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THE TOPOLOGICAL DEGREE OF A-PROPER MAPPING IN THE MENGER PN-SPACE (II)

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In the paper "The topological degree of A-proper mapping in the Menger PN-space (I)", the new concept of A-proper topological degree has been given. Now, utilising the new concept, we give the corresponding definitions of convex A-proper, P_*-compact and P_γ-compact in Menger PN-space. As an application of these new concepts, we prove the existence of solution for some equations.

1. INTRODUCTION

In this paper, utilising A-proper properties, we discuss the existence of solution for some equations. For the sake of convenience, we recall some definitions and properties of PN-space.

DEFINITION 1: (Chang [1].) A probabilistic normed space (shortly a PN-space) is an ordered pair (E,F), where E is a real linear space, F is a mapping of E into D (D is the set of all distribution functions. We shall denote the distribution function $F(x)$ by F_x, $F_x(t)$ denotes the value F_x for $t \in R$.) satisfying the following conditions:

(PN-1) $F_x(0) = 0$;

(PN-2) $F_x(t) = H(t)$ for all t $\in$ R if and only if $x = \theta$, where H(t)=0 when $t \leqslant 0$, and H(t)=1 when $t > 0$;

(PN-3) For all $\alpha \neq 0$, $F_{\alpha x}(t) = F_x\big(t/|\alpha|\big)$;

(PN-4) For any $x, y \in E$ and $t_1, t_2 \in R$, if $F_x(t_1) = 1$ and $F_y(t_2) = 1$, then we have $F_{x+y}(t_1 + t_2) = 1$.

LEMMA 1. (Chang [1].) *Let (E, F, Δ) be a Menger PN-space with a continuous t-norm Δ, then $x_n \subset E$ is said to be convergent to $x \in E$ if for any $t > 0$, we have $\lim\limits_{n\to\infty} F_{x_n - x}(t) = H(t)$.*

LEMMA 2. *The generalised topological degree* $\operatorname{Deg}(f, \Omega, p)$ *has the following properties:*

(i) $\operatorname{Deg}(I, \Omega, p) = 1, \forall p \in \Omega$, *where I is an identity operator;*

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(ii) *If* $\mathrm{Deg}(f,\Omega,p) \neq \{0\}$, *then the equation* $f(x) = p$ *has a solution in* Ω;

(iii) *If* $L : [0,1]\times\overline{\Omega} \to E$ *is continuous and for any fixed* $t \in [0,1]$, $L(t,.) : \overline{\Omega} \to E$ *is an A-proper mapping satisfying*

$$\lim_{t\to t_0} \inf_{x\in\overline{\Omega}} F_{L(t,x)-L(t_0,x)}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0.$$

Let $p \notin h_t(\partial\Omega)$, $0 \leqslant t \leqslant 1$, *where* $h_t(x) = L(t,x)$, *then we have*

$$\mathrm{Deg}(h_t,\Omega,p) = \mathrm{Deg}(h_0,\Omega,p), \quad \forall\, 0 \leqslant t \leqslant 1;$$

(iv) *If* Ω_0 *is an open subset of* Ω *and* $p \notin f(\overline{\Omega}\backslash\Omega_0)$, *then we have*

$$\mathrm{Deg}(f,\Omega,p) = \mathrm{Deg}(f,\Omega_0,p);$$

(v) *If* $\Omega_{(1)}$ *and* $\Omega_{(2)}$ *are two disjoint open subsets of* Ω *and*

$$p \notin f\big(\overline{\Omega} \setminus (\Omega_{(1)} \cup \Omega_{(2)})\big),$$

then

$$\mathrm{Deg}(f,\Omega,p) \subseteq \mathrm{Deg}(f,\Omega_{(1)},p) + \mathrm{Deg}(f,\Omega_{(2)},p).$$

If either $\mathrm{Deg}(f,\Omega_{(1)},p)$ *or* $\mathrm{Deg}(f,\Omega_{(2)},p)$ *is single-valued, then*

$$\mathrm{Deg}(f,\Omega,p) = \mathrm{Deg}(f,\Omega_{(1)},p) + \mathrm{Deg}(f,\Omega_{(2)},p);$$

(vi) *If* $p \notin f(\partial\Omega)$, *then* $\mathrm{Deg}(f,\Omega,p) = \mathrm{Deg}(f - p,\Omega,\theta)$;

(vii) *If* p *varies on every connected component of* $E \setminus f(\partial\Omega)$, *then* $\mathrm{Deg}(f,\Omega,p)$ *is a constant.*

2. Main Results

Lemma 3. *Let* (E,F,Δ) *be a projected complete Menger PN-space,* Δ *is a continuous t-norm, and* $f : \overline{\Omega} \to E$ *is an A-proper mapping. Then* λf *is also an A-proper mapping (* $\lambda \neq 0$*).*

Proof: For any sequence $\{x_{n_k}\} \in \overline{\Omega}_{n_k}$, we have

$$\lim_{k\to\infty} F_{Q_{n_k}\lambda f(x_{n_k})-Q_{n_k}(y)}(t) = H(t) \qquad \forall\, t > 0.$$

Because $y \in E$, E is a linear space and $\lambda \neq 0$, then we have $y/\lambda \in E$. Hence the above is equal to

$$\lim_{k\to\infty} F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(y/\lambda)}(t/\lambda) = H(t) = H(t/\lambda)$$

Because f is an A-proper mapping, then by the definition, there exists a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \to x \in \overline{\Omega}$, $f(x) = y/\lambda$. So $\lambda f(x) = y$. Therefore λf is an A-proper mapping. ∎

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Lemma 4. *Let (E,F,Δ) be a projected complete Menger PN-space, Δ be a continuous t-norm, $f:\overline{\Omega}\to E$ be an A-proper mapping, $C:\overline{\Omega}\to E$ be a continuous compact mapping, and $(f+C)(x)=f(x)+C(x)$. Then $f+C$ is an A-proper mapping.*

Proof: If for any sequence $\{x_{n_k}\}\in\overline{\Omega}_{n_k}$, we have

$$\lim_{k\to\infty} F_{Q_{n_k}\left(C(x_{n_k})+f(x_{n_k})\right)-Q_{n_k}(y)}(t)=H(t)\qquad \forall t>0$$

Because C is a continuous compact mapping and $x_{n_k}\in\overline{\Omega}_{n_k}\subset\overline{\Omega}$, then there exists a subsequence(shortly, we assume that it is $\{x_{n_k}\}$ itself) such that $C(x_{n_k})\to y_0\in E$. Because Q_{n_k} is continuous and linear, then we have $Q_{n_k}C(x_{n_k})\to Q_{n_k}(y_0)$. Because

$$\begin{aligned}F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(y-y_0)}(t)&=F_{Q_{n_k}f(x_{n_k})+Q_{n_k}C(x_{n_k})-Q_{n_k}C(x_{n_k})-Q_{n_k}(y-y_0)}(t)\\&\geqslant\Delta\left(F_{Q_{n_k}f(x_{n_k})+Q_{n_k}C(x_{n_k})-Q_{n_k}y}\left(\frac{t}{2}\right),F_{Q_{n_k}y_0-Q_{n_k}C(x_{n_k})}\left(\frac{t}{2}\right)\right),\end{aligned}$$

taking limit between the two sides, we have

$$\lim_{k\to\infty}F_{Q_{n_k}f(x_{n_k})-Q_{n_k}(y-y_0)}(t)=H(\frac{t}{2})=H(t).$$

By the A-proper properties of f, there must exist a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}}\to x\in\overline{\Omega}$ and $f(x)=y-y_0$. By the continuity of C, we have $C(x_{n_{k_i}})\to C(x)$. Therefore $C(x)=y_0$ and $f(x)+C(x)=y$. Hence $f+C$ is an A-proper mapping. ∎

Lemma 5. *I is an A-proper mapping.*

Proof: If for any sequence $\{x_{n_k}\}\in\overline{\Omega}_{n_k}$, we have $\lim\limits_{k\to\infty}F_{Q_{n_k}I(x_{n_k})-Q_{n_k}(y)}(t)=H(t)$, then $\lim\limits_{k\to\infty}\left(Q_{n_k}(x_{n_k})-Q_{n_k}(y)\right)=0$. Because $Q_nx\to x,(n\to\infty)$, then we have $\lim\limits_{k\to\infty}(x_{n_k}-y)=0$ that is, $\lim\limits_{k\to\infty}x_{n_k}=y$, hence there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}}\to y$. Because $I(y)=y$, then I is an A-proper mapping. ∎

Lemma 6. *A and B are two nonempty number sets. If for any $x\in A$ and $y\in B$, we have $x<y$, then $\operatorname{Sup}A\leqslant\inf B$. In particular, when $A=\{a\}$ the conclusion still holds.*

Proof: It is obvious. ∎

Theorem 1. *Let (E,F,Δ) be a projected complete Menger PN-space, Δ be a continuous t-norm, $f:\overline{\Omega}\to E$ be an A-proper mapping, $C:\overline{\Omega}\to E$ be a continuous compact mapping and $(f-C)(x)=f(x)-C(x)$, $p\notin f(\partial\Omega)$, $p\notin C(\partial\Omega)$. For any $\varepsilon>0$, $\lambda\in R$, $x\in\partial\Omega$, $F_{f(x)+\lambda C(x)}(\varepsilon)\geqslant F_p(\varepsilon)$. Then $\operatorname{Deg}(f,\Omega,p)=\operatorname{Deg}(C,\Omega,p)$.*

Proof: Let $L(t,x)=tf(x)+(1-t)C(x)$. Because f and C are continuous, then $L(t,x)$ is also continuous. Because f is an A-proper mapping, by Lemma 3, tf is an A-proper mapping. Because C is continuous and compact, then $(1-t)C$ is still continuous

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and compact. By Lemma 4, $tf(x) + (1-t)C(x)$ is an A-proper mapping. Because f is an A-proper mapping and C is continuous and compact, then $f - C$ is continuous and bounded. Hence when $t \to t_0$, we have $t\big(f(x) - C(x)\big) \to t_0\big(f(x) - C(x)\big)$. Hence

$$\lim_{t \to t_0} F_{t(f(x)-C(x))-t_0(f(x)-C(x))}(\varepsilon) = H(\varepsilon), \quad \forall \varepsilon > 0, \quad \forall x \in \overline{\Omega}.$$

Then for any $\lambda > 0$, we have $F_{t(f(x)-C(x))-t_0(f(x)-C(x))}(\varepsilon) > 1 - \lambda$ $(t \to t_0)$. By Lemma 6, we have $\inf_{x \in \overline{\Omega}} F_{t(f(x)-C(x))-t_0(f(x)-C(x))}(\varepsilon) \geqslant 1 - \lambda$. By the arbitrariness of λ, we have

$$\inf_{x \in \overline{\Omega}} F_{t(f(x)-C(x))-t_0(f(x)-C(x))}(\varepsilon) = 1 \ (t \to t_0).$$

Therefore

$$\lim_{t \to t_0} \inf_{x \in \overline{\Omega}} F_{t(f(x)-C(x))-t_0(f(x)-C(x))}(\varepsilon) = H(\varepsilon).$$

Next, we prove $p \notin h_t(\partial\Omega)$. Using reduction to absurdity, we assume there exist $x_0 \in \partial\Omega$ and $t_0 \in [0,1]$ such that $h_{t_0}(x_0) = p$. Because $p \notin f(\partial\Omega)$, then $t_0 \neq 1$. Because $p \notin C(\partial\Omega)$, then $t_0 \neq 0$. Hence $t_0 \in (0,1)$. By $p = t_0 f(x_0) + (1-t_0)C(x_0)$, we have $p/t_0 = f(x_0) + (1-t_0)/(t_0)C(x_0)$. Taking $\lambda = (1-t_0)/(t_0)$, we have

$$F_{f(x_0)+\lambda C(x_0)}(\varepsilon) = F_{f(x_0)+(1-t_0)/(t_0)C(x_0)}(\varepsilon) = F_{(p/t_0)}(\varepsilon) = F_p(\varepsilon t_0) < F_p(\varepsilon).$$

It contradicts known conditions. Hence we have $p \notin h_t(\partial\Omega)$. By the Lemma 2 (iii), we have $\mathrm{Deg}(h_1, \Omega, p) = \mathrm{Deg}(h_0, \Omega, p)$. Hence $\mathrm{Deg}(f, \Omega, p) = \mathrm{Deg}(C, \Omega, p)$. ∎

DEFINITION 2: Let (E, F, Δ) be a projected complete Menger PN-space. Δ is a continuous t-norm. Ω is the bounded open set of E:

(i) If for any $\lambda \geqslant 0$, $f + \lambda I : \overline{\Omega} \to E$ is A-proper, then f is called P_*-compact mapping;

(ii) For given $\gamma > 0$, if for any $\lambda \geqslant \gamma$, $f - \lambda I : \overline{\Omega} \to E$ is A-proper, then f is called P_γ-compact mapping.

DEFINITION 3: Let (E, F, Δ) be a projected complete Menger PN-space. Δ is a continuous t-norm. Ω is a bounded open neighbourhood of E, which is symmetric about $\theta \in \Omega$. $f : \overline{\Omega} \to E$ is said to be a convex A-proper mapping, if for $L : [0,1] \times \overline{\Omega} \to E$, we have $L(t,x) = h_t(x) = (1/1+t)f(x) - (t/1+t)f(-x)$ is A-proper.

THEOREM 2. *Let (E, F, Δ) be a projected complete Menger PN-space. Δ is a continuous t-norm. Ω is a bounded open set of E, $f : \overline{\Omega} \to E$ is a P_*-compact mapping, $\theta \in \Omega$, we assume that $F_{x-f(x)}(\varepsilon) > F_x(\varepsilon), \forall \varepsilon > 0, \forall x \in \partial\Omega$, then there must exist an $x^* \in \Omega$ such that $f(x^*) = \theta$.*

PROOF: Let $h_t(x) = L(t,x) = (1-t)f(x) + tx$, then $L : [0,1] \times \overline{\Omega} \to E$ is continuous. When $t \neq 1$, we have $h_t = (1-t)f + tI = (1-t)(f + (t/1-t)I)$. Because $(t/1-t) \geqslant 0$, by the P_*-compact property of f, $h_t : \overline{\Omega} \to E$ is A-proper. Because $h_1 = I$ is A-proper, then for any $t \in [0,1]$, h_t is A-proper. Because f is a P_*−compact mapping, then $f(x)$ is bounded. Because $\overline{\Omega}$ is a bounded closed set, $I(x)$ is bounded. Hence $f(x)-x$ is bounded. Thus for any $t, t_0 \in [0,1]$, $x \in \overline{\Omega}$, when $t \to t_0$, we have $t(f(x)-x) \to t_0(f(x)-x)$. Thus $\lim\limits_{t \to t_0} F_{t(f(x)-x)-t_0(f(x)-x)}(\varepsilon) = H(\varepsilon)$. By Lemma 6, we have

$$\lim_{t \to t_0} \inf_{x \in \overline{\Omega}} F_{t(f(x)-x)-t_0(f(x)-x)}(\varepsilon) = H(\varepsilon).$$

In the following, we prove $\theta \notin h_t(\partial\Omega)$ $\big(t \in [0,1]\big)$. Assuming there exist an $x_0 \in \partial\Omega$ and a $t_0 \in [0,1]$ such that $h_{t_0}(x_0) = \theta$ that is, $(1-t_0)f(x_0) + t_0 x_0 = \theta$. Because $\theta \in \Omega$, then $t_0 \neq 1$. Thus

$$f(x_0) = (t_0/t_0 - 1)x_0, \; F_{x_0 - f(x_0)}(\varepsilon) = F_{x_0 - (t_0/t_0 - 1)x_0}(\varepsilon) = F_{(1/1-t_0)x_0}(\varepsilon) = F_{x_0}\big((1-t_0)\varepsilon\big).$$

Because $t_0 \in [0,1)$, then $(1-t_0)\varepsilon \leqslant \varepsilon$. By the properties of distribution function, we have $F_{x_0}\big((1-t_0)\varepsilon\big) \leqslant F_{x_0}(\varepsilon)$. It contradicts known conditions. Thus $\theta \notin h_t(\partial\Omega)$. By the Lemma 2 (iii), we have

$$\mathrm{Deg}(f,\Omega,\theta) = \mathrm{Deg}(h_0,\Omega,\theta) = \mathrm{Deg}(h_1,\Omega,\theta) = \mathrm{Deg}(I,\Omega,\theta) = \{1\}.$$

Therefore, there exists an $x^* \in \Omega$ such that $f(x^*) = \theta$. ∎

Theorem 3. *Let (E,F,Δ) be a projected complete Menger PN-space. Δ is a continuous t-norm. Ω is a bounded open neighbourhood of E, which is symmetric about $\theta \in \Omega$. If $f : \overline{\Omega} \to E$ is an A-proper mapping and*

$$f(-x) = -f(x), \; f(x) \neq \theta, \quad \forall x \in \partial\Omega,$$

then there exists an $x_0 \in \Omega$ such that $f(x_0) = \theta$.

Proof: Because $f(-x) = -f(x)$, then $f(-x) + f(x) = 0$. Because Q_{n_k} is continuous and linear, then $Q_{n_k}\big(f(-x) + f(x)\big) = 0$ and $Q_{n_k} f(-x) = -Q_{n_k} f(x)$. Hence $Q_{n_k} f$ is an odd mapping. By the properties of topological degree in finite dimensional space, $\deg(Q_{n_k} f, \Omega_{n_k}, \theta)$ is an odd integer (see Chang [**2**]). By the definition $\mathrm{Deg}(f,\Omega,\theta) = \{\gamma \in Z^* \mid$ there exists a subsequence $\{n_k\}$of $\{n\}$ such that $\deg_R(Q_{n_k} f, \Omega_{n_k}, \theta) \to \gamma\}$, then there exists an $x_0 \in \Omega$ such that $f(x_0) = \theta$. ∎

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