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MEROMORPHIC FUNCTIONS WITH SHARED LIMIT VALUES OF POSITIVE LINEAR MEASURE

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Abstract We prove that if two transcendental meromorphic functions share all limit values from a set of positive linear measure on a rectifiable Jordan arc, then they share all limit values.

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1. Introduction and results

If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a transcendental meromorphic function then, by the Casorati–Weierstrass theorem for every $a \in \hat{\mathbb{C}}$, there exist sequences $z_n \to \infty$ such that $f(z_n) \to a$. In [**7**] we introduced the notion of *shared limit values* for meromorphic functions f and g in the plane: f and g share the limit value $a \in \hat{\mathbb{C}}$ if, for all sequences, $z_n \to \infty$ the expression

$$f(z_n) \to a \Longleftrightarrow g(z_n) \to a$$

holds. Suppose that f and g share all limit values contained in a set $L \subset \hat{\mathbb{C}}$. We consider the following question: which assumptions on L imply that f and g share *all* limit values in $\hat{\mathbb{C}}$? A set L with this property will be called a *limit uniqueness set*. In [**7**, Theorem 8.4] we proved the following theorem.

Theorem 1.1. *Every open set is a limit uniqueness set.*

In this paper we will prove the following.

Theorem 1.2. *Every set of positive linear measure on a rectifiable Jordan arc is a limit uniqueness set.*

For the definition and properties of linear measure we refer the reader to [**4**, Chapter 6] and to [**1**]. We note here merely that, for rectifiable Jordan arcs, length and linear measure are the same.

It is easy to show that every continuum has positive linear measure and it follows from the basic results in geometric measure theory that every continuum of finite linear measure can be decomposed as the union of countably many rectifiable Jordan arcs and a set of linear measure zero (see [**1**, Theorem 3.14]). At least one of these rectifiable Jordan arcs satisfies the assumptions of Theorem 1.2, which gives the following corollary.

Corollary 1.3. *Every continuum of finite linear measure is a limit uniqueness set.*

In the language of geometric measure theory, a regular 1-set is a set of finite positive linear measure with a regular behaviour concerning certain density properties (see [**1**, Chapter 2]). Each regular 1-set is contained in the union of countably many rectifiable Jordan arcs and a set of linear measure zero [**1**, Theorem 3.25]. Thus, the next result follows immediately from Theorem 1.2.

Corollary 1.4. *Every regular* 1*-set is a limit uniqueness set.*

If the assumption of rectifiability is dropped, we require the whole arc to consist of shared limit values. Simple modifications of the proof of Theorem 1.2 show the following (see the remarks at the end of the paper).

Theorem 1.5. *Every Jordan arc is a limit uniqueness set.*

2. Preliminaries for the proof of Theorem 1.2

First we state a simple technical lemma, which is proved by a standard compactness argument. As usual we denote by $\mathbb{D}_r$ the set of all $z \in \mathbb{C}$ with $|z| < r$.

Lemma 2.1. *Let $f, g : \mathbb{C} \to \hat{\mathbb{C}}$ be meromorphic functions that share the limit values from a compact set $M \subset \mathbb{C}$. Then $f - g \to 0$ uniformly on $f^{-1}(M)$, i.e. for every $\varepsilon > 0$ there exists $r > 0$ such that $|f - g| < \varepsilon$ on $f^{-1}(M) \setminus \mathbb{D}_r$.*

If φ is bounded and analytic in the unit disc $\mathbb{D}$, i.e. if $\varphi \in H^\infty(\mathbb{D})$, then by Fatou's theorem [**9**, Theorem IV.8] the radial boundary function φ^* exists almost everywhere (a.e.) on the boundary $\mathbb{T}$ of the unit disc and is integrable. Furthermore, by the Riesz uniqueness theorem [**9**, Theorem IV.9], $\varphi^* = 0$ at most on a set of measure zero unless $\varphi \equiv 0$. The following is a Möbius invariant statement of these facts. Here G^{c} denotes the complement of G.

Lemma 2.2. *Let $G \subset \hat{\mathbb{C}}$ be a domain such that G^{c} contains an inner point and let $\varphi : \mathbb{D} \to G$ be meromorphic. Then the radial boundary function φ^* exists a.e. on $\mathbb{T}$, is measurable and $\varphi^* = c \in \hat{\mathbb{C}}$ at most on a set of measure zero for any constant $c \in \hat{\mathbb{C}}$ unless $\varphi \equiv c$.*

By ω we denote harmonic measure. For its definition and properties we refer the reader to [**2**, **4**, **5**, **9**]. It is well known that analytic maps increase harmonic measure [**5**, Theorem 4.3.8]. The statements of this principle of harmonic measure usually require that the analytic map is continuous on the measured part of the boundary. In the unit disc radial limits are sufficient for an estimate that fits for our purposes. As usual we call a domain such that each boundary point (including ∞) is a regular point for the Dirichlet problem (see, for example, [**9**]) a regular domain.

Lemma 2.3. *Let $G \subset \mathbb{C}$ be a regular domain such that G^{c} contains an inner point and let $\varphi : \mathbb{D} \to G$ be holomorphic. Let $A \subset \partial G$ be a Borel set and let $B := (\varphi^*)^{-1}(A)$. Then $\omega(z, B, \mathbb{D}) \leqslant \omega(\varphi(z), A, G)$ for all $z \in \mathbb{D}$.*

Proof. Let F be a compact set in $\partial G \setminus A$ and let V be open on $\mathbb{T}$ with $\mathbb{T} \setminus B \subset V$ and consider the bounded harmonic function $\Omega(z) := \omega(\varphi(z), F, G) - \omega(z, V, \mathbb{D})$ on $\mathbb{D}$. For $z \to \mathrm{e}^{\mathrm{i}\theta} \in V$ radially we have $\omega(z, V, \mathbb{D}) \to 1$. Also for almost every $\mathrm{e}^{\mathrm{i}\theta} \in V^{\mathrm{c}} \subset B$ it follows that the radial limit $\varphi^*(\mathrm{e}^{\mathrm{i}\theta}) \in A$ exists and therefore $\omega(\varphi(z), F, G) \to 0$ as $z \to \mathrm{e}^{\mathrm{i}\theta}$ radially. We conclude that $\Omega^* \leqslant 0$ a.e. on $\mathbb{T}$. It is now follows from the Poisson formula and Fatou's theorem [**9**, Chapter IV] that $\Omega(z) \leqslant 0$ for all $z \in \mathbb{D}$ and hence $\omega(\varphi(z), F, G) \leqslant \omega(z, V, \mathbb{D})$ on $\mathbb{D}$. From the measure-theoretic properties of harmonic measure we obtain $\omega(\varphi(z), \partial G \setminus A, G) \leqslant \omega(z, \mathbb{T} \setminus B, \mathbb{D})$. Taking complements, the claim follows. □

The next statement is a variant of a convergence theorem for bounded analytic functions. We refer the reader to [**3**]. Here μ denotes normalized Lebesque measure on $\mathbb{T}$ and $\|\cdot\|_\infty$ is the sup-norm on $H^\infty(\mathbb{D})$. In contrast to more usual theorems, e.g. Vitali's theorem on the convergence of normal sequences, it is assumed that the sequence of functions converges on the boundary.

Lemma 2.4. *Let $\varphi_n : \mathbb{D} \to \mathbb{C}$ be a sequence in $H^\infty(\mathbb{D})$ with $\|\varphi_n\|_\infty \leqslant K$ for all n. Suppose for every n there exists $B_n \subset \mathbb{T}$ measurable with $\mu(B_n) > c > 0$, where c is independent of n, and such that $S_n := \sup_{z \in B_n} |\varphi_n^*(z)|$ converges to 0. Then φ_n converges to 0 in $\mathbb{D}$ locally uniformly.*

Proof. Exactly as in the first inequality on p. 86 of [**3**], we obtain from Jensen's formula for $|z| \leqslant r$:

$$\log|\varphi_n(z)| \leqslant \frac{1+r}{1-r} \log K + c\frac{1-r}{1+r} \log S_n.$$

It follows that $\log|\varphi_n(z)| \to -\infty$ uniformly in $|z| \leqslant r$. □

If $\varphi \in H^\infty(\mathbb{D})$ with $|\varphi^*| = 1$ a.e. on $\mathbb{T}$, then φ is called an inner function. It is well known that for inner functions φ with $\varphi(0) = 0$ the radial boundary function $\varphi^* : \mathbb{T} \to \mathbb{T}$ is measure preserving. A surprisingly short proof of this can be found in [**6**]. In fact, the proof given there yields a more general statement (see [**6**, Theorem 1.3 and Remark 1.4]).

Lemma 2.5. *Let φ be an inner function and let $A \subset \mathbb{T}$ be measurable. Then*

$$\mu((\varphi^*)^{-1}(A)) = \omega(\varphi(0), A, \mathbb{D}).$$

3. Proof of Theorem 1.2

Let Γ be the rectifiable Jordan arc and denote by M the set of positive linear measure $\Lambda(M)$ on Γ that consists of shared limit values of f and g. From the basic properties of linear measure [**4**, Proposition 6.3] it follows that by decreasing M slightly we can assume that M is closed and still has positive linear measure $c := \Lambda(M)$. From the density properties of linear measure [**1**, Corollary 2.5] it follows that we can choose two points a_1, a_2 in M, distinct from the endpoints, and small discs $\mathbb{D}_\varepsilon(a_i)$ around the a_i such that $0 < \Lambda(M \cap \mathbb{D}_\varepsilon(a_i)) < \frac{1}{3}c$. Now $\Gamma \cap \mathbb{D}_\varepsilon(a_i)$ consists of at most countably many components Γ_k^i and for ε small enough each of these is a cross-cut of $\mathbb{D}_\varepsilon(a_i)$. For $i = 1, 2$ choose one of these cross-cuts, say Γ_1 and Γ_2, that contain a part of M of positive linear measure. Then $\mathbb{D}_\varepsilon(a_i) \setminus \Gamma_i$ for $i = 1, 2$ are four Jordan domains $G_1, \dots, G_4$ with rectifiable boundary, each boundary containing a part of M of positive linear measure. By decreasing ε if needed, we can assume that the linear measure of the part of M which lies on the boundaries of the G_i, which is less than $\frac{2}{3}c + 4\pi\varepsilon$, is strictly less than c. Hence, there exists a point $b \in M$ outside the closures of the G_i. Now we apply a Möbius transformation to f, g and the considered sets such that b is mapped to ∞, so that ∞ is a shared limit value.

It is easy to see that the unbounded components of at least one of two preimages $P_i := f^{-1}(G_i)$ all have an unbounded boundary component. The same holds, of course, for preimages of g, so that we find at least one in three of the G_i such that each unbounded component of $f^{-1}(G_i)$ and $g^{-1}(G_i)$ has an unbounded boundary component. We will denote this P_i and the related image domain G_i with boundary J_i simply by P, G and J. We may now summarize this first step of the proof as follows. We can assume without loss of generality that the set of shared limit values M has positive linear measure, is compact and lies on a rectifiable Jordan curve J in $\mathbb{C}$ with inner domain G. Furthermore, the unbounded components of the preimages of G with respect to f and g all have an unbounded boundary component and ∞ is a shared limit value of f and g.

Let $z_n \to \infty$ be a sequence with $f(z_n) \to a \in G$. We will prove $g(z_n) \to a$, so that Theorem 1.1 shows the assertion.

For large n it follows that $z_n \in P$ and we denote by P_n the component of P that contains z_n. Furthermore, let $\varphi_n : \mathbb{D} \to P_n$ be a universal cover map, such that $\varphi_n(0) = z_n$. We can apply Lemma 2.2, and hence the boundary function φ_n^* exists a.e. on $\mathbb{T}$ and $\varphi_n^* = \infty$ at most on a set of measure zero. The basic properties of covering maps show that radial limits of φ_n lie in the boundary of P_n, so that φ_n^* maps a set of full measure to the finite boundary of P_n.

Next let $\psi : G \to \mathbb{D}$ be a conformal map with $\psi(a) = 0$. By the Carathéodory theorem [**4**, Theorem 2.6], ψ has a homeomorphic boundary function $\psi^* : J \to \mathbb{T}$ and since J is rectifiable it follows from the Riesz–Privalov theorem [**4**, Theorem 6.8] that $A := \psi^*(M)$ is a set of positive measure on $\mathbb{T}$.

Now consider the map $F_n := \psi \circ f \circ \varphi_n : \mathbb{D} \to \mathbb{D}$. From the properties of ψ, f and φ_n we conclude that the boundary function F_n^* exists a.e. on $\mathbb{T}$ with values in $\mathbb{T}$, i.e. F_n is an inner function. By Lemma 2.5, the sets $B_n := (F_n^*)^{-1}(A) \subset \mathbb{T}$ satisfy $\mu(B_n) = \omega(\psi \circ f(z_n), A, \mathbb{D}) \to \omega(0, A, \mathbb{D}) = \mu(A) > 0$.

The next step is to prove that the boundary functions Φ_n^* of the sequence $\Phi_n := (f - g) \circ \varphi_n : \mathbb{D} \to \mathbb{C}$ converge to 0 uniformly on $\tilde{B}_n := B_n \setminus E_n$ with $\mu(E_n) \to 0$. For this we consider two cases.

In the first case we assume that in each component of P there lie at most finitely many z_n. This implies that the domains P_n tend to ∞ and therefore that $f - g$ is uniformly bounded on P_n for large n. Assume to the contrary that there is a sequence $w_n \to \infty$ in P_n such that $(f-g)(w_n) \to \infty$. Since f is bounded on P, it follows that $g(w_n) \to \infty$, which contradicts the fact that ∞ is a shared limit value. Lemma 2.2 shows that Φ_n^* is defined a.e. on $\mathbb{T}$ for large n. Furthermore, Lemma 2.1 shows that Φ_n^* converges uniformly to 0 on $\tilde{B}_n := B_n \setminus E_n$, where, according to Lemma 2.2, the E_n can be chosen such that $\mu(E_n) = 0$.

In the second case we can assume that all z_n lie in one unbounded component P' of P. Since ∞ is a shared limit value, it follows from the arguments of the first case that g has at most finitely many poles in P'. If g has poles in P', we consider $\tilde{g} := R \cdot g$ instead of g with a rational function R with $R(\infty) = 1$ and such that $\tilde{g}$ has no poles in P'. Note that this does not affect the asymptotic behaviour of g. Thus, we can assume that g is bounded on P', so that Φ_n is holomorphic and Φ_n^* exists a.e. on $\mathbb{T}$. Since $\partial P'$ contains an unbounded component, ∞ is a regular point for the Dirichlet problem in P' [**9**, Theorem I.11]. (All finite points of $\partial P'$ are also contained in continua of $\partial P'$ so that P' is a regular domain on the sphere.) Hence, for any bounded Borel subset C of $\partial P'$ we have $\omega(z_n, C, P') \to 0$ for $n \to \infty$. It follows that, for large n, bounded parts of $M' := \partial P' \cap f^{-1}(M)$ can be neglected in the following sense: there exists a sequence $r_n \to \infty$ such that $\omega(z_n, \bar{\mathbb{D}}_{r_n} \cap M', P') \to 0$. We set $E_n := (\varphi_n^*)^{-1}(\bar{\mathbb{D}}_{r_n} \cap M')$. Then, by Lemma 2.3, we get $\mu(E_n) = \omega(0, E_n, \mathbb{D}) \leqslant \omega(\varphi_n(0), \bar{\mathbb{D}}_{r_n} \cap M', P') = \omega(z_n, \bar{\mathbb{D}}_{r_n} \cap M', P') \to 0$ and Φ_n^* converges to 0 uniformly on $\tilde{B}_n := B_n \setminus E_n$ by Lemma 2.1.

For large n we have $\mu(\tilde{B}_n) > \frac{1}{2}\mu(A)$ and it follows from Lemma 2.4 that Φ_n converges to 0 locally uniformly in $\mathbb{D}$. In particular, $\Phi_n(0) = (f-g)(z_n) \to 0$, so that $f(z_n) \to a \in G$ implies $g(z_n) \to a$. In the first part of the proof we have arranged the situation such that it is symmetric with respect to f and g. Hence, the same argument shows that, for every $z_n \to \infty$, $g(z_n) \to a \in G$ implies $f(z_n) \to a$. Theorem 1.1 thus proves Theorem 1.2.

4. Remarks

Inspection of the above proof shows that it is important for our method that M is mapped by ψ to a set of positive measure on $\mathbb{T}$. In Theorem 1.2 this is ensured by the Riesz–Privalov theorem. If the assumption of rectifiability is dropped, there seems to be no substitute for this argument: there exist quasi-discs Q such that the boundary homeomorphism of any conformal map $\psi : Q \to \mathbb{D}$ maps the boundary of Q minus a set of linear measure zero to a set of measure zero on $\mathbb{T}$ (combine Proposition 4.12, Proposition 6.22 and Corollary 6.26 from [**4**]). But of course the boundary homeomorphism of a Jordan domain maps open arcs to open sets on $\mathbb{T}$, which are of positive measure. This justifies Theorem 1.5. (The construction of a suitable domain G is even simpler if the whole arc consists of shared limit values.)

Note that a continuum of infinite linear measure need not contain a continuum of finite linear measure. For example, the von Koch curve [**1**, § 8.3] is a Jordan arc of infinite linear measure that contains no subarc of finite length. (It is easy to check that subcontinua of Jordan arcs are subarcs.) This means that our method, in general, does not cover continua of infinite linear measure, so that we cannot delete the assumption of finite linear measure in Corollary 1.3. (Of course, the von Koch curve is a limit uniqueness set by Theorem 1.5.)

The classes of sets which are considered in our results are well-known examples of sets that are not removable for bounded analytic functions (see, for example, [**8**] and the references therein). Therefore, it is natural to require a connection between removability and limit uniqueness sets. We believe that there is no such connection. Any subset of $\hat{\mathbb{C}}$ with at least five elements should be a limit uniqueness set.

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