ON ABSOLUTE SUMMABILITY BY RIESZ AND GENERALIZED CESÀRO MEANS. II

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1. We will use the terminology of part I [**9**], including the general assumptions of [**9**, § 1]. In that paper we had proved that $|R, \lambda, \kappa| = |C, \lambda, \kappa|$ in case that κ is an integer. Now, we turn to non-integral orders κ.

As to *ordinary* summation, the following inclusion relations (in the customary sense; see [**9**, end of § 1]) for non-integral κ have been established so far. (Since we are comparing Riesz methods of the same type λ and order κ only, (R, λ, κ) is written (R), etc., for the moment.) $(R) \subseteq (C)$ is a result by Borwein and Russell [**2**]. $(C) \subseteq (R)$ was proved by Jurkat [**3**] in the case $0 < \kappa < 1$, and, after Borwein [**1**], it holds in the case $1 < \kappa < 2$ if

$$\frac{\lambda_{n+1}}{\lambda_n} \searrow \quad \text{for large } n, \tag{1}$$

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_n - \lambda_{n-1}} \searrow \quad \text{for large } n \tag{2}$$

(i.e. decreases in the wide sense).

In the present note we deal with the counterparts of these inclusions for *absolute* summation. In our proof of $|R| \subseteq |C|$ for non-integral κ we need the restriction

$$\max_{\nu \leqq n} \Lambda_\nu \lambda_\nu^\vartheta = O(\Lambda_n \lambda_n^\vartheta), \qquad \Lambda_n = \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_n}, \quad \vartheta = \frac{[\kappa]}{\kappa}, \tag{3}$$

which is weaker than (1). If $0 < \kappa < 1$, we know that $|C| \subseteq |R|$ holds under (3), $\vartheta = 0$; this result is contained in [**5**, *Satz* 3] and follows from [**5**, *Satz* 2; **8**] (also cf. [**4**, *Satz* 1, *Satz* 4]). For $1 < \kappa < 2$, we prove that $|C| \subseteq |R|$ under the assumptions required by Borwein in the case of ordinary summation, that is, under (1), (2).

For all inclusions which we are concerned with in the theorems of this paper, the analogues for ordinary summability are known to hold at least under the restrictions made on λ and κ. Therefore we need not pay attention

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to the values of summation when proving our equivalence theorems. Throughout,

$$\sum a_n = \sum_{n=0}^{\infty} a_n$$

denotes a series with complex terms a_n.

2. First, we deal with the inclusion that is easier to handle.

THEOREM 1. *Assume* (3). *Then* $|R, \lambda, \kappa| \subseteq |C, \lambda, \kappa|$ *holds.*

Proof. For integral κ, the relation holds even without (3), as was shown in [**9**]; we write $k = [\kappa]$, $\delta = \kappa - [\kappa]$ and assume that $\delta > 0$. The case $0 < \kappa < 1$ is trivial; therefore we also assume that $k \geqq 1$. Property (3) implies that

$$\sup_{n \geqq \nu} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right) = O\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right) \tag{4}$$

and thus

$$\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+p}} = O\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right), \qquad p = 1, 2, \ldots \text{ fixed.} \tag{5}$$

Let $\sum a_n$ be summable $|R, \lambda, \kappa|$. On account of (3), the theorem [**5**, *Satz* 2] is applicable and yields

$$\sum \Lambda_n^{-\kappa} |a_n| < \infty. \tag{6}$$

We write (see [**9**, § 1]; $a_\nu^{(\kappa)} = \lambda_\nu^\kappa a_\nu$) the generalized Cesàro means

$$\tau_{n-1}^{(\kappa)} = \sum_{\nu=0}^{n-1} l_{n\nu} \cdots l_{n+k-1,\nu} l_{n+k,\nu}^{\delta} a_\nu^{(\kappa)}, \qquad l_{p\nu} = \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right) - \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_{n+k}}\right),$$

in the form

$$\sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^{k+\delta} a_\nu^{(\kappa)} + \sum_{j=1}^{k} \alpha_{nj} \sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^{k-j+\delta} a_\nu^{(\kappa)},$$

with certain combinations α_{nj} from

$$\frac{1}{\lambda_p} - \frac{1}{\lambda_{n+k}} \quad \left(\leqq \frac{1}{\lambda_n} - \frac{1}{\lambda_{n+k}}\right), \qquad p = n, \ldots, n + k - 1.$$

This provides the following representation:

$$\begin{aligned} \tau_{n-1}^{(\kappa)} = \sigma^{(\kappa)}(\lambda_{n+k}) &- \sum_{\nu=n}^{n+k-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^{\kappa} a_\nu^{(\kappa)} \\ &+ \sum_{j=1}^{k} \alpha_{nj} \sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^{\kappa-j} a_\nu^{(\kappa)} \\ = s_n^{(1)} + s_n^{(2)} + s_n^{(3)}&, \end{aligned}$$

say, and $\sum a_n$ becomes summable $|C, \lambda, \kappa|$ if each of the sequences $(s_n^{(i)})$ has

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bounded variation. The case $i = 1$ is immediately clear, and we (even) obtain $\sum |s_n^{(2)}| < \infty$ from (6), since (5) implies that

$$\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^\kappa \lambda_\nu^\kappa = O(\Lambda_\nu^{-\kappa}), \qquad \nu = n, \ldots, n+k-1.$$

As to the case $i = 3$, we note that $\alpha_{nj} = O(\lambda_n^{-1} - \lambda_{n+1}^{-1})^j$ by (5); therefore, the proof will be complete if we can (even) show that

$$(7) \qquad \sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^j \left|\sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+k}}\right)^{\kappa-j} a_\nu^{(\kappa)}\right| < \infty, \qquad j = 1, \ldots, k.$$

To this end we subdivide the interval $[1/\lambda_{n+1}, 1/\lambda_n]$ by

$$\frac{1}{\mu_i} = \frac{1}{\mu_{ni}} = \frac{1}{\lambda_n} - \frac{i}{K}\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right), \qquad i = 0, \ldots, k+1 = K,$$

and form differences of the orders $1, \ldots, K$ from the numbers

$$\sigma^{(\kappa)}(\mu_0), \sigma^{(\kappa)}(\mu_1) - \left(1 - \frac{\lambda_n}{\mu_1}\right)^\kappa a_n, \ldots, \sigma^{(\kappa)}(\mu_K) - \left(1 - \frac{\lambda_n}{\mu_K}\right)^\kappa a_n,$$

namely

$$\alpha_n^{(J)} = \sum_{i=0}^{J} (-1)^i \binom{J}{i} \sum_{\nu=0}^{n-1} \left(1 - \frac{\lambda_\nu}{\mu_{ni}}\right)^\kappa a_\nu, \qquad J = 1, \ldots, K.$$

By virtue of $\sum |\sigma^{(\kappa)}(\mu_{n,i-1}) - \sigma^{(\kappa)}(\mu_{ni})| < \infty$, $i = 1, \ldots, K$, and of (6), we have $\sum |\alpha_n^{(J)}| < \infty$, $J = 1, \ldots, K$. Through that, we arrive at the hypotheses

$$(8) \qquad \sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^J \left|\sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{\theta_{nJ}}\right)^{\kappa-J} a_\nu^{(\kappa)}\right| < \infty, \qquad J = 1, \ldots, K,$$

with certain mean values $\theta_{nJ} \in (\lambda_n, \lambda_{n+1})$, according to [**10**, *Kap.* 1, (29)].

Consider (7) for a fixed j. In terms of

$$S_n^{(\kappa-i)}(x) = \sum_{\nu=0}^{n-1} \left(\frac{1}{\lambda_\nu} - \frac{1}{x}\right)^{\kappa-i} a_\nu^{(\kappa)}, \qquad x > \lambda_{n-1},$$

the inner sum of (7) is

$$S_n^{(\kappa-j)}(\lambda_{n+k}) = S_n^{(\kappa-j)}(\theta_{nj}) + (\kappa - j)\left(\frac{1}{\theta_{nj}} - \frac{1}{\lambda_{n+k}}\right) S_n^{(\kappa-j-1)}(\theta'_{nj})$$

with certain $\theta'_{nj} \in (\theta_{nj}, \lambda_{n+k})$. Thus, the proof of (7) for fixed j can be reduced by (8), with $J = j$, and (5) to the proof of

$$\sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^{j+1} |S_n^{(\kappa-j-1)}(\theta'_{nj})| < \infty.$$

We now "approximate" $S_n^{(\kappa-j-1)}(\theta'_{nj})$ by $S_n^{(\kappa-j-1)}(\theta_{n,j+1})$, then apply (8), with

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$J = j + 1$, and (5). After introducing (8), with $J = j, j + 1, \ldots, K$, in the same way, it remains to prove

$$\sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^{K+1} |S_n^{(\kappa-K-1)}(\vartheta_{nj})| < \infty$$

for certain $\vartheta_{nj} \in (\lambda_n, \lambda_{n+k})$.

We may begin with the estimate

$$|S_n^{(\kappa-K-1)}(\vartheta_{nj})| \leqq \sum_{\nu=0}^{n-2} \left(\frac{1}{\lambda_\nu} - \frac{1}{\vartheta_{nj}}\right)^{\kappa-K-1} |a_\nu^{(\kappa)}| + \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\vartheta_{nj}}\right)^{\kappa-K-1} |a_{n-1}^{(\kappa)}|$$
$$= A_{nj}^{(1)} + A_{nj}^{(2)},$$

say. Observing (4), we have:

$$\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^{K+1} A_{nj}^{(2)} = O(\Lambda_{n-1}^{-\kappa})|a_{n-1}|,$$

and can apply (6). Due to (4),

$$\sum_n \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)^{K+1} A_{nj}^{(1)} < \infty$$

holds if

$$(9) \qquad \sum_\nu |a_\nu|\lambda_\nu^\kappa \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right)^K \sum_{n \geqq \nu+2} \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\left(\frac{1}{\lambda_\nu} - \frac{1}{\vartheta_{nj}}\right)^{\kappa-K-1} < \infty$$

is true. By means of $\zeta_{n\nu} \in (\lambda_{n-1}, \lambda_n)$ such that

$$\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n-1}}\right)^{\kappa-K} - \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_n}\right)^{\kappa-K} = \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)(K - \kappa)\left(\frac{1}{\lambda_\nu} - \frac{1}{\zeta_{n\nu}}\right)^{\kappa-K-1},$$

the inner sum of (9) is seen to be

$$O(1)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right)^{\kappa-K} = O(1)\lambda_\nu^{-\kappa}\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right)^{-K} \Lambda_\nu^{-\kappa}.$$

Hence, (9) results from (6) as well.

3. Regarding non-integral orders, the proof of Theorem 1 relied upon (6), i.e. an information on the "order of magnitude" of series summable $|R, \lambda, \kappa|$. On the other hand, the problem $|C, \lambda, \kappa| \subseteq |R, \lambda, \kappa|$, $0 < \kappa < 1$, was also attacked via (6) (see [**4**; **5**]). Since we are now concerned with this inclusion in the case of $1 < \kappa < 2$, we will first provide a *limitation theorem* for summability $|C, \lambda, \kappa|$, too. The problem bears analogy to the problem $|R^*, \lambda, \kappa| \subseteq |R, \lambda, \kappa|$, $1 < \kappa < 2$, where another matrix method takes the place of $|C, \lambda, \kappa|$. Thus, the limitation theorem we need here will, to some extent, be obtained on the line of the proofs for the limitation theorems in [**6**; **7**]. It is prepared by the following key lemma, in the course of whose proof we will express the generalized Cesàro means in terms of an auxiliary function h, namely by

$$\tau_n^{(\kappa)} = \sum_{\nu=0}^{n} \left(\frac{1}{\lambda_\nu} - \frac{1}{h(\lambda_{n+2})}\right)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+2}}\right)^{\delta} a_\nu^{(\kappa)},$$

similar to the procedure used by Borwein [**1**]. (For higher orders we suggest representations $\lambda_{n+1} = h(h(\lambda_{n+3}))$ and the like.)

LEMMA. *Suppose that* $0 < \lambda_n < \lambda_{n+1} \to \infty$ $(n = 0, 1, \ldots)$, $0 < \delta < 1$, *and let*

$$a_{n\nu} = f_n\left(\frac{1}{\lambda_{\nu+1}}\right) - f_n\left(\frac{1}{\lambda_\nu}\right) \qquad (n = 0, 1, \ldots; \nu = 0, \ldots, n),$$

where

$$f_n(t) = \left(t - \frac{1}{\lambda_n}\right)\left(t - \frac{1}{\lambda_{n+1}}\right)^{\delta} - \left(t - \frac{1}{\lambda_{n+1}}\right)\left(t - \frac{1}{\lambda_{n+2}}\right)^{\delta} \qquad \left(t \geqq \frac{1}{\lambda_{n+1}}\right).$$

Then $a_{n\nu} > 0$ *holds for all* $n \geqq 0$, $0 \leqq \nu \leqq n$.

If, moreover, (1) *and* (2) *are satisfied, then*

$$\frac{a_{n\nu}}{a_{n-1,\nu}} \geqq \frac{a_{n,\nu+1}}{a_{n-1,\nu+1}} \qquad (0 \leqq \nu \leqq n-2) \tag{10}$$

holds for large $n \geqq 2$.

Proof. Let us write

$$f'_{n-1}(t) = g(\lambda_n, t) - g(\lambda_{n+1}, t) \qquad \left(t > \frac{1}{\lambda_n}, n \geqq 1\right),$$

where

$$g(u, t) = \delta\left(t - \frac{1}{h(u)}\right)\left(t - \frac{1}{u}\right)^{\delta-1} + \left(t - \frac{1}{u}\right)^{\delta} \qquad \left(t > \frac{1}{u}, u \geqq \lambda_1\right),$$

$$h(u) = p_n(u - \lambda_n) + \lambda_{n-1}, \qquad p_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \qquad (\lambda_n \leqq u < \lambda_{n+1}, n \geqq 1)$$

(note the graph of h to be the polygon joining the points $(\lambda_n, \lambda_{n-1})$). The partial derivative of $g(u, t)$, $\lambda_n < u < \lambda_{n+1}$, with respect to u, i.e.

$$\frac{\delta}{u^2}\left(t - \frac{1}{u}\right)^{\delta-1} \gamma(u, t), \qquad \gamma(u, t) = (1 - \delta)\frac{1/h - 1/u}{t - 1/u} + \left(\frac{u}{h}\right)^2 h' + \delta,$$

is positive since $h(u) < u$, $h' > 0$, and thus we have $f'_n(t) < 0$ for $t > 1/\lambda_{n+1}$, $n \geqq 0$. From this we conclude, in the first place, that $a_{n\nu} > 0$ for $n \geqq 0$, $0 \leqq \nu \leqq n$.

(10) may be written in the form

$$\frac{f'_n(t_\nu)}{f'_{n-1}(t_\nu)} \geqq \frac{f'_n(t_{\nu+1})}{f'_{n-1}(t_{\nu+1})} \qquad (0 \leqq \nu \leqq n-2) \tag{10'}$$

with mean values $t_\mu = t_{\mu n} \in (1/\lambda_{\mu+1}, 1/\lambda_\mu)$, $\mu = \nu, \nu + 1$ (here, and in what follows, we omit irrelevant indications). All terms in (10′) have the same sign. Hence, (10′) is equivalent to

$$\frac{g(\lambda_{n+1}, t_\nu) - g(\lambda_{n+2}, t_\nu)}{g(\lambda_{n+1}, t_{\nu+1}) - g(\lambda_{n+2}, t_{\nu+1})} \geqq \frac{g(\lambda_n, t_\nu) - g(\lambda_{n+1}, t_\nu)}{g(\lambda_n, t_{\nu+1}) - g(\lambda_{n+1}, t_{\nu+1})} \qquad (0 \leqq \nu \leqq n-2),$$

and this, in terms of mean values $\theta_m = \theta_{m\nu n} \in (\lambda_m, \lambda_{m+1})$, $m = n, n+1$, assumes the form

$$(10'') \qquad T_{n+1}\frac{\gamma(\theta_{n+1}, t_\nu)}{\gamma(\theta_{n+1}, t_{\nu+1})} \geqq T_n \frac{\gamma(\theta_n, t_\nu)}{\gamma(\theta_n, t_{\nu+1})} \qquad (0 \leqq \nu \leqq n-2),$$

where

$$T_m = T_{m\nu n} = \left(\frac{t_\nu - 1/\theta_m}{t_{\nu+1} - 1/\theta_m}\right)^{\delta-1} = \left(\frac{t_\nu - t_{\nu+1}}{t_{\nu+1} - 1/\theta_m} + 1\right)^{\delta-1}, \qquad m = n, n+1.$$

(Note that $t_{\nu n} > t_{\nu+1,n}$ ($> 1/\lambda_{\nu+2} \geqq 1/\lambda_n$) for all $n \geqq \nu + 2$ and $\theta_{n+1,\nu,n} > \theta_{n\nu n}$ ($> \lambda_n$) for all $\nu = 0, \ldots, n-2$.) All terms in (10″) are positive, and since $T_{n+1} > T_n$ always, it suffices to prove that, from a certain n,

$$\frac{\partial}{\partial t}\log\frac{\gamma(\theta_{n+1}, t)}{\gamma(\theta_n, t)} \geqq 0 \qquad (t_{\nu+1} < t < t_\nu)$$

is satisfied for each $\nu = 0, \ldots, n-2$. This holds if

$$(11) \quad G(\theta_n, t) \leqq G(\theta_{n+1}, t) \qquad (t_{\nu+1} < t < t_\nu,\ 0 \leqq \nu \leqq n-2) \quad \text{for large } n,$$

where

$$G(u, t) = \frac{\gamma(u, t)}{-(\partial/\partial t)\gamma(u, t)} = \left(t - \frac{1}{u}\right) + \frac{1}{1-\delta}\frac{(u/h)^2 h' + \delta}{u/h - 1}u\left(t - \frac{1}{u}\right)^2 \qquad (u \neq \lambda_0, \lambda_1, \ldots).$$

Condition (1) implies that

$$(12) \qquad \frac{h(u)}{u} \nearrow \quad \text{for large } u$$

(i.e. increases in the wide sense), since

$$(13) \qquad \frac{h(u)}{u} = p_n - \frac{\lambda_{n-1}}{u}L_n \qquad (\lambda_n \leqq u < \lambda_{n+1},\ n \geqq 1)$$

with

$$L_n = \left(\frac{\lambda_n}{\lambda_{n-1}} - 1\right)\Big/\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) - 1 \geqq 0 \quad \text{for large } n.$$

Condition (2) takes the form

$$h'(\theta_n) = p_n \leqq p_{n+1} = h'(\theta_{n+1}) \quad \text{for large } n.$$

By these reasons, (11) reduces to

$$(14) \qquad F(u,t) = \frac{(u/h)^2 u (t - 1/u)^2}{u/h - 1} \nearrow \quad \text{for large } u.$$

Let $u \to \infty$; by (12) we have $x(u) = h(u)/u \to q \leqq 1$, say. If $q = 1$ (in fact, the argument applies to $1/2 < q \leqq 1$) we write

$$F(u,t) = u\left(t - \frac{1}{u}\right)^2 \frac{1}{x - x^2}.\dagger$$

In this case, (14) follows from $d(x - x^2)/dx \to 1 - 2q < 0$. If $q < 1$ (the so-called "high indices" case), we write

$$F(u,t) = \frac{(t - 1/u)^2}{1 - x} \frac{u}{x}.$$

Now, $p_n \to q$ (>0), $L_n \to 0$, and hence, in view of (13), we have $d(x(u)/u)/du < 0$ for large $u \neq \lambda_0, \lambda_1, \ldots$. This completes the proof.

The limitation theorem we need is the following.

THEOREM 2. *Let* $1 < \kappa < 2$, *and assume* (1), (2). *If* $\sum a_n$ *is summable* $|C, \lambda, \kappa|$, *then*

$$(15) \qquad \sum_n \Lambda_n^{-\kappa} \lambda_n^{-\kappa} \left| \sum_{\nu=0}^{n} \lambda_\nu^\kappa a_\nu \right| < \infty, \qquad \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

Proof. We set $\delta = \kappa - 1$,

$$P_{n\nu} = \left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_n}\right)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right)^\delta, \qquad r_\nu = \sum_{\mu=0}^{\nu} \lambda_\mu^\kappa a_\mu,$$

and thus have

$$\tau_{n-1} = \tau_{n-1}^{(\kappa)} = \sum_{\nu=0}^{n-1} (P_{n\nu} - P_{n,\nu+1}) r_\nu = \sum_{\nu=0}^{n-1} Q_{n\nu} r_\nu,$$

say (note that $Q_{nn} = 0$). This leads us to the transformation

$$\Delta\tau_n = \tau_n - \tau_{n-1} = \sum_{\nu=0}^{n} (Q_{n+1,\nu} - Q_{n\nu}) r_\nu = \sum_{\nu=0}^{n} a_{n\nu} r_\nu, \qquad n \geqq 1,$$

with $a_{n\nu}$, $n \geqq 1$, the same as in the Lemma. Define $\Delta\tau_0 = a_{00} r_0$ for arbitrary $a_{00} > 0$.

By the Lemma we have $a_{n\nu} > 0$ for $n \geqq 1$, $0 \leqq \nu \leqq n$, and may assume (10) to apply for $n \geqq 2$. (If (10) holds for $n \geqq m > 2$ only, set $a_{n\nu} = a_{m-1,\nu}$ $(\nu = 0, \ldots, m - 2; n = \nu, \ldots, m - 2)$, for example, thus possibly changing finitely many $\Delta\tau_n$; then resume the former notation.) Resulting from [**11**, Theorem 5], for instance (also cf. [**12**, p. 261, *Bemerkung* 2]), the inverse

$$r_n = \sum_{\nu=0}^{n} a'_{n\nu} \Delta\tau_\nu \qquad (n \geqq 0)$$

†I wish to thank the referee for his useful hint with regards to the representation of $F(u, t)$, which helped simplify the argument in this case.

of the transformation above has the property $a'_{n\nu} \leqq 0$ for $n \geqq 1, 0 \leqq \nu \leqq n-1$ ($a'_{nn} > 0$), and this implies that

$$(16) \qquad \sum_{n \geqq \nu+1} |a'_{n\nu}| \sum_{m \geqq n} a_{mn} \leqq \frac{1}{a_{\nu\nu}} \sum_{n \geqq \nu+1} a_{n\nu} \qquad (\nu \geqq 0),$$

by virtue of [**4**, Lemma 2].

To prove (15), we start from

$$(\Lambda_n\lambda_n)^{-\kappa}|r_n| \leqq (\Lambda_n\lambda_n)^{-\kappa} \sum_{\nu=0}^{n-1} |a'_{n\nu}|\,|\Delta\tau_\nu| + |\Delta\tau_n| = S_n + |\Delta\tau_n|,$$

say (observe that $a'_{nn} = 1/a_{nn}$, $a_{nn} = P_{n+1,n}$). By hypothesis, $\sum |\Delta\tau_n| < \infty$. Due to (1), $\sum S_n < \infty$ follows from

$$(17) \qquad \sum_\nu |\Delta\tau_\nu|\,\Lambda_\nu^{-\delta} \sum_{n \geqq \nu+1} |a'_{n\nu}| \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\lambda_n^{1-\kappa} < \infty.$$

Through mean values $\theta_n \in (\lambda_n, \lambda_{n+1})$ satisfying

$$(18) \qquad \lambda_n^{-\kappa} - \lambda_{n+1}^{-\kappa} = \kappa\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\theta_n^{1-\kappa},$$

we realize (note that $\theta_n = O(\lambda_n)$, by (1)) that

$$\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\lambda_n^{1-\kappa} = O(\lambda_n^{-\kappa} - \lambda_{n+1}^{-\kappa}).$$

Now,

$$\sum_{n \geqq \nu} a_{n\nu} = \lambda_\nu^{-\kappa} - \lambda_{\nu+1}^{-\kappa},$$

whence the inner sum of (17) turns out to be

$$O(1) \sum_{n \geqq \nu+1} |a'_{n\nu}| \sum_{m \geqq n} a_{mn}$$

and thus, in view of (16),

$$O(1)\frac{1}{a_{\nu\nu}} \sum_{n \geqq \nu} a_{n\nu} = O(1)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\right)^{-\kappa}(\lambda_\nu^{-\kappa} - \lambda_{\nu+1}^{-\kappa}).$$

But the latter is $O(\Lambda_\nu^\delta)$, by (18), and we conclude (17).

Remark. In case of integral orders k, all series summable $|C, \lambda, k|$ are subject to (15), without any additional assumptions on λ. This follows from [**5**, footnote 2; **9**, Theorem].

We are now in the position to prove that $|C, \lambda, \kappa| \subseteq |R, \lambda, \kappa|$, $1 < \kappa < 2$, for λ according to (1), (2). Let $\sum a_n$ be summable $|C, \lambda, \kappa|$. Then Theorem 2 yields (15) which, since $\Lambda_n\lambda_n = O(\Lambda_{n+1}\lambda_{n+1})$ by (1), implies that

$$(19) \qquad \sum_n \Lambda_n^{-\kappa}\lambda_n^{-\kappa}\left|\sum_{\nu=0}^{n-1} \lambda_\nu^\kappa a_\nu\right| < \infty.$$

From (15) and (19) we obtain (6). Now, the Tauberian theorem [8] asserts that, under a condition on λ less restrictive than (1), $\sum a_n$ is summable $|R, \lambda, \kappa|$ if (6) and

$$\sum |\sigma^{(\kappa)}(\lambda_n) - \sigma^{(\kappa)}(\lambda_{n+1})| < \infty \tag{20}$$

hold. Furthermore, the condition

$$\frac{1}{\lambda_n}\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right) \geqq \frac{1}{\lambda_{n+1}}\left(1 - \frac{\lambda_{n+1}}{\lambda_{n+2}}\right) \quad \text{for large } n \tag{21}$$

is weaker than (1). Thus it suffices to prove the following result.

THEOREM 3. *Let* $1 < \kappa < 2$, *and assume* (21). *If* $\sum a_n$ *is summable* $|C, \lambda, \kappa|$ *and satisfies* (15), *then it satisfies* (20).

Proof. We have (with $\delta = \kappa - 1$)

$$\begin{aligned} \tau_{n-1}^{(\kappa)} &= \sum_{\nu=0}^{n} \left[\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right) - \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\right]\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right)^{\delta} a_\nu^{(\kappa)} \\ &= \sigma^{(\kappa)}(\lambda_{n+1}) - \sum_{\nu=0}^{n} p_{n\nu} a_\nu^{(\kappa)}, \qquad p_{n\nu} = \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\left(\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{n+1}}\right)^{\delta}, \end{aligned}$$

and (with $d_{n\nu} = p_{n\nu} - p_{n+1,\nu}$)

$$\tau_{n-1}^{(\kappa)} - \tau_n^{(\kappa)} = \sigma^{(\kappa)}(\lambda_{n+1}) - \sigma^{(\kappa)}(\lambda_{n+2}) - \sum_{\nu=0}^{n+1} d_{n\nu} a_\nu^{(\kappa)}.$$

In order to verify that

$$\sum_n \left|\sum_{\nu=0}^{n+1} d_{n\nu} a_\nu^{(\kappa)}\right| < \infty,$$

we write

$$\sum_{\nu=0}^{n+1} d_{n\nu} a_\nu^{(\kappa)} = \sum_{\nu=0}^{n} (d_{n\nu} - d_{n,\nu+1}) r_\nu + d_{n,n+1} r_{n+1},$$

with r_ν as in the proof of Theorem 2. Since $d_{n,n+1} = -(\Lambda_{n+1}\lambda_{n+1})^{-\kappa}$, we may apply (15), and it remains to show that

$$\sum_{n \geqq 0} \sum_{\nu=0}^{n} |d_{n\nu} - d_{n,\nu+1}|\, |r_\nu| < \infty. \tag{22}$$

When $0 \leqq \nu \leqq n$, $1/\lambda_{\nu+1} \leqq t \leqq 1/\lambda_\nu$, we set

$$p_n(t) = \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\left(t - \frac{1}{\lambda_{n+1}}\right)^{\delta}, \qquad d_n(t) = p_n(t) - p_{n+1}(t)$$

(note that $d_n(1/\lambda_\nu) = d_{n\nu}$), so that

$$\frac{1}{\delta} d_n'(t) = \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\left(t - \frac{1}{\lambda_{n+1}}\right)^{\delta-1} - \left(\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_{n+2}}\right)\left(t - \frac{1}{\lambda_{n+2}}\right)^{\delta-1}$$

if $1/\lambda_{\nu+1} < t < 1/\lambda_\nu$. By (21) we obtain $d_n'(t) > 0$, and hence $d_{n\nu} - d_{n,\nu+1} > 0$ for $n \geqq n_1$, say. Thus, apart from an additive constant, the series of (22) is

$$\sum_{\nu \geqq n_1} |r_\nu| \sum_{n \geqq \nu} \{(p_{n\nu} - p_{n,\nu+1}) - (p_{n+1,\nu} - p_{n+1,\nu+1})\} = \sum_{\nu \geqq n_1} p_{\nu\nu}|r_\nu|,$$

and (15) yields (22).

4. We have previously obtained $|R, \lambda, \kappa| = |C, \lambda, \kappa|$, $0 < \kappa < 1$, (3) with $\vartheta = 0$ (cf. § 1). From Theorem 1, from what was proved in the foregoing section, and observing (1) to imply (3), we finally arrive at the following result.

THEOREM 4. *Let* $1 < \kappa < 2$, *and assume* (1), (2). *Then* $|R, \lambda, \kappa| = |C, \lambda, \kappa|$ *holds.*

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