \bigcup\limits_{n=0}^{\infty} f^n(\sigma_{p_0+j_0+1})$, then for any $z \in \Omega$, there exist a point $z' \in \sigma_{p_0+j_0+1}$ and a natural number n such that $f^n(z') = z$. By (6) we get

$$|f(z)| < k|z|^t, \quad z \in \Omega.$$

Noting $\Omega \supset \{z \mid r_j/b < |z| < br_j\}$ for sufficiently large j, we have

$$M(r_j, g) = M(r_j, f) + o(1) = O(r_j^t) \quad (r_j \to \infty).$$

This contradicts the transcendence of $g(z)$. Therefore $F(f)$ has only bounded components.

Denote the component of $F(f)$ containing B_{p_0} by D_0. Since $B_{p_0} \cap I(f) \neq \emptyset$, so $f^n(z) \to \infty$ for $z \in D_0$. It follows from (5) and the boundedness of D_0 that $f^n(\partial D_0) \to \infty$, and thus $\partial D_0 \subset I(f) \cap J(f)$. The proof of Theorem 1 is complete. $\square$

3. The proof of Theorem 2

Denote the Nevanlinna characteristic function of $f(z)$ by $T(r, f)$ [5]. Since f is of positive lower order and finite order, there exists a constant $\alpha > 1$ such that $T(2r, f) < T^\alpha(r, f)$ for sufficiently large r. Therefore, Theorem 2 is the corollary of the following result:

Theorem 3. *Let $f(z)$ be a transcendental meromorphic function of lower order $\mu \in (0, \infty)$. If*

$$\varlimsup_{r\to\infty} \frac{\log T(2r, f)}{\log T(r, f)} < \infty,$$

then there exists a point $a \in I(f) \cap J(f)$ such that each limiting direction of $O^+(a)$ is a Borel direction of order at least μ.

In order to prove Theorem 3, we need the following lemmas:

Lemma 2. [5] *Let f be a transcendental meromorphic function. If R is sufficiently large to satisfy*

$$T(R, f) \geq \max\left\{240, \frac{240 \log(2R)}{\log k}, 12T(r, f), \frac{12T(kr, f)}{\log k} \log \frac{2R}{r}\right\},$$

then there exists a point z_j lying in $r < |z| < 2R$ such that in the domain

$$\Gamma: \quad |z - z_j| < \frac{4\pi}{q}|z_j|,$$

f takes every complex value at least

$$n = c^* \frac{T(R, f)}{q^2 \left(\log \frac{r}{R}\right)^2}$$

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times except for those complex values which can be contained in two spherical disks each with radius e^{-n}, *where* $k > 1$ *is a constant,* q *is a sufficiently large integer, and* $c^* > 0$ *is an absolute constant. The disk* Γ *is called a filling disk of* $f(z)$.

Lemma 3. [4] *Let* $T(r)$ *be a positive, increasing and continuous function, and* $T(r) \to +\infty\,(r \to +\infty)$. *If*

$$\varlimsup_{r\to\infty} \frac{\log T(r)}{\log r} \le \nu < +\infty,$$

then for any two numbers $\tau_1 > 1$, $\tau_2 > 1$, *the lower logrithmic density of the set* $\{r \mid T(\tau_1 r) \le \tau_2 T(r)\}$ *is not less than* $1 - (\nu \log \tau_1)/(\log \tau_2)$.

Lemma 4. *Let* $T(r)$ *be a positive, increasing and continuous function, and* $T(r) \to +\infty\ (r \to +\infty)$. *If*

$$\varliminf_{r\to\infty} \frac{\log T(r)}{\log r} \ge \omega > 0,$$

where $\tau_1 > 1$, $\tau_2 > 1$ *are two constants satisfying* $\tau_2 < \tau_1^{\omega}$, *then for any constant* $m > 1/\big(1 - (\log \tau_2)/(\omega \log \tau_1)\big)$, *there exists a constant* $R_0 > 0$ *such that*

$$\{\ t \ \mid\ \tau_2 T(t) \le T(\tau_1 t)\} \cap \Big[r, T^{-1}\big(T^m(r)\big)\Big] \ne \emptyset$$

when $r > R_0$.

The proof of Lemma 4: Put $s = T(r)$, $T_0(s) = T^{-1}(s)$. Then $T_0(s)$ is a positive, increasing and continuous function, and $T_0(s) \to +\infty\ (s \to +\infty)$. Obviously,

$$\varlimsup_{s\to\infty} \frac{\log T_0(s)}{\log s} \le \frac{1}{\omega} < +\infty.$$

By Lemma 3,

$$\text{lower-logdens}\{s \mid T_0(\tau_2 s) \le \tau_1 T_0(s)\} \ge 1 - \frac{\log \tau_2}{\omega \log \tau_1}.$$

Therefore, there exists $s_0 \in [s, s^m]$ such that $T_0(\tau_2 s_0) \le \tau_1 T_0(s_0)$ for sufficiently large s. Put $r_0 = T^{-1}(s_0)$, then $r_0 = T_0(s_0)$, $T(r_0) = s_0$. Thus $\tau_2 T(r_0) \le T(\tau_1 r_0)$. Since $r_0 \ge r$, $T(r_0) = s_0 \le s^m = T^m(r)$, we deduce $r_0 \in \big[r, T^{-1}(T^m(r))\big]$. The proof of Lemma 4 is complete. ∎

The proof of Theorem 3: Choose two constants $k > 1$ and $\tau_1 > 1$ such that

$$\frac{12}{\log k} \log(2k\tau_1) < \tau_1^{\mu}.$$

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Put

$$\tau_2 = \frac{12}{\log k}\log(2k\tau_1) \quad\text{and}\quad \alpha = \varlimsup_{r\to\infty}\frac{\log T(2r,f)}{\log T(r,f)}.$$

Choose a natural number m such that

$$m > \max\left(\frac{1}{1-(\log\tau_2)/(\mu\log\tau_1)}, 2\alpha\right). \tag{7}$$

For convenience, we put $T(r,f) = T(r)$. It is obvious that there exists a constant $M_0 > 0$ such that

$$M_0 > \max\{R_0, e^{8\pi}\}, \tag{8}$$

$$T(r) > \max\left\{\frac{1}{K}(\log r)^{2m^{4p+1}+2}, \ \frac{240\log(2r)}{\log k}\right\}, \tag{9}$$

$$T(2r,f) < T^{2\alpha}(r,f), \tag{10}$$

$$c\cdot c^*\frac{\tau_1^{\mu/2}r^{\mu/4}}{(\log(k\tau_1)\log r)^2} > 1 \tag{11}$$

when $r \geq M_0$, where $R_0 > 0$ is the constant stated in Lemma 4, $c^* > 0$ is the constant stated in Lemma 2, and

$$c = \frac{1}{1+9\tau_1^2}, \quad K = \frac{c^{\mu/2}(\mu/2)^{m^{2p+1}+1}}{(m^{4p+1}+1)!}\frac{(c^*)^{m^{2p+1}+1}}{(\log(k\tau_1))^{2m^{4p+1}+2}}, \quad p = \left[\frac{\log(6k\tau_1)}{\log 2}\right]+2, \tag{12}$$

(where [.] denotes the integral part). Put $r^* = \max\{M_0, M_0^{4/\mu}\}$. From (11) we deduce that

$$c\cdot c^*\frac{(\tau_1 r)^{\mu/2}}{(\log(k\tau_1)\log r)^2} > r^{\mu/4} \geq M_0 \tag{13}$$

for $r \geq r^*$.

By Lemma 4, there exists $r_0 \in \left[r^*, T^{-1}(T^m(r^*))\right]$ such that

$$\tau_2 T(r_0) \leq T(\tau_1 r_0).$$

Put $r_1 = r_0/k$, $R_1 = \tau_1 r_0$, then

$$\frac{12}{\log k}\log\frac{2R_1}{r_1}T(kr_1) \leq T(R_1), \tag{14}$$

and

$$12T(r_1) \leq \frac{12}{\log k}\log\frac{2R_1}{r_1}T(kr_1) \leq T(R_1). \tag{15}$$

By (8), (9), (14), (15) and Lemma 2, there exists z_0 lying in $r_1 < |z| < 2R_1$ such that in the disk

$$\Gamma_0 : |z - z_0| < \frac{4\pi}{\log r^*}|z_0|$$

f takes every complex value a at least

$$n_0 = c^* \frac{T(R_1)}{(\log r^*)^2 \left(\log (k\tau_1)^2\right)}$$

times except for those complex values which can be contained in two spherical disks γ_0' and γ_0'' with radius e^{-n_0}, that is, Γ_0 is a filling disk of $f(z)$. Obviously,

$$n_0 \geq \frac{c^*}{(\log k\tau_1)^2} \frac{T\left(1/2|z_0|\right)}{\left(\log\left(k|z_0|\right)\right)^2} \geq \left(|z_0|\right)^{\mu - \varepsilon\left(|z_0|\right)}, \tag{16}$$

where $\varepsilon(r) > 0$, and $\varepsilon(r) \to 0$ as $r \to \infty$. It can be easily verified from (8) that

$$\Gamma_0 \subset \left\{ z \;\middle|\; \frac{1}{2k} r^* < |z| < 3\tau_1 T^{-1}\left(T^m(r^*)\right) \right\}.$$

Put $t_j = T^{-1}\left(T^{m^j}(r^*)\right)$. It is obvious that $t_0 = r^*$, $\{t_j\}$ is an increasing sequence and $t_j \to \infty$. So the sequence of annuli $A_j = \{z \mid t_j/2k < |z| < 3\tau_1 t_{j+1}\}$ tends to infinity as $j \to \infty$ and $\Gamma_0 \subset A_0$. By $T(t_{j+1}) = T^m(t_j)$, (7) and (10) we get

$$T(t_{j+2}) = T^m(t_{j+1}) > T^{2\alpha}(t_{j+1}) > T(2t_{j+1}).$$

It follows that $t_{j+2} > 2t_{j+1}$, so $t_{j+p} > 2^{p-1} t_{j+1}$, and thus $t_{j+p} > 6k\tau_1 t_{j+1}$. Therefore,

$$A_j \cap A_{j+p} = \emptyset \quad (j = 0, 1, 2, \cdots).$$

Next we prove that there is at least one in five annuli $A_p, A_{2p}, A_{3p}, A_{4p}, A_{5p}$ which does not meet $\gamma_0' \cup \gamma_0''$. Assume γ_0' (or γ_0'') meet both A_{jp} and $A_{(j+2)p}$ ($j \in \{1, 2, 3\}$). Then we have

$$e^{-n_0} \geq \frac{3\tau_1 t_{(j+1)p+1} - \frac{1}{2k} t_{(j+1)p}}{\sqrt{1 + 9\tau_1^2 t_{(j+1)p+1}^2} \sqrt{1 + \frac{1}{4k^2} t_{(j+1)p}^2}} \geq \frac{c}{t_{(j+1)p+1}},$$

where $c > 0$ is the constant in (12). This means

$$T^{m^{(j+1)p+1}}(r^*) \geq T(ce^{n_0}). \tag{17}$$

On the other hand, by (10) and (13) we have

$$ce^{n_0} > cn_0 > c \cdot c^* \frac{(\tau_1 r^*)^{\mu/2}}{(\log(k\tau_1)\log r^*)^2} \geq M_0,$$

and hence

(18)

$$T(ce^{n_0}) > c^{\mu/2} e^{(\mu/2)n_0} > c^{\mu/2} \frac{(\mu/2)^{m^{(j+1)p+1}+1}}{(m^{(j+1)p+1}+1)!} n_0^{m^{(j+1)p+1}+1} > K \frac{T^{m^{(j+1)p+1}+1}(r^*)}{(\log r^*)^{2m^{4p+1}+2}},$$

where $K > 0$ is the constant in (12). By (17) and (18) we have

$$T(r^*) < \frac{1}{K}(\log r^*)^{2m^{4p+1}+2}.$$

This contradicts (9). Therefore, γ_0' (or γ_0'') can not meet both A_{jp} and $A_{(j+2)p}$ $(j \in \{1,2,3\})$. It follows immediately that there exists at least one in five annuli A_p, A_{2p}, A_{3p}, A_{4p}, A_{5p} which does not meet γ_0' or γ_0''. Denote this annulus by A_0^1. So $f(\Gamma_0) \supset A_0^1$.

By the same discussion, we can deduce that there exists a filling disk $\Gamma_1 \subset A_0^1$ and an annulus $A_0^2 \in \{A_j \mid j \in \mathbf{N}\}$ such that $f(\Gamma_1) \supset A_0^2$. Repeating this construction, we obtain a sequence of filling disks Γ_j such that

$$f(\Gamma_j) \supset \overline{\Gamma_{j+1}},\ \Gamma_j \to \infty\ (j \to \infty). \tag{19}$$

Denote the centre of Γ_j by z_j. From (16) we know that each limiting point of $\{\arg z_j \mid j = 1, 2, \cdots\}$ is a Borel direction of order at least μ (see [5]). It follows (19) that there is a sequence of domains $B_j \subset A_0$ such that $f^{j-1}(B_j) = \Gamma_j$ and $\Gamma_0 \supset B_j \supset \overline{B_{j+1}}$.

Now, we prove $\left(\bigcap_{j=1}^{\infty} \overline{B_j}\right) \cap J(f) \neq \emptyset$: Otherwise, there exists a natural number j_0 such that $B_j \subset F(f)$ when $j \geq j_0$. Since Γ_0 is a filling disk, we have $f^j(B_j) = f(\Gamma_j) \supset \overline{\mathbf{C}} \setminus (\gamma_j' \cup \gamma_j'')$ (where γ_j' and γ_j'' are two spherical disks each with radius e^{-n_j} and $n_j \to \infty$ as $j \to \infty$), so $J(f) \subset \gamma_j' \cup \gamma_j''$ when $j \geq j_0$. This implies $J(f)$ contains at most two points. This is a contradiction [2].

For a point $a \in \left(\bigcap_{j=1}^{\infty} \overline{B_j}\right) \cap J(f)$, we have $a \in I(f)$ and each limiting direction of $O^+(a)$ is a Borel direction of order at least μ. The proof of Theorem 3 is complete. □

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