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ON DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE PRIME SQUARED PLUS ONE

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Abstract

Groups with the property of the title were considered by Chillag (1977); this paper completes his results by showing that, with known exceptions, they are triply transitive.

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THEOREM A. *Let G be a 2-transitive permutation group of degree p^2+1, where p is prime. Then one of the following occurs:*

(a) *G is 3-transitive;*

(b) $PSL(2,p^2) \leqslant G \leqslant P\Gamma L(2,p^2)$;

(c) *G is the Frobenius group of order 20, with $p=2$.*

Thus, conclusion (d) of Chillag's Corollary (asserting that the stabilizer of two points has orbit lengths 1, 1, $2(p-1)$, $(p-1)^2$) does not occur; or, more precisely, groups in case (d) occur already in case (b). This is a consequence of the following result.

THEOREM B. *Let G be a 2-transitive permutation group on X, of degree n^2+1 $(n>1)$. Suppose that, for $x, y \in X$, $x \neq y$, G_{xy} has orbit lengths 1, 1, $2(n-1)$, $(n-1)^2$. Then $n=3$, $G = PSL(2,9)$ or $P\Sigma L(2,9)$.*

PROOF. The results of Higman (1970) imply that G_x is a subgroup of S_n wr S_2; so G_x has an imprimitive subgroup $N(x)$ of index 2. For $y \neq x$, $K = N(x) \cap G_y$ has orbit lengths 1, 1, $n-1$, $n-1$, $(n-1)^2$, and the orbit of length $(n-1)^2$ is isomorphic (as K-space) to the direct product of the two orbits of length $n-1$.

Now $N(y) \cap G_x = K'$ is a subgroup of index 2 in G_{xy} with the same orbit lengths as K. If $K \neq K'$, then $K \cap K'$ has four orbits of length $\frac{1}{2}(n-1)$, so the K-orbit of length $(n-1)^2$ splits into four orbits of length $\frac{1}{4}(n-1)^2$ of $K \cap K'$. This is impossible since $|K: K \cap K'| = 2$. We conclude that

$$N(y) \cap G_x = N(x) \cap G_y \leqslant N(x),$$

whence $N(x)$ is strongly closed in G_x with respect to G.

By the "two-graph transfer theorem" (see Taylor (1977), Theorem 6.1), either G has a subgroup N of index 2 with $N \cap G_x = N(x)$, or G is an automorphism group of a non-trivial regular two-graph. In the first case, if B is either orbit of length $n-1$ of N_{xy}, then $B \cup \{y\}$ is a block of imprimitivity for N_x, and so the setwise stabilizer L of $B \cup \{x, y\}$ acts 2-transitively on it. Then

$$|N:L| = (n^2+1)\,n^2/(n+1)\,n;$$

but this is not an integer for $n > 1$.

In the second case we use the fact that, if H is a rank 3 group whose parameters (in Higman's (1970) sense) are k, l, λ, μ, and G is a transitive extension of H which acts on a regular two-graph, then $k = 2\mu$. (See Taylor (1977), Proposition 2.3.) Here $k = 2(n-1)$, $\mu = 2$; so we have $n = 3$. The rest of the theorem is clear.

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