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Abstract

A conjecture of Manin predicts the distribution of rational points on Fano varieties. We provide a framework for proofs of Manin's conjecture for del Pezzo surfaces over imaginary quadratic fields, using universal torsors. Some of our tools are formulated over arbitrary number fields. As an application, we prove Manin's conjecture over imaginary quadratic fields K for the quartic del Pezzo surface S of singularity type $\mathbf{A}_3$ with five lines given in $\mathbb{P}^4_K$ by the equations $x_0x_1 - x_2x_3 = x_0x_3 + x_1x_3 + x_2x_4 = 0$.

Contents

1. Introduction

Let S be a del Pezzo surface defined over a number field K with only **ADE**-singularities, let H be a height function on $S(K)$ given by an anticanonical embedding, and let U be the subset

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obtained by removing the lines in S. If $S(K)$ is Zariski-dense in S, we are interested in the counting function

$$N_{U,H}(B) := |\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leqslant B\}|. \tag{1.1}$$

In this setting, Manin's conjecture [FMT89, BM90] (generalized in [BT98b] to include our singular del Pezzo surfaces) predicts an asymptotic formula of the form

$$N_{U,H}(B) = c_{S,H} B(\log B)^{\rho-1}(1 + o(1)), \tag{1.2}$$

where ρ is the rank of the Picard group of a minimal desingularization of S. The positive constant $c_{S,H}$ was made explicit by Peyre [Pey95, Pey03] and Batyrev–Tschinkel [BT98b].

Over $\mathbb{Q}$, Manin's conjecture is known for several del Pezzo surfaces and some other classes of varieties. To the best of the authors' knowledge, all currently known cases of Manin's conjecture over number fields beyond $\mathbb{Q}$ concern varieties with a suitable action of an algebraic group and can be proved via harmonic analysis on adelic points (e.g. flag varieties [FMT89], toric varieties [BT98a], and equivariant compactifications of additive groups [CLT02]; this includes some del Pezzo surfaces, classified in [DL10]).

In this article, we provide a framework for proofs of the above formula over imaginary quadratic fields for del Pezzo surfaces without such a special structure. Where no additional efforts are required, our results are formulated for arbitrary number fields.

These methods are then applied to prove Manin's conjecture for the del Pezzo surfaces over arbitrary imaginary quadratic fields K of degree 4 and type $\mathbf{A}_3$ with five lines, with respect to their anticanonical embeddings in $\mathbb{P}^4_K$ given by the equations

$$x_0x_1 - x_2x_3 = x_0x_3 + x_1x_3 + x_2x_4 = 0. \tag{1.3}$$

This is the first proof of Manin's conjecture over number fields beyond $\mathbb{Q}$ for varieties where the harmonic analysis approach cannot be applied.

Similar applications of our framework allow the treatment of at least the split quartic del Pezzo surfaces of types $\mathbf{A}_3 + \mathbf{A}_1$, $\mathbf{A}_4$, $\mathbf{D}_4$, $\mathbf{D}_5$ over imaginary quadratic fields [DF14a]. For a cubic surface of type $\mathbf{E}_6$, see [DF14b].

1.1 Background

Apart from the general results mentioned above for varieties with large group actions, Manin's conjecture is known for smooth complete intersections over the rationals[1] whose dimension is large enough compared with their degree, via the Hardy–Littlewood circle method [Bir62, Pey95].

For low-dimensional varieties without such actions of algebraic groups, Manin's conjecture is known so far only in isolated cases over $\mathbb{Q}$, for heights given by specific anticanonical embeddings. In particular, the case of del Pezzo surfaces has been investigated from the beginning (see, e.g., [BM90, Proposition 5.4], [Pey95, § 8–11] for some toric del Pezzo surfaces of degree $d \geqslant 6$ over $\mathbb{Q}$, and [FMT89, Appendix] and [PT01] for computational evidence in degree 3 over $\mathbb{Q}$).

The most important technique is the use of universal torsors, which were invented by Colliot–Thélène and Sansuc (see [CTS87], for example) and first applied to Manin's conjecture by Salberger (see [Pey98, Sal98]). The testing ground was a new proof in the case of split toric varieties over $\mathbb{Q}$ (see [Sal98]).

[1] After submission of the present article, Loughran [Lou13] showed how to derive this over arbitrary number fields from the work of Skinner [Ski97].

The central milestones beyond toric varieties were the first examples of possibly singular del Pezzo surfaces of degrees 5 (see [dlB02]), 4 (see [dlBB07b]), 3 (see [dlBBD07]), and 2 (see [BB13]) that are not covered by [BT98a] or [CLT02]. A long series of further examples followed, all of them over $\mathbb{Q}$, each dealing with difficulties not encountered before. Also all higher-dimensional results involving universal torsors concern varieties over $\mathbb{Q}$ (specific cubic hypersurfaces of dimension 3 (see [dlB07]) and 4 (see [BBS14])).

A relatively general strategy has emerged for split singular del Pezzo surfaces over $\mathbb{Q}$ whose universal torsors are open subsets of affine hypersurfaces, as classified in [Der14]. This is summarized in [Der09]. In that basic form, it turns out to be sufficient for quartic del Pezzo surfaces over $\mathbb{Q}$ of types $\mathbf{D}_5$ (see [dlBB07a]), $\mathbf{D}_4$ (see [DT07]), $\mathbf{A}_4$ (see [BD09b]), $\mathbf{A}_3 + \mathbf{A}_1$ (see [Der09]) and $\mathbf{A}_3$ with five lines (see Theorem 9.11).

For the cubic surfaces of types $\mathbf{E}_6$ (see [dlBBD07]), $\mathbf{D}_5$ (see [BD09a]) and $\mathbf{A}_5 + \mathbf{A}_1$ (see [BD13]) over $\mathbb{Q}$, the strategy of [Der09] goes through when combined with significant further analytic input. In other cases such as [LB13], larger deviations from [Der09] seem necessary.

Over number fields beyond $\mathbb{Q}$, we have the classical result of Schanuel [Sch79] for projective spaces (which are toric) that can be interpreted as a basic case of the universal torsor approach, and a new proof of Manin's conjecture via universal torsors for the toric singular cubic surface of type $3\mathbf{A}_2$ (see [DJ13] over imaginary quadratic fields of class number 1 and [Fre13] over arbitrary number fields).

Our goal is to generalize the universal torsor approach towards Manin's conjecture to non-toric varieties over number fields other than $\mathbb{Q}$. The two main general challenges arise from the unavailability of unique factorization (if the class number is greater than 1) and from difficulties in regard to counting lattice points (if K has more than one Archimedean place, whence the unit group of its ring of integers is infinite). Furthermore, the existing results over $\mathbb{Q}$ often combine the universal torsor method with subtle applications of deep results from analytic number theory that are only available over $\mathbb{Q}$ in their full strength. To mitigate these additional difficulties, it seems natural to focus on singular quartic del Pezzo surfaces first.

1.2 Results

Our main results are the techniques presented in §§ 4–8, which are described in slightly more detail below.

They allow a rather straightforward treatment of the split quartic del Pezzo surfaces of types $\mathbf{A}_3$ with five lines (see Theorem 1.1), $\mathbf{A}_3 + \mathbf{A}_1$, $\mathbf{A}_4$, $\mathbf{D}_4$, $\mathbf{D}_5$ (see [DF14a]) and are an important ingredient in the proof for the $\mathbf{E}_6$ cubic surface [DF14b] over imaginary quadratic fields. They should also be enough for some del Pezzo surfaces of higher degree (e.g. in degree 5, that of type $\mathbf{A}_2$ treated over $\mathbb{Q}$ in [Der07b] and, in degree 6, those of type $\mathbf{A}_2$ in [Lou10] and of type $\mathbf{A}_1$ with three lines in [Bro09]). We expect that an application to the other cubic cases mentioned above or to other quartic del Pezzo surfaces (such as those of type $\mathbf{A}_3$ with four lines treated over $\mathbb{Q}$ in [LB12a], of types $3\mathbf{A}_1$ and $\mathbf{A}_2 + \mathbf{A}_1$ in [LB12b], of types $2\mathbf{A}_1$ with eight lines in [dlBBP12, dlBB12, dlBT13, Des13, Lou12], and the smooth quartic del Pezzo surface of [dlBB11]) would require additional work.

In § 9 we demonstrate how to apply our techniques by proving the following case of Manin's conjecture.

Let $K \subset \mathbb{C}$ be an imaginary quadratic field with ring of integers $\mathcal{O}_K$, discriminant Δ_K, class number h_K, and with $\omega_K := |\mathcal{O}_K^\times|$ units. On $\mathbb{P}^4_K(K)$, we use the (exponential) Weil height given by

$$H(x_0 : \cdots : x_4) := \frac{\max\{\|x_0\|_\infty, \ldots, \|x_4\|_\infty\}}{\mathfrak{N}(x_0\mathcal{O}_K + \cdots + x_4\mathcal{O}_K)}, \tag{1.4}$$

where $\|x_i\|_\infty := |x_i|^2$ for the usual complex absolute value $|\cdot|$ and $\mathfrak{N}\mathfrak{a}$ denotes the absolute norm of a fractional ideal $\mathfrak{a}$.

Let $S \subset \mathbb{P}^4_K$ be the del Pezzo surface of degree 4 defined by (1.3). Up to isomorphism, it is the unique split del Pezzo surface that contains a singularity of type $\mathbf{A}_3$ and five lines.

Theorem 1.1. *Let K be an imaginary quadratic field. Let U be the complement of the lines in the del Pezzo surface $S \subset \mathbb{P}^4_K$ defined over K by (1.3). For $B \geqslant 3$, we have*

$$N_{U,H}(B) = c_{S,H} B(\log B)^5 + O(B(\log B)^4 \log\log B),$$

with

$$c_{S,H} = \frac{1}{4320} \cdot \frac{(2\pi)^6 h_K^6}{\Delta_K^4 \omega_K^6} \cdot \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right) \cdot \omega_\infty,$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of $\mathcal{O}_K$ and

$$\omega_\infty = \frac{12}{\pi} \int_{\max\{\|z_0 z_2^2\|_\infty, \|z_1 z_2^2\|_\infty, \|z_2^3\|_\infty, \|z_0 z_1 z_2\|_\infty, \|z_0 z_1 (z_0+z_1)\|_\infty\} \leqslant 1} dz_0\, dz_1\, dz_2.$$

Since S is split, its minimal desingularization $\widetilde{S}$ is a blow-up of $\mathbb{P}^2_K$ in five rational points in almost general position, hence $\rho = \operatorname{rk}\operatorname{Pic}(\widetilde{S}) = 6$, so our result agrees with Manin's conjecture. Our leading constant appears to be that predicted by Peyre. See Theorem 9.11 for the analogous result over $\mathbb{Q}$.

It would be interesting to see an explicit application of [Lou13, Theorem 1.1] giving Manin's conjecture for the family of fourfolds over $\mathbb{Q}$ obtained by Weil restriction of our surfaces over varying imaginary quadratic fields K.

1.3 Techniques and plan of the paper

What follows is a short description of our main results and how they should be applied to prove Manin's conjecture for some split del Pezzo surfaces S over imaginary quadratic fields. How this works in the specific case of S defined by (1.3) is shown in our proof of Theorem 1.1 in § 9.

In § 2, we investigate sums of two classes of arithmetic functions over general number fields.

In § 3, we consider the problem of asymptotically counting lattice points in certain bounded subsets of $\mathbb{C} = \mathbb{R}^2$ given by inequalities of the form $\|f_i(z)\|_\infty \leqslant \|g_i(z)\|_\infty$, with polynomials f_i, $g_i \in \mathbb{C}[X]$. We use the notion of sets of *class m* introduced by Schmidt [Sch95] and reduce our counting problems to a classical result of Davenport [Dav51]. Moreover, we prove a tameness result for parametric integrals over semialgebraic functions, which can be applied to show that certain volume functions arising in partial summations do not oscillate too much.

In § 4, we describe a strategy to parameterize, up to a certain action of a power of the unit group, K-rational points on U of bounded height by points $(\eta_1, \dots, \eta_t)$ on a universal torsor $\mathcal{T}$ over a minimal desingularization $\widetilde{S}$ of S with coordinates η_i in certain fractional ideals $\mathcal{O}_i$ of K and satisfying certain coprimality and height conditions. If K is $\mathbb{Q}$ or imaginary quadratic, we propose a parameterization (Claim 4.1) that is closely related to the geometry of $\widetilde{S}$. We expect this to work whenever $\mathcal{T}$ is an open subset of a hypersurface in affine space $\mathbb{A}^t_K$ provided that the anticanonical embedding $S \subset \mathbb{P}^4_K$ is chosen favorably. In [Der14], all such del Pezzo surfaces are classified and suitable models are given.

It is usually straightforward to prove Claim 4.1 in special cases by induction over a chain of blow-ups of $\mathbb{P}^2_K$ giving $\widetilde{S}$. Using the structure of $\operatorname{Pic}(\widetilde{S})$, we show that certain steps in this induction hold in general. To deal with the lack of unique factorization in $\mathcal{O}_K$, we apply arguments introduced by Dedekind and Weber.

In § 5, we provide the tools to sum the result of our parameterization in § 4 over two variables η_{t-1}, η_t, using our lattice point counting results from § 2. Unavailability of unique factorization leads to difficulties of a technical nature. The results of this and the next section are specific to imaginary quadratic fields.

In § 6, we provide a general tool to sum the main term in the result of § 5 over a further variable η_{t-2}. Depending on the form of the equation defining the universal torsor $\mathcal{T}$ in a specific application, this result will be applied in two different ways.

In applications to specific del Pezzo surfaces, it still remains to estimate the error terms in the first and second summations. This is straightforward for some singular del Pezzo surfaces of degree 4 and higher, but much harder for del Pezzo surfaces of lower degree that are smooth or have mild singularities. To handle additional cases, the most elementary trick is to choose different orders of summations depending on the relative sizes of the variables. Our results are compatible with this trick, and indeed it is heavily applied in the proof of Theorem 1.1 (with four different orders of summations; fortunately, two of them can be handled by symmetry).

In § 7, we prove a result handling the summations over all of the remaining variables $\eta_1, \dots, \eta_{t-3}$ at once, under certain assumptions on the main term after the second summation. The results in this section are formulated in terms of ideals instead of elements, which appears to be the natural way to generalize the respective versions over $\mathbb{Q}$. It seems interesting to point out that in our applications, we find an opportunity to pass from sums over elements to sums over ideals right after the second summation (cf. Lemmas 9.4 and 9.7 in the $\mathbf{A}_3$-case).

To prove the analog of Theorem 1.1 over arbitrary number fields K, one can also start with a bijection between the rational points on U and orbits of integral points on universal torsors under an action of $(\mathcal{O}_K^\times)^6$, see Claim 4.2. If $\mathcal{O}_K^\times$ is infinite, one must work with a fundamental domain for this action. The main difficulties are, on the one hand, to construct such a fundamental domain in a way that facilitates counting integral points in it, and, on the other hand, to find techniques to conduct this counting with acceptable error terms.

1.4 Notation

The symbol K will always denote a fixed number field, which is in some sections arbitrary and in some sections imaginary quadratic or $\mathbb{Q}$. We denote the degree of K by d, and the number of real (respectively complex) places of K by s_1 (respectively s_2). By $\mathcal{C}$, we denote a fixed system of integral representatives for the ideal classes of K, i.e. $\mathcal{C}$ contains exactly one integral ideal from each class.

When we use Vinogradov's $\ll$-notation or Landau's O-notation, the implied constants may always depend on K. In cases where they may depend on other objects as well, we mention this, for example by writing $\ll_C$ or O_C if the constant may depend on C.

In addition to the notation introduced before Theorem 1.1, we use R_K to denote the regulator of K and $\mathcal{I}_K$ to denote the monoid of non-zero ideals of $\mathcal{O}_K$. The symbol $\mathfrak{a}$ (respectively $\mathfrak{p}$) always denotes an ideal (respectively non-zero prime ideal) of $\mathcal{O}_K$, and $v_\mathfrak{p}(\mathfrak{a})$ is the non-negative integer such that $\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})} \mid \mathfrak{a}$ and $\mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})+1} \nmid \mathfrak{a}$. We extend this in the usual way to fractional ideals (with $v_\mathfrak{p}(\{0\}) := \infty$), and for $x \in K$, write $v_\mathfrak{p}(x) := v_\mathfrak{p}(x\mathcal{O}_K)$ for the usual $\mathfrak{p}$-adic exponential valuation.

We say that $x \in K$ is *defined modulo* $\mathfrak{a}$ (respectively *invertible modulo* $\mathfrak{a}$) if $v_\mathfrak{p}(x) \geqslant 0$ (respectively $v_\mathfrak{p}(x) = 0$) for all $\mathfrak{p} \mid \mathfrak{a}$. If x is defined modulo $\mathfrak{a}$, then it has a well-defined residue class modulo $\mathfrak{a}$, and we write $x \equiv_\mathfrak{a} y$ if the residue classes of x, y coincide or, equivalently, $v_\mathfrak{p}(x - y) \geqslant v_\mathfrak{p}(\mathfrak{a})$ for all $\mathfrak{p} \mid \mathfrak{a}$.

Sums and products indexed by (prime) ideals always run over non-zero (prime) ideals. For simplicity, we define

$$\rho_K := \frac{2^{s_1}(2\pi)^{s_2}R_K}{\omega_K\sqrt{|\Delta_K|}}.$$

By $\tau_K(\mathfrak{a})$ (respectively $\omega_K(\mathfrak{a})$), we denote the number of distinct divisors (respectively distinct prime divisors) of $\mathfrak{a} \in \mathcal{I}_K$, and μ_K is the Möbius function on $\mathcal{I}_K$. Moreover, ϕ_K is Euler's ϕ-function for $\mathcal{I}_K$, and $\phi_K^*(\mathfrak{a}) := \phi_K(\mathfrak{a})/\mathfrak{N}\mathfrak{a} = \prod_{\mathfrak{p}\mid\mathfrak{a}}(1 - 1/\mathfrak{N}\mathfrak{p})$.

2. Arithmetic functions

In this section, K can be any number field of degree $d \geqslant 2$ (for $d = 1$, see [Der09]). We will need to deal with sums involving certain coprimality conditions, which are encoded by arithmetic functions of the following type, analogous to [Der09, Definition 6.6].

Definition 2.1. Let $\mathfrak{b} \in \mathcal{I}_K$ and $C_1, C_2, C_3 \geqslant 1$. Then $\Theta(\mathfrak{b}, C_1, C_2, C_3)$ is the set of all functions $\vartheta : \mathcal{I}_K \to \mathbb{R}_{\geqslant 0}$ such that there exist functions $A_\mathfrak{p} : \mathbb{Z}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ satisfying

$$\vartheta(\mathfrak{a}) = \prod_{\mathfrak{p}} A_\mathfrak{p}(v_\mathfrak{p}(\mathfrak{a}))$$

for all $\mathfrak{a} \in \mathcal{I}_K$, where:

(1) for all $\mathfrak{p}$ and $n \geqslant 1$,

$$|A_\mathfrak{p}(n) - A_\mathfrak{p}(n-1)| \leqslant \begin{cases} C_1 & \text{if } \mathfrak{p}^n \mid \mathfrak{b}, \\ C_2\mathfrak{N}\mathfrak{p}^{-n} & \text{if } \mathfrak{p}^n \nmid \mathfrak{b}; \end{cases}$$

(2) for all $\mathfrak{a} \in \mathcal{I}_K$, we have $\prod_{\mathfrak{p}\nmid\mathfrak{a}} A_\mathfrak{p}(0) \leqslant C_3$.

We say that the functions $A_\mathfrak{p}$ *correspond to* ϑ.

The following lemma describes some elementary properties of the functions defined above.

Lemma 2.2. *Let $\vartheta \in \Theta(\mathfrak{b}, C_1, C_2, C_3)$ with corresponding functions $A_\mathfrak{p}$. Then:*

(1) *for any $\mathfrak{a} \in \mathcal{I}_K$,*

$$(\vartheta * \mu_K)(\mathfrak{a}) = \prod_{p\nmid a} A_\mathfrak{p}(0) \prod_{\mathfrak{p}\mid\mathfrak{a}} (A_\mathfrak{p}(v_\mathfrak{p}(\mathfrak{a})) - A_\mathfrak{p}(v_\mathfrak{p}(\mathfrak{a}) - 1));$$

(2) *for any $t \geqslant 0$,*

$$\sum_{\mathfrak{N}\mathfrak{a}\leqslant t} |(\vartheta * \mu_K)(\mathfrak{a})| \cdot \mathfrak{N}\mathfrak{a} \ll_{C_2} \tau_K(\mathfrak{b})(C_1C_2)^{\omega_K(\mathfrak{b})}C_3 t \log(t+2)^{C_2-1};$$

(3) *if ϑ is not the zero function and $\mathfrak{q} \in \mathcal{I}_K$, then the infinite sum and the infinite product*

$$\sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ \mathfrak{a}+\mathfrak{q}=\mathcal{O}_K}} \frac{(\vartheta * \mu_K)(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} \quad \text{and} \quad \prod_{\mathfrak{p}\nmid\mathfrak{q}}\left(\left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)\sum_{n=0}^{\infty}\frac{A_\mathfrak{p}(n)}{\mathfrak{N}\mathfrak{p}^n}\right)\prod_{\mathfrak{p}\mid\mathfrak{q}} A_\mathfrak{p}(0)$$

converge to the same real number.

Proof. The proof of [Der09, Proposition 6.8] holds almost verbatim in our case. □

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For $\vartheta \in \Theta(\mathfrak{b}, C_1, C_2, C_3)$ and $\mathfrak{q} \in \mathcal{I}_K$, we define

$$\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{q}) := \sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ \mathfrak{a}+\mathfrak{q}=\mathcal{O}_K}} \frac{(\vartheta * \mu_K)(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}$$

and $\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}) := \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K)$. Proposition 2.3 below shows that $\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a})$ can be seen as the average value of ϑ with respect to the variable $\mathfrak{a}$, which we mention explicitly to avoid confusion when dealing with multiple variables, see Corollary 2.7.

Lemma 2.2(3) provides an alternative form. In the simple case when ϑ has corresponding functions $A_\mathfrak{p}$ satisfying $A_\mathfrak{p}(n) = A_\mathfrak{p}(1)$ for all prime ideals $\mathfrak{p}$ and all $n \geqslant 1$, we have

$$\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{q}) = \prod_{\mathfrak{p}\nmid\mathfrak{q}} \left(\left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right) A_\mathfrak{p}(0) + \frac{1}{\mathfrak{N}\mathfrak{p}} A_\mathfrak{p}(1) \right) \prod_{\mathfrak{p}\mid\mathfrak{q}} A_\mathfrak{p}(0). \tag{2.1}$$

Proposition 2.3. *Let $\mathfrak{k}$ be an ideal class of K. For $\vartheta \in \Theta(\mathfrak{b}, C_1, C_2, C_3)$, we have*

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{k}\cap\mathcal{I}_K \\ \mathfrak{N}\mathfrak{a}\leqslant t}} \vartheta(\mathfrak{a}) = \rho_K \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}) t + O_{C_2}(\tau_K(\mathfrak{b})(C_1C_2)^{\omega_K(\mathfrak{b})} C_3 t^{1-1/d}),$$

for $t \geqslant 0$.

Proof. This follows immediately from Lemmas 2.2(1) and 2.5 below. □

Lemma 2.4. *Let $C \geqslant 0$, $c_\vartheta > 0$, and let $\vartheta : \mathcal{I}_K \to \mathbb{R}_{\geqslant 0}$ such that, for $t \geqslant 0$,*

$$\sum_{\mathfrak{N}\mathfrak{a}\leqslant t} \vartheta(\mathfrak{a}) \leqslant c_\vartheta t (\log(t+2))^C.$$

For any $\kappa \in \mathbb{R}$ and $1 \leqslant t_1 \leqslant t_2$, we have

$$\sum_{t_1\leqslant\mathfrak{N}\mathfrak{a}\leqslant t_2} \frac{\vartheta(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^\kappa} \ll_{C,\kappa} c_\vartheta \cdot \begin{cases} t_2^{1-\kappa}(\log(t_2+2))^C & \text{if } \kappa < 1, \\ \log(t_2+2)^{C+1} & \text{if } \kappa = 1, \\ t_1^{1-\kappa}(\log(t_1+2))^C \ll_{C,\kappa} 1 & \text{if } \kappa > 1. \end{cases}$$

Proof. We apply Abel's summation formula to $\vartheta'(n) := c_\vartheta^{-1} \sum_{\mathfrak{N}\mathfrak{a}=n} \vartheta(\mathfrak{a})$; see also [Der09, Lemma 3.4]. □

The next lemma completes the proof of Proposition 2.3.

Lemma 2.5. *Let $\mathfrak{k}$ be an ideal class of K, and let $\vartheta : \mathcal{I}_K \to \mathbb{R}$ such that*

$$\sum_{\mathfrak{N}\mathfrak{a}\leqslant t} |(\vartheta * \mu_K)(\mathfrak{a})| \cdot \mathfrak{N}\mathfrak{a} \ll c_\vartheta t (\log(t+2))^C,$$

for some $C \geqslant 0$, $c_\vartheta > 0$ and for all $t \geqslant 0$. Then

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{k}\cap\mathcal{I}_K \\ \mathfrak{N}\mathfrak{a}\leqslant t}} \vartheta(\mathfrak{a}) = \rho_K \sum_{\mathfrak{a}\in\mathcal{I}_K} \frac{(\vartheta * \mu_K)(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}} t + O_C(c_\vartheta t^{1-1/d}).$$

Proof. By Lemma 2.4, $\sum_{\mathfrak{a}\in\mathcal{I}_K}(\vartheta*\mu_K)(\mathfrak{a})/\mathfrak{N}\mathfrak{a} \ll_C c_\vartheta$, so the lemma holds for $t<1$. Now assume that $t \geqslant 1$. Since $\vartheta = (\vartheta*\mu_K)*1$, we have

$$\sum_{\substack{\mathfrak{a}\in\mathfrak{k}\cap\mathcal{I}_K\\ \mathfrak{N}\mathfrak{a}\leqslant t}} \vartheta(\mathfrak{a}) = \sum_{\substack{\mathfrak{a}\in\mathfrak{k}\cap\mathcal{I}_K\\ \mathfrak{N}\mathfrak{a}\leqslant t}} \sum_{\mathfrak{b}\mid\mathfrak{a}} (\vartheta*\mu_K)(\mathfrak{b}) = \sum_{\mathfrak{N}\mathfrak{b}\leqslant t}(\vartheta*\mu_K)(\mathfrak{b}) \sum_{\substack{\mathfrak{a}'\in[\mathfrak{b}]^{-1}\mathfrak{k}\cap\mathcal{I}_K\\ \mathfrak{N}\mathfrak{a}'\leqslant t/\mathfrak{N}\mathfrak{b}}} 1.$$

By the ideal theorem (see, e.g., [Lan94, VI, Theorem 3]), the inner sum is $\rho_K t/\mathfrak{N}\mathfrak{b} + O((t/\mathfrak{N}\mathfrak{b})^{(d-1)/d})$, so our sum is equal to

$$\rho_K \sum_{\mathfrak{b}\in\mathcal{I}_K} \frac{(\vartheta*\mu_K)(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} t + O\biggl(t\sum_{\mathfrak{N}\mathfrak{b}>t}\frac{|(\vartheta*\mu_K)(\mathfrak{b})|}{\mathfrak{N}\mathfrak{b}} + t^{(d-1)/d}\sum_{\mathfrak{N}\mathfrak{b}\leqslant t}\frac{|(\vartheta*\mu_K)(\mathfrak{b})|}{\mathfrak{N}\mathfrak{b}^{(d-1)/d}}\biggr).$$

By Lemma 2.4, the first part of the error term is $\ll_C c_\vartheta(\log(t+2))^C$ and the second part is $\ll_C c_\vartheta t^{1-1/d}$. $\square$

We introduce a class of multivariate arithmetic functions, similar to [Der09, Definition 7.8]. When fixing all variables but one, these functions are a special case of those discussed above.

Definition 2.6. Let $C \geqslant 1$, $r \in \mathbb{Z}_{\geqslant 0}$. Then $\Theta'_r(C)$ is the set of all functions $\theta : \mathcal{I}_K^r \to \mathbb{R}_{\geqslant 0}$ of the following shape: with $J_\mathfrak{p}(\mathfrak{a}_1,\dots,\mathfrak{a}_r) := \{i\in\{1,\dots,r\} : \mathfrak{p}\mid\mathfrak{a}_i\}$, we have

$$\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r) = \prod_\mathfrak{p} \theta_\mathfrak{p}(J_\mathfrak{p}(\mathfrak{a}_1,\dots,\mathfrak{a}_r)),$$

for functions $\theta_\mathfrak{p} : \{J \mid J\subset\{1,\dots,r\}\} \to [0,1]$ with

$$\theta_\mathfrak{p}(J) \geqslant \begin{cases} 1 - C\mathfrak{N}\mathfrak{p}^{-2} & \text{if } |J| = 0,\\ 1 - C\mathfrak{N}\mathfrak{p}^{-1} & \text{if } |J| = 1.\end{cases}$$

Let $\theta\in\Theta'_r(C)$, fix $\mathfrak{a}_1,\dots,\mathfrak{a}_{r-1}$, and let $\vartheta(\mathfrak{a}_r) := \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r-1},\mathfrak{a}_r)$. Then the factors $\theta_\mathfrak{p}(J_\mathfrak{p}(\mathfrak{a}_1,\dots,\mathfrak{a}_{r-1},\mathfrak{a}_r))$ depend only on $v_\mathfrak{p}(\mathfrak{a}_r)$, and we immediately obtain $\vartheta(\mathfrak{a}_r)\in\Theta(\prod_{\mathfrak{p}\mid\mathfrak{a}_1\cdots\mathfrak{a}_{r-1}}\mathfrak{p}, 1, C, 1)$. The following result follows immediately from Proposition 2.3.

Corollary 2.7. *Let $\theta\in\Theta'_r(C)$ and $\mathfrak{a}_1,\dots,\mathfrak{a}_{r-1}\in\mathcal{I}_K$. For $t\geqslant 0$, we have*

$$\sum_{\mathfrak{N}\mathfrak{a}_r\leqslant t} \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r) = \rho_K h_K \mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r),\mathfrak{a}_r)t + O_C((2C)^{\omega_K(\mathfrak{a}_1\cdots\mathfrak{a}_{r-1})}t^{1-1/d}).$$

By (2.1),

$$\mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r),\mathfrak{a}_r) = \prod_\mathfrak{p}\theta_\mathfrak{p}^{(r)}(J_\mathfrak{p}(\mathfrak{a}_1,\dots,\mathfrak{a}_{r-1})),$$

with

$$\theta_\mathfrak{p}^{(r)}(J) := \biggl(1-\frac{1}{\mathfrak{N}\mathfrak{p}}\biggr)\theta_\mathfrak{p}(J) + \frac{1}{\mathfrak{N}\mathfrak{p}}\theta_\mathfrak{p}(J\cup\{r\}).$$

If $r\geqslant 1$, we conclude that $\mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r),\mathfrak{a}_r)\in\Theta'_{r-1}(2C)$. This allows us to define, for $l\in\{1,\dots,r\}$,

$$\mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r),\mathfrak{a}_r,\dots,\mathfrak{a}_l) := \mathcal{A}(\cdots\mathcal{A}(\mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_r),\mathfrak{a}_r),\mathfrak{a}_{r-1})\cdots,\mathfrak{a}_l).$$

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Lemma 2.8. *Let $\theta \in \Theta'_r(C)$. Then*

$$\mathcal{A}(\theta(\mathfrak{a}_1, \dots, \mathfrak{a}_r), \mathfrak{a}_r, \dots, \mathfrak{a}_l) = \prod_{\mathfrak{p}} \theta_{\mathfrak{p}}^{(r,\dots,l)}(J_{\mathfrak{p}}(\mathfrak{a}_1, \dots, \mathfrak{a}_{l-1})),$$

where, for $J \subset \{1, \dots, l-1\}$,

$$\theta_{\mathfrak{p}}^{(r,\dots,l)}(J) := \sum_{L \subset \{l,\dots,r\}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{r+1-l-|L|} \left(\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{|L|} \theta_{\mathfrak{p}}(J \cup L).$$

In particular, for $l = 1$,

$$\mathcal{A}(\theta(\mathfrak{a}_1, \dots, \mathfrak{a}_r), \mathfrak{a}_r, \dots, \mathfrak{a}_1) = \prod_{\mathfrak{p}} \sum_{L \subset \{1,\dots,r\}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{r-|L|} \left(\frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{|L|} \theta_{\mathfrak{p}}(L). \tag{2.2}$$

Proof. This follows easily by induction. □

For our error estimates, we frequently need the following lemma.

Lemma 2.9. *Let $C \geqslant 0$. For $t \geqslant 0$, we have*

$$\sum_{\mathfrak{N}\mathfrak{a} \leqslant t} (C+1)^{\omega_K(\mathfrak{a})} \ll_C t(\log(t+2))^C.$$

Proof. This is clear if $t < 1$, so assume $t \geqslant 1$. Write $\vartheta(\mathfrak{a}) := (C+1)^{\omega_K(\mathfrak{a})}$. For any $\mathfrak{p}$, we have

$$(\vartheta * \mu_K)(\mathfrak{p}^n) = \begin{cases} 1 & \text{if } n = 0, \\ C & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

Since $\vartheta = (\vartheta * \mu_K) * 1$,

$$\sum_{\mathfrak{N}\mathfrak{a} \leqslant t} \vartheta(\mathfrak{a}) = \sum_{\mathfrak{N}\mathfrak{a} \leqslant t} \sum_{\mathfrak{b} | \mathfrak{a}} (\vartheta * \mu_K)(\mathfrak{b}) \ll t \sum_{\mathfrak{N}\mathfrak{b} \leqslant t} \frac{(\vartheta * \mu_K)(\mathfrak{b})}{\mathfrak{N}\mathfrak{b}} \leqslant t \prod_{\mathfrak{N}\mathfrak{p} \leqslant t} \left(1 + \frac{C}{\mathfrak{N}\mathfrak{p}}\right),$$

where $\mathfrak{p}$ runs over all non-zero prime ideals of $\mathcal{O}_K$ with norm bounded by t. By the prime ideal theorem (e.g. [Nar90, Corollary 1 after Proposition 7.10]) and Abelian partial summation, we obtain

$$\prod_{\mathfrak{N}\mathfrak{p} \leqslant t} \left(1 + \frac{C}{\mathfrak{N}\mathfrak{p}}\right) \leqslant \exp\left(\sum_{\mathfrak{N}\mathfrak{p} \leqslant t} \frac{C}{\mathfrak{N}\mathfrak{p}}\right) \ll_C (\log(t+2))^C.$$ □

The following lemma allows us to replace certain sums with integrals. It is a crucial tool for the results in §§ 6 and 7.

Lemma 2.10. *Let $\mathfrak{k}$ be an ideal class of K and $\vartheta : \mathcal{I}_K \to \mathbb{R}$ be a function such that*

$$\sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} \leqslant t}} \vartheta(\mathfrak{a}) - ct \ll \sum_{i=1}^{m} c_i t^{b_i} \log(t+2)^{k_i}, \tag{2.3}$$

with $m \in \mathbb{Z}_{>0}$, $c > 0$, c_i, $b_i \geqslant 0$, $k_i \in \mathbb{Z}_{\geqslant 0}$, holds for all $t \geqslant 0$.

Let $1 \leqslant t_1 \leqslant t_2$, and let $g : [t_1, t_2] \to \mathbb{R}$ such that there exists a partition of $[t_1, t_2]$ into at most $R(g) \geqslant 1$ intervals on whose interior g is continuously differentiable and monotonic. Moreover, we assume that there are $a \leqslant 0$, $c_g \geqslant 0$ such that $g(t) \ll c_g t^a$ for all $t \in [t_1, t_2]$. Then

$$\sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ t_1 < \mathfrak{N}\mathfrak{a} \leqslant t_2}} \vartheta(\mathfrak{a}) g(\mathfrak{N}\mathfrak{a}) = c \int_{t_1}^{t_2} g(t)\, dt + \mathcal{E}(t_1, t_2), \tag{2.4}$$

where

$$\mathcal{E}(t_1, t_2) \ll_{a, b_i, k_i} R(g) \sum_{i=1}^{m} c_g c_i \begin{cases} t_2^{b_i} \log(t_2 + 2)^{k_i} & \text{if } a = 0, \\ \sup_{t_1 \leqslant t \leqslant t_2} (t^{a+b_i} \log(t+2)^{k_i}) & \text{if } a + b_i \neq 0, \\ \log(t_2 + 2)^{k_i + 1} & \text{if } a + b_i = 0. \end{cases} \tag{2.5}$$

An analogous formula holds for $\sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ t_1 \leqslant \mathfrak{N}\mathfrak{a} \leqslant t_2}} \vartheta(\mathfrak{a}) g(\mathfrak{N}\mathfrak{a})$.

Proof. For any $t \in \mathbb{Z} \cap [t_1, t_2]$, $\varepsilon \in (0, 1)$, we have

$$\sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} = t}} \vartheta(\mathfrak{a}) = \sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} \leqslant t}} \vartheta(\mathfrak{a}) - \sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} \leqslant t - \varepsilon}} \vartheta(\mathfrak{a}) \ll c\varepsilon + \sum_{i=1}^{m} c_i t^{b_i} \log(t+2)^{k_i}.$$

Letting $\varepsilon \to 0$, we see that the contribution of the ideals $\mathfrak{a}$ with $\mathfrak{N}\mathfrak{a} = t$ is dominated by the error term.

Hence, it is enough to consider the case $R(g) = 1$ and to assume that g is continuously differentiable and monotonic on $[t_1, t_2]$. We denote

$$E(t) := \sum_{\substack{\mathfrak{a} \in \mathfrak{k} \cap \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} \leqslant t}} \vartheta(\mathfrak{a}) - ct$$

and start with a similar strategy as in the proof of [Der09, Lemma 3.1]. Let $S(t_1, t_2)$ be the sum on the left-hand side of (2.4). With Abel's summation formula and integration by parts, we obtain

$$S(t_1, t_2) = c \int_{t_1}^{t_2} g(t)\, dt + E(t_2) g(t_2) - E(t_1) g(t_1) - \int_{t_1}^{t_2} E(t) g'(t)\, dt.$$

By linearity, we may assume that $m = 1$, so $|E(t)| \leqslant c_1 t^{b_1} \log(t+2)^{k_1}$. Clearly, the $E(t_i) g(t_i)$ satisfy (2.5). Then

$$\int_{t_1}^{t_2} E(t) g'(t)\, dt \ll c_1 \left| \int_{t_1}^{t_2} t^{b_1} \log(t+2)^{k_1} g'(t)\, dt \right|. \tag{2.6}$$

The bound for $a = 0$ follows by estimating the integrand by $t_2^{b_1} \log(t_2 + 2)^{k_1} g'(t)$. Moreover, if $b_1 = k_1 = 0$, the term on the right-hand side of (2.6) is clearly $\ll c_1 |[g(t)]_{t_1}^{t_2}| \ll c_g c_1 t_1^a$. Otherwise, we use integration by parts to further estimate the integral by

$$\begin{aligned} &\ll_{b_1, k_1} c_1 |[t^{b_1} \log(t+2)^{k_1} g(t)]_{t_1}^{t_2}| + c_1 \left| \int_{t_1}^{t_2} t^{b_1 - 1} \log(t+2)^{k_1} g(t)\, dt \right| \\ &\ll c_g c_1 \sup_{t_1 \leqslant t \leqslant t_2} t^{a+b_1} \log(t+2)^{k_1} + c_g c_1 \int_{t_1}^{t_2} t^{a+b_1-1} \log(t+2)^{k_1}\, dt. \end{aligned}$$

A simple computation shows that the last integral is $\ll \log(t_2 + 2)^{k_1 + 1}$ if $a + b_1 = 0$, and k_1-fold integration by parts shows that it is $\ll_{a, b_1, k_1} |[t^{a+b_1} \log(t+2)^{k_1}]_{t_1}^{t_2}|$ otherwise. $\square$

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3. Lattice points and integrals

Whenever we talk about integrals or lattices, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z \mapsto (\Re z, \Im z)$. For a lattice Λ in $\mathbb{R}^n$ (by which we mean the $\mathbb{Z}$-span of n linearly independent vectors in $\mathbb{R}^n$) and a 'nice' bounded subset $S \subset \mathbb{R}^n$, one usually approximates $|\Lambda \cap S|$ by the quantity $\mathrm{vol}(S)/\det(\Lambda)$. To make this precise, we need to define 'nice' sets in our context. We follow an approach developed by Davenport [Dav51] and Schmidt [Sch95]. For a comparison with a different approach using Lipschitz-parameterizability, see [Wid12].

Definition 3.1 [Sch95, p. 347]. A compact subset $S \subset \mathbb{R}^n$ is *of class m* if every line intersects S in at most m single points and intervals and if the same holds for all projections of S on all linear subspaces.

In particular, the sets of class 1 are the compact convex sets. In our applications, we consider sets as in the following lemma.

Lemma 3.2. *Let l, $D \in \mathbb{Z}_{>0}$. For $j \in \{1, \dots, l\}$, let f_j, $g_j \in \mathbb{C}[X]$ be polynomials of degree at most D, and let $\prec_j \in \{\leqslant, =\}$. Moreover, assume that the set*

$$S := \{z \in \mathbb{C} \mid \|f_j(z)\|_\infty \prec_j \|g_j(z)\|_\infty \text{ for all } 1 \leqslant j \leqslant l\}$$

is bounded. Then S is of class m, for some effective constant m depending only on l and D.

Proof. The set S is clearly closed, so it is compact. Write $z = x + iy$, with x, $y \in \mathbb{R}$. Then S is defined by the polynomial (in)equalities $h_j(x, y) \prec_j 0$, for $1 \leqslant j \leqslant l$, with

$$h_j(X, Y) := f_j(X + iY)\overline{f_j}(X - iY) - g_j(X + iY)\overline{g_j}(X - iY),$$

where $\bar{\ }$ denotes complex conjugation of the coefficients. Hence, $h_j \in \mathbb{R}[X, Y]$ and $\deg h_j \leqslant 2D$. We conclude that S has $O_{l,D}(1)$ connected components (see, e.g., [Cos00, Proposition 4.13]). Therefore, every projection of S to a linear subspace has $O_{l,D}(1)$ connected components, that is single points and intervals.

The intersection of S with a line is defined by the (in)equalities $h_j(x, y) \prec_j 0$ and a linear equality, so once again it has $O_{l,D}(1)$ connected components, that is single points and intervals. □

Let $K \subset \mathbb{C}$ be an imaginary quadratic field, and let $S \subset \mathbb{C}$ be as in Lemma 3.2. We use the following lemma, inspired by [Sch95, Lemma 1], to count the elements of a given fractional ideal of K that lie in S.

Lemma 3.3. *Let $\mathfrak{a}$ be a fractional ideal of an imaginary quadratic field $K \subset \mathbb{C}$, let $\beta \in K$, and let $S \subset \mathbb{C}$ be a subset of class m that is contained in the union of k closed balls $B_{p_i}(R)$ of radius R, centered at arbitrary points $p_i \in \mathbb{C}$. Then*

$$|(\beta + \mathfrak{a}) \cap S| = \frac{2\,\mathrm{vol}(S)}{\sqrt{|\Delta_K|}\mathfrak{N}\mathfrak{a}} + O_{m,k}\left(\frac{R}{\sqrt{\mathfrak{N}\mathfrak{a}}} + 1\right).$$

Proof. After translation by $-\beta$, we may assume that $\beta = 0$. The ideal $\mathfrak{a}$ is a lattice in $\mathbb{C}$ of determinant $\det \mathfrak{a} = 2^{-1}\sqrt{|\Delta_K|}\mathfrak{N}\mathfrak{a}$. Denote its successive minima (with respect to the unit ball) by $\lambda_1 \leqslant \lambda_2$. Then $\lambda_1 \geqslant \sqrt{\mathfrak{N}\mathfrak{a}}$ (see, e.g., [MV07, Lemma 5]). By [Cas97, Lemmas VIII.1 and V.8], there is a basis $\{u_1, u_2\}$ of $\mathfrak{a}$ with $|u_j| = \lambda_j$. Let $\psi : \mathbb{C} \to \mathbb{C}$ be the linear automorphism given by $\psi(u_1) = 1$, $\psi(u_2) = i$. Then $\psi(\mathfrak{a}) = \mathbb{Z}[i]$ and, with respect to the standard basis, ψ is represented

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by the matrix

$$\frac{1}{\det \mathfrak{a}}\begin{pmatrix} \Im u_2 & -\Re u_2 \\ -\Im u_1 & \Re u_1 \end{pmatrix},$$

so its operator norm $|\psi|$ is bounded by $2\lambda_2/\det \mathfrak{a}$. By Minkowski's second theorem and the facts from the beginning of this proof, we obtain $|\psi| \ll 1/\sqrt{\mathfrak{N}\mathfrak{a}}$.

Clearly, $|\mathfrak{a} \cap S| = |\mathbb{Z}[i] \cap \psi(S)|$, and $\psi(S)$ is still of class m. In particular, it satisfies the conditions I and II from [Dav51], so by [Dav51, Theorem],

$$|\mathbb{Z}[i] \cap \psi(S)| = \operatorname{vol}\psi(S) + O(mV_1 + m^2),$$

where V_1 is the sum of the volumes of the projections of $\psi(S)$ to $\mathbb{R}$ and $i\mathbb{R}$. Since $\det\psi = 1/\det\mathfrak{a}$, the main term is as claimed in the lemma. Since $\psi(S) \subset \bigcup_i \psi(B_{p_i}(R))$, the volume of the projection of $\psi(S)$ to $\mathbb{R}$ or $i\mathbb{R}$ is bounded by

$$\sum_{1\leqslant i\leqslant k} \operatorname{diam}(\psi(B_{p_i}(R))) \leqslant \sum_{1\leqslant i\leqslant k} |\psi| \operatorname{diam}(B_{p_i}(R)) \ll \frac{kR}{\sqrt{\mathfrak{N}\mathfrak{a}}}. \qquad \square$$

For meaningful applications of Lemma 3.3 to a set S as in Lemma 3.2, we need R to be sufficiently small. The following two lemmas provide such values of R for certain sets S and list some consequences analogous to [Der09, Lemma 5.1, (4)–(6)] and [Der09, Lemma 5.1, (1)–(3)], respectively. For positive x, y, we interpret the expression $\min\{x, y/0\}$ as x.

Lemma 3.4. *Let $a \in \mathbb{C}\smallsetminus\{0\}$, $b \in \mathbb{C}$, $k > 1$. With $R := \min\{|a|^{-1/2}, 2|b|^{-1}\}$, we have:*

(1) $\{z \in \mathbb{C} \mid \|az^2 + bz\|_\infty \leqslant 1\} \subset B_0(R) \cup B_{-b/a}(R)$;

(2) $\operatorname{vol}\{z \in \mathbb{C} \mid \|az^2 + bz\|_\infty \leqslant 1\} \ll R^2 \ll \min\{\|a\|_\infty^{-1/2}, \|b\|_\infty^{-1}\}$.

If, in addition, $b \neq 0$, we have:

(3) $\operatorname{vol}\{(z,u) \in \mathbb{C}^2 \mid \|az^2 + bzu^k\|_\infty \leqslant 1\} \ll \|a\|_\infty^{-(k-1)/(2k)}\|b\|_\infty^{-1/k}$;

(4) $\operatorname{vol}\{(z,u) \in \mathbb{C}^2 \mid \|az^2u + bzu^2\|_\infty \leqslant 1\} \ll \|ab\|_\infty^{-1/3}$;

(5) $\operatorname{vol}\{(z,t) \in \mathbb{C}\times\mathbb{R}_{\geqslant 0} \mid \|az^2 + bzt^{k/2}\|_\infty \leqslant 1\} \ll \|a\|_\infty^{-(k-1)/(2k)}\|b\|_\infty^{-1/k}$;

(6) $\operatorname{vol}\{(z,t) \in \mathbb{C}\times\mathbb{R}_{\geqslant 0} \mid \|az^2t^{1/2} + bzt\|_\infty \leqslant 1\} \ll \|ab\|_\infty^{-1/3}$.

Proof. For part (1), we note that $|z||z + b/a| \leqslant |a|^{-1}$ implies

$$z \in B_0(|a|^{-1/2}) \cup B_{-b/a}(|a|^{-1/2}).$$

Suppose now that $b \neq 0$, $|az^2 + bz| \leqslant 1$, $|b||z| > 2$ and $|b||az + b| > 2|a|$ hold. Then

$$|b||az + b| > 2|a| \geqslant 2|a||z||az + b|,$$

so $|b| > 2|az|$ and thus $|az+b| > |az|$. This in turn implies that $|az^2| < 1$, so $|az^2+bz| > 2-1 > 1$, a contradiction. This proves parts (1) and (2). The volume in part (3) is

$$\begin{aligned} &\ll \int_{u\in\mathbb{C}} \min\{\|a\|_\infty^{-1/2}, \|bu^k\|_\infty^{-1}\}\,du \\ &\ll \int_{\|u\|_\infty \leqslant (\|a\|_\infty^{1/2}\|b\|_\infty^{-1})^{1/k}} \|a\|_\infty^{-1/2}\,du + \int_{\|u\|_\infty > (\|a\|_\infty^{1/2}\|b\|_\infty^{-1})^{1/k}} \|bu^k\|_\infty^{-1}\,du \\ &\ll \|a\|_\infty^{-(k-1)/(2k)}\|b\|_\infty^{-1/k}. \end{aligned}$$

The proof of part (4) is another elementary computation similar to the proof of part (3), and parts (5) and (6) are analogous to parts (3) and (4). $\square$

LEMMA 3.5. *Let $a \in \mathbb{C}\smallsetminus\{0\}$, $b \in \mathbb{C}$, $k > 1$. With $R := \min\{|a|^{-1/2}, |ab|^{-1/2}\}$, we have:*

(1) $\{z \in \mathbb{C} \mid \|az^2 - b\|_\infty \leqslant 1\} \subset B_{\sqrt{b/a}}(R) \cup B_{-\sqrt{b/a}}(R)$;

(2) $\mathrm{vol}\{z \in \mathbb{C} \mid \|az^2 - b\|_\infty \leqslant 1\} \ll R^2 \leqslant \min\{\|a\|_\infty^{-1/2}, \|ab\|_\infty^{-1/2}\}$.

If, in addition, $b \neq 0$, we have:

(3) $\mathrm{vol}\{(z,u) \in \mathbb{C}^2 \mid \|az^2 - bu^k\|_\infty \leqslant 1\} \ll \|a\|_\infty^{-1/2}\|b\|_\infty^{-1/k}$ *if $k > 2$;*

(4) $\mathrm{vol}\{(z,u) \in \mathbb{C}^2 \mid \|az^2u - bu^k\|_\infty \leqslant 1\} \ll (\|a\|_\infty\|b\|_\infty^{1/k})^{-1/2}$;

(5) $\mathrm{vol}\{(z,t) \in \mathbb{C} \times \mathbb{R}_{\geqslant 0} \mid \|az^2 - bt^{k/2}\|_\infty \leqslant 1\} \ll \|a\|_\infty^{-1/2}\|b\|_\infty^{-1/k}$ *if $k > 2$;*

(6) $\mathrm{vol}\{(z,t) \in \mathbb{C} \times \mathbb{R}_{\geqslant 0} \mid \|az^2t^{1/2} - bt^{k/2}\|_\infty \leqslant 1\} \ll (\|a\|_\infty\|b\|_\infty^{1/k})^{-1/2}$.

Proof. Using the substitution $t = z - \sqrt{b/a}$, part (1) is an immediate consequence of Lemma 3.4(1). Moreover, part (2) follows from part (1), and parts (3)–(6) follow from part (2) similarly to Lemma 3.4. □

The following lemma provides an easy way to prove uniform boundedness of quantities such as $R(V_\mathbf{y})$ in Lemma 2.10, for families $V_\mathbf{y}$ of certain volume functions. This is relevant for applications of our methods from §§ 6 and 7. We use the language of semialgebraic geometry (see, e.g., [Cos00]). The proof uses o-minimal structures, as presented in [vdD98].

LEMMA 3.6. *Let $k, n \in \mathbb{Z}_{\geqslant 0}$, let $M \subset \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n$ be a semialgebraic set, and let $f : M \to \mathbb{R}$ be a semialgebraic function. Assume that for all $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, $t \in \mathbb{R}$, the function $f(\mathbf{y}, t, \cdot)$ is integrable on the fiber*

$$M_{\mathbf{y},t} := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (y_1, \ldots, y_k, t, x_1, \ldots, x_n) \in M\}.$$

Then there exists a constant $C \in \mathbb{Z}_{>0}$, such that for all $\mathbf{y} \in \mathbb{R}^k$ there is a partition of $\mathbb{R}$ into at most C intervals on whose interior the function $V_\mathbf{y} : \mathbb{R} \to \mathbb{R}$ defined by

$$V_\mathbf{y}(t) := \int_{\mathbf{x} \in M_{\mathbf{y},t}} f \, d\mathbf{x}$$

is continuously differentiable and monotonic.

Proof. The function $V : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$, $(\mathbf{y}, t) \mapsto V_\mathbf{y}(t)$ is definable in an o-minimal structure. Indeed, by [LR98], parametric integrals of global subanalytic functions are definable in the expansion $(\mathbb{R}_{\mathrm{an}}, \exp)$ of the structure of global subanalytic sets $\mathbb{R}_{\mathrm{an}}$ by the global exponential function, which is o-minimal. (In [Kai13], a smaller structure is constructed which is sufficient for parametric integrals of semialgebraic functions.)

Let $\mathcal{D}$ be a decomposition of $\mathbb{R}^k \times \mathbb{R}$ into C^1-cells such that the restriction of V to each cell D of $\mathcal{D}$ is C^1 (see [vdD98, Theorem 7.3.2]).

For each cell D of $\mathcal{D}$, there is a definable open set $D \subset U_D \subset \mathbb{R}^k \times \mathbb{R}$ and a definable C^1-function $V_D : U_D \to \mathbb{R}$ such that $V_D|_D = V|_D$. Let $\mathcal{E}$ be a decomposition of $\mathbb{R}^k \times \mathbb{R}$ into C^1-cells partitioning the definable sets

$$A_D^+ := \{(\mathbf{y}, t) \in D \mid \partial V_D/\partial t \geqslant 0\} \quad \text{and} \quad A_D^- := \{(\mathbf{y}, t) \in D \mid \partial V_D/\partial t \leqslant 0\} \quad \text{for } D \in \mathcal{D}.$$

We note that $\bigcup_D (A_D^+ \cup A_D^-) = \mathbb{R}^k \times \mathbb{R}$, so each cell E of $\mathcal{E}$ is contained in some A_D^+ or A_D^-.

Let $\pi : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$ be the projection on the first k coordinates. Let $\mathbf{y} \in \mathbb{R}^k$. For cells E of $\mathcal{E}$ with $\mathbf{y} \in \pi(E)$, the sets $E_\mathbf{y} := \{t \in \mathbb{R} \mid (\mathbf{y}, t) \in E\}$ are the cells of a decomposition $\mathcal{E}_\mathbf{y}$ of $\mathbb{R}$ (see [vdD98, Proposition 3.3.5]). On cells $E_\mathbf{y}$ that are open intervals, $V'_\mathbf{y}(t)$ is defined and coincides with $\partial V_D/\partial t(\mathbf{y}, t)$ (if $E \subset A_D^+$ or $E \subset A_D^-$). Therefore, $V_\mathbf{y}$ is continuously differentiable and monotonic on $E_\mathbf{y}$. The observation that $|\mathcal{E}_\mathbf{y}| \leqslant |\mathcal{E}|$ completes our proof. □

4. Passage to a universal torsor

In this section, we describe a strategy to parameterize rational points on a split singular del Pezzo surface by integral points on a universal torsor. This generalizes [DT07, § 4] from $\mathbb{Q}$ to imaginary quadratic fields with arbitrary class number. In [DJ13, § 4] and [Fre13], a similar strategy is used in the easier case of a toric split singular cubic surface, where a universal torsor is an open subset of affine space.

Let K be a number field. Let S be a non-toric split singular del Pezzo surface defined over K whose minimal desingularization $\widetilde{S}$ has a universal torsor that is an open subset of a hypersurface in affine space. Up to isomorphism, there are only finitely many del Pezzo surfaces satisfying these properties. Together with an explicit description of all of their properties used below, their classification can be found in [Der14]. For del Pezzo surfaces with more complicated universal torsors, we expect that a similar strategy can be used, but that several complications may appear.

We assume for simplicity that $\deg(S) \in \{3, \dots, 6\}$; the adaptation to $\deg(S) \in \{1, 2\}$ is straightforward. To count K-rational points on S, we use the Weil height given by an anticanonical embedding $S \subset \mathbb{P}_K^{\deg(S)}$ satisfying the following assumptions.

(1) Let $r := 9 - \deg(S)$. By our assumption on a universal torsor of $\widetilde{S}$, its Cox ring $\mathrm{Cox}(\widetilde{S})$ has a minimal system of $r+4$ generators $\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}$ that are homogeneous (with respect to the natural $\mathrm{Pic}(\widetilde{S})$-grading of $\mathrm{Cox}(\widetilde{S})$), are defined over K (since S is split), correspond to curves $E_1, \dots, E_{r+4}$ on $\widetilde{S}$, and satisfy one homogeneous relation

$$R(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}) = 0, \tag{4.1}$$

which we call the *torsor equation*. Possibly after replacing some $\tilde{\eta}_i$ by scalar multiples, we may assume that all coefficients in R are ± 1.

(2) The choice of a basis $s_0, \dots, s_{\deg(S)}$ of $H^0(\widetilde{S}, \mathcal{O}(-K_{\widetilde{S}}))$ defines a map $\pi : \widetilde{S} \to \mathbb{P}_K^{\deg(S)}$ whose image is an anticanonical embedding $S \subset \mathbb{P}_K^{\deg(S)}$. Since $H^0(\widetilde{S}, \mathcal{O}(-K_{\widetilde{S}})) \subset \mathrm{Cox}(\widetilde{S})$, we may choose each s_i as a monic monomial

$$\Psi_i(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}) \tag{4.2}$$

in the generators of $\mathrm{Cox}(\widetilde{S})$, for $i = 0, \dots, \deg(S)$.

To describe our expected parameterization of K-rational points of bounded height on S in Claims 4.1 and 4.2 below, we introduce the following notation.

(1) The split generalized del Pezzo surface $\widetilde{S}$ is a blow-up $\rho : \widetilde{S} \to \mathbb{P}_K^2$ in r points *in almost general position*, i.e. a composition of r blow-ups

$$\widetilde{S} = \widetilde{S}_r \xrightarrow{\rho_r} \widetilde{S}_{r-1} \to \cdots \to \widetilde{S}_1 \xrightarrow{\rho_1} \widetilde{S}_0 = \mathbb{P}_K^2, \tag{4.3}$$

where each $\rho_i : \widetilde{S}_i \to \widetilde{S}_{i-1}$ is the blow-up of a point p_i not lying on a (-2)-curve on $\widetilde{S}_{i-1}$. Let ℓ_0 be the class of $\rho^*(\mathcal{O}_{\mathbb{P}_K^2}(1))$ and ℓ_i the class of the total transform of the exceptional divisor of ρ_i, for $i = 1, \dots, r$. Then $\ell_0, \dots, \ell_r$ form a basis of $\mathrm{Pic}(\widetilde{S})$, so

$$[E_j] = a_{j,0}\ell_0 + \cdots + a_{j,r}\ell_r \in \mathrm{Pic}(\widetilde{S}) \tag{4.4}$$

for some $a_{j,i} \in \mathbb{Z}$, for $j = 1, \dots, r+4$.

For any $\mathbf{C} = (C_0, \dots, C_r) \in \mathcal{C}^{r+1}$ (see § 1.4), we use the integers $a_{j,i}$ to define the fractional ideals

$$\mathcal{O}_j := C_0^{a_{j,0}} \cdots C_r^{a_{j,r}}, \tag{4.5}$$

and their subsets

$$\mathcal{O}_{j*} := \begin{cases} (\mathcal{O}_j)^{\neq 0}, & ([E_j],[E_j]) < 0, \\ \mathcal{O}_j & \text{otherwise.} \end{cases}$$

(2) For $\eta_j \in \mathcal{O}_j$, consider the ideals

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

Via the configuration of $E_1, \ldots, E_{r+4}$, we define *coprimality conditions*

$$\sum_{j \in J} I_j = \mathcal{O}_K \text{ for all minimal } J \subset \{1, \ldots, r+4\} \text{ with } \bigcap_{j \in J} E_j = \emptyset. \tag{4.6}$$

We observe from the classification in [Der14] that these minimal J have the form $J = \{j, j'\}$ for non-intersecting $E_j, E_{j'}$ (encoded in the extended Dynkin diagram) or $J = \{j, j', j''\}$ for pairwise intersecting $E_j, E_{j'}, E_{j''}$ that do not meet in a common point.

(3) Assume that K is an imaginary-quadratic field or $K = \mathbb{Q}$. We consider K as a subset of $K_\infty \in \{\mathbb{R}, \mathbb{C}\}$, its completion at the infinite place, with $\| \cdot \|_\infty$ the usual real absolute value respectively the square of the usual complex one.

Let $\mathcal{R}(B)$ be the set of all $(\eta_1, \ldots, \eta_{r+4}) \in K_\infty^{r+4}$ satisfying the *height conditions*

$$\|\Psi_i(\eta_1, \ldots, \eta_{r+4})\|_\infty \leqslant B, \tag{4.7}$$

for $i = 0, \ldots, \deg(S)$, where Ψ_i is the monic monomial from (4.2).

For any $\mathbf{C} \in \mathcal{C}^{r+1}$, we define $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_r^{-1})$, corresponding to the anticanonical class $[-K_{\widetilde{S}}] = 3\ell_0 - \ell_1 - \cdots - \ell_r$. Let $M_{\mathbf{C}}(B)$ be the set of all

$$(\eta_1, \ldots, \eta_{r+4}) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{r+4*}$$

lying in the set $\mathcal{R}(u_{\mathbf{C}} B)$ defined by the height conditions and satisfying the torsor equation (4.1) and the coprimality conditions (4.6).

Claim 4.1. *Let K be an imaginary quadratic field or $K = \mathbb{Q}$. Let $S \subset \mathbb{P}_K^{\deg(S)}$ be a split singular del Pezzo surface of degree $3, \ldots, 6$ over K whose universal torsors are open subsets of hypersurfaces, with an anticanonical embedding satisfying the assumptions above. Let U be the complement of its lines. Let $N_{U,H}(B)$ be defined as in (1.1), with the usual Weil height H on $\mathbb{P}_K^{\deg(S)}(K)$. With the notation introduced above, for $B > 0$, we have*

$$N_{U,H}(B) = \frac{1}{\omega_K^{10-\deg(S)}} \sum_{\mathbf{C} \in \mathcal{C}^{10-\deg(S)}} |M_{\mathbf{C}}(B)|.$$

Motivated by the geometry of S, we propose a strategy to prove Claim 4.1 by induction, via the closely related Claim 4.2 below, for $i = 0, \ldots, r$. The starting point is a parameterization of rational points via the birational map $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S$. In each step $i = 1, \ldots, r$, the rational points are parameterized by variables η_j corresponding to curves on $\widetilde{S}_{i-1}$; if ρ_i is the blow-up of the intersection point of some of these curves, we introduce a new variable essentially as the greatest common divisor of the variables corresponding to those curves to obtain the next step of the parameterization.

From here on, we work again over an arbitrary number field K. To set up the induction in Claim 4.2, we need more notation. For $i = 0, \ldots, r$ and $j = 1, \ldots, r+4$, let

$$E_j^{(i)} := (\rho_{i+1} \circ \cdots \circ \rho_r)(E_j)$$

be the projection of E_j on $\widetilde{S}_i$. If $E_j^{(i)}$ is a curve on $\widetilde{S}_i$, then E_j is its strict transform on $\widetilde{S}$. Possibly after rearranging the generators of $\mathrm{Cox}(\widetilde{S})$, we may assume that $E_1^{(0)}, E_2^{(0)}, E_3^{(0)}$ are lines in $\mathbb{P}^2_K$, that $E_4^{(0)}$ is a curve of some degree D in $\mathbb{P}^2_K$, and that $E_{i+4}^{(i)}$ is the exceptional divisor of ρ_i, so

$$a_{1,0} = a_{2,0} = a_{3,0} = 1, \quad a_{4,0} = D, \quad a_{i+4,0} = \cdots = a_{i+4,i-1} = 0, \quad a_{i+4,i} = 1, \tag{4.8}$$

for $i = 1, \dots, r$. By [Der14, Lemma 12], we may assume (possibly by a linear change of coordinates y_0, y_1, y_2 on $\mathbb{P}^2_K$) that

$$E_1^{(0)} = \{y_0 = 0\}, \quad E_2^{(0)} = \{y_1 = 0\}, \quad E_3^{(0)} = \{y_2 = 0\}, \quad E_4^{(0)} = \{R'(y_0, y_1, y_2) = 0\}$$

in $\mathbb{P}^2_K$, where R' is a homogeneous polynomial of degree D satisfying

$$Y_3 - R'(Y_0, Y_1, Y_2) = R(Y_0, \dots, Y_3, 1, \dots, 1). \tag{4.9}$$

Via the natural embeddings $\mathrm{Pic}(\mathbb{P}^2_K) \subset \mathrm{Pic}(\widetilde{S}_1) \subset \cdots \subset \mathrm{Pic}(\widetilde{S}_{r-1}) \subset \mathrm{Pic}(\widetilde{S})$, we may view $\ell_0, \dots, \ell_i$ as a basis of $\mathrm{Pic}(\widetilde{S}_i)$. Then

$$[E_j^{(i)}] = a_{j,0}\ell_0 + \cdots + a_{j,i}\ell_i \in \mathrm{Pic}(\widetilde{S}_i) \tag{4.10}$$

with the integers $a_{j,i}$ from (4.4), for any $i = 0, \dots, r$ and $j = 1, \dots, i+4$.

For $i = 0, \dots, r$ and any $(C_0, \dots, C_i) \in \mathcal{C}^{i+1}$, we define analogously to (4.5)

$$\mathcal{O}_j^{(i)} := C_0^{a_{j,0}} \cdots C_i^{a_{j,i}}, \quad \mathcal{O}_{j*}^{(i)} := \begin{cases} (\mathcal{O}_j^{(i)})^{\neq 0}, & ([E_j], [E_j]) < 0, \\ \mathcal{O}_j^{(i)} & \text{otherwise,} \end{cases}$$

and, for $\eta_j \in \mathcal{O}_j^{(i)}$,

$$I_j^{(i)} := \eta_j (\mathcal{O}_j^{(i)})^{-1}$$

for $j = 1, \dots, i+4$.

We use the monomials $\Psi_i(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4})$ from (4.7) to define the map

$$\Psi : K^{r+4} \to K^{\deg(S)+1}, \quad (\eta_1, \dots, \eta_{r+4}) \mapsto (\Psi_i(\eta_1, \dots, \eta_{r+4}))_{i=0,\dots,\deg(S)}.$$

Claim 4.2. *Let K be a number field. Assume that $U \subset S \subset \mathbb{P}^{\deg(S)}_K$ are as in Claim 4.1. Assume that $\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}$ are ordered in such a way that $E_{i+4}^{(i)}$ is the exceptional divisor of ρ_i, for $i = 1, \dots, r$. For any $i \in \{0, \dots, r\}$, we have a map $(\eta_1, \dots, \eta_{i+4}) \mapsto \Psi(\eta_1, \dots, \eta_{i+4}, 1, \dots, 1)$ from the disjoint union*

$$\bigcup_{C_0,\dots,C_i \in \mathcal{C}} \left\{ \begin{aligned} &(\eta_1, \dots, \eta_{i+4}) \in \mathcal{O}_{1*}^{(i)} \times \cdots \times \mathcal{O}_{i+4*}^{(i)} : R(\eta_1, \dots, \eta_{i+4}, 1, \dots, 1) = 0, \\ &\sum_{j \in J} I_j^{(i)} = \mathcal{O}_K \text{ for all minimal } J \subset \{1, \dots, i+4\} \text{ with } \bigcap_{j \in J} E_j^{(i)} = \emptyset \end{aligned} \right\}$$

to $U(K)$. This induces a bijection between the orbits under the natural free action of $(\mathcal{O}_K^\times)^{i+1}$ on the former set and $U(K)$.

Here, the natural action of $(\lambda_0, \dots, \lambda_i) \in (\mathcal{O}_K^\times)^{i+1}$ on these subsets of K^{i+4} is explicitly given via the $\mathrm{Pic}(\widetilde{S}_i)$-degrees of $\tilde{\eta}_1, \dots, \tilde{\eta}_{i+4}$ (see (4.10)):

$$(\lambda_0, \dots, \lambda_i) \cdot (\eta_1, \dots, \eta_{i+4}) := (\lambda_0^{a_{1,0}} \cdots \lambda_i^{a_{1,i}} \eta_1, \dots, \lambda_0^{a_{i+4,0}} \cdots \lambda_i^{a_{i+4,i}} \eta_{i+4}).$$

Freeness of this action follows immediately from (4.8), the assumption that E_j is a negative curve for all $j \in \{5, \dots, r+4\}$, and the fact that there are at least $r+1$ negative curves on any generalized del Pezzo surface of degree $d \leqslant 7$. Also, Ψ induces a well-defined map on the orbits because all $\Psi_j(\tilde{\eta}_1, \dots, \tilde{\eta}_{i+4}, 1, \dots, 1)$ have the same degree $[-K_{\widetilde{S}_i}]$.

Assume that we have established Claim 4.2 for $i = r$. To deduce Claim 4.1 in specific cases over number fields K with finite $\mathcal{O}_K^\times$, it remains to lift the height function via Ψ. By the definition of the Weil height as in (1.4), this depends essentially on the norm of

$$\Psi_0(\eta_1, \dots, \eta_{r+4})\mathcal{O}_K + \cdots + \Psi_{\deg(S)}(\eta_1, \dots, \eta_{r+4})\mathcal{O}_K.$$

For $(\eta_1, \dots, \eta_{r+4}) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{r+4*}$, this is a multiple of $u_{\mathbf{C}} = \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_r^{-1})$ since the $\Psi_i(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4})$ have degree $[-K_{\widetilde{S}}] = 3\ell_0 - \ell_1 - \cdots - \ell_r$. We expect that it is indeed equal to $u_{\mathbf{C}}$ under (4.6). Then $H(\Psi(\eta_1, \dots, \eta_{r+4})) \leqslant B$ if and only if $(\eta_1, \dots, \eta_{r+4}) \in \mathcal{R}(u_{\mathbf{C}} B)$, and Claim 4.1 follows.

The following two lemmas turn out to be sufficient to prove Claim 4.2 for the quartic surface of type $\mathbf{A}_3$ with five lines defined by (1.3). For other surfaces, some induction steps must be done by hand. In particular, it may be necessary to use the relation R to deduce the new set of coprimality conditions. We note that the assumption on ψ in the first lemma holds for every example in [Der14].

LEMMA 4.3. *The birational map $\pi \circ \rho^{-1} : \mathbb{P}^2_K \dashrightarrow S$ induces an isomorphism between an open subset $V \subset \mathbb{P}^2_K$ and $U \subset S$. The homogeneous cubic polynomials*

$$\psi_i(Y_0, Y_1, Y_2) := \Psi_i(Y_0, Y_1, Y_2, R'(Y_0, Y_1, Y_2), 1, \dots, 1), \tag{4.11}$$

for $i = 0, \dots, \deg(S)$, define a rational map

$$\psi : \mathbb{P}^2_K \dashrightarrow S, \quad (y_0 : y_1 : y_2) \mapsto (\psi_0(y_0, y_1, y_2) : \cdots : \psi_{\deg(S)}(y_0, y_1, y_2)). \tag{4.12}$$

If ψ represents $\pi \circ \rho^{-1}$ on V, then Claim 4.2 holds for $i = 0$.

Proof. Let $V \subset \mathbb{P}^2_K$ be the complement of all $E_j^{(0)}$ with $j \in \{1, \dots, 4\}$ such that E_j is a negative curve on $\widetilde{S}$. Let W be the complement of the negative curves on $\widetilde{S}$. Then $\pi(W) = U$ since π maps the (-1)-curves to the lines and the (-2)-curves to the singularities on S (each lying on a line for any singular del Pezzo surface except for the Hirzebruch surface F_2, which is excluded since it is toric), and $\rho(W) = V$ since ρ contracts the negative curves $E_5, \dots, E_{r+4}$ to points lying on the negative curves among $E_1^{(0)}, \dots, E_4^{(0)}$ (since the extended Dynkin diagram of negative curves on $\widetilde{S}$ is connected and there are at least $r+1$ negative curves). Therefore, the birational map $\pi \circ \rho^{-1}$ induces an isomorphism between V and U.

For $i = 0, \dots, \deg(S)$, we note that ψ_i is a cubic polynomial, by considering coefficients $(a_{1,0}, \dots, a_{r+4,0}) = (1, 1, 1, D, 0, \dots, 0)$ of ℓ_0 from (4.8) and the degree of Ψ_i. Since Ψ_i are monomials, ψ is defined at least on the complement of $E_1^{(0)}, \dots, E_4^{(0)}$. Its image lies in S since for any equation $F \in K[X_0, \dots, X_{\deg(S)}]$ defining $S \subset \mathbb{P}_K^{\deg(S)}$, we know that

$$F(\Psi_0(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}), \dots, \Psi_{\deg(S)}(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4}))$$

is a multiple of $R(\tilde{\eta}_1, \dots, \tilde{\eta}_{r+4})$, so that $F(\psi_0(Y_0, Y_1, Y_2), \dots, \psi_{\deg(S)}(Y_0, Y_1, Y_2))$ is a multiple of $R(Y_0, Y_1, Y_2, R'(Y_0, Y_1, Y_2), 1, \dots, 1)$, which is trivial by (4.9).

To prove Claim 4.2 for $i = 0$, we note that $\pi \circ \rho^{-1}$ induces a bijection between $V(K)$ and $U(K)$ that is explicitly given by ψ by assumption.

Any element of $\mathbb{P}^2_K(K)$ is represented uniquely up to multiplication by scalars from $\mathcal{O}_K^\times$ by $(y_0, y_1, y_2) \in \mathcal{O}_K^3 \smallsetminus \{0\}$ with $y_0\mathcal{O}_K + y_1\mathcal{O}_K + y_2\mathcal{O}_K \in \mathcal{C}$ (and, in particular, y_0, y_1, y_2 in the same element of $\mathcal{C}$, say C_0). Therefore, ψ induces a bijection between the orbits of the action of $\mathcal{O}_K^\times$ by scalar multiplication on the disjoint union

$$\bigcup_{C_0 \in \mathcal{C}} \left\{ (y_0, y_1, y_2) \in C_0^3 \;\middle|\; \begin{array}{l} y_0 C_0^{-1} + y_1 C_0^{-1} + y_2 C_0^{-1} = \mathcal{O}_K, \\ y_{i-1} \neq 0 \text{ if } E_i \text{ is a negative curve, for } i = 1, 2, 3, \\ R'(y_0, y_1, y_2) \neq 0 \text{ if } E_4 \text{ is a negative curve} \end{array} \right\}$$

and $U(K)$.

We rename (y_0, y_1, y_2) to (η_1, η_2, η_3) and introduce an additional variable $\eta_4 := R'(\eta_1, \eta_2, \eta_3)$, which is equivalent to $R(\eta_1, \dots, \eta_4, 1, \dots, 1) = 0$ by (4.9). By (4.11), this substitution turns ψ into $\Psi(\eta_1, \dots, \eta_4, 1, \dots, 1)$. We note $(\mathcal{O}_1^{(0)}, \dots, \mathcal{O}_4^{(0)}) = (C_0, C_0, C_0, C_0^D)$ by (4.8) and that the action of $\lambda_0 \in \mathcal{O}_K^\times$ on (η_1, η_2, η_3) by scalar multiplication leads to an action on η_4 by multiplication by λ_0^D.

It remains to show that the coprimality condition for η_1, η_2, η_3 is equivalent to the system of coprimality conditions described in Claim 4.2. Since any two curves in $\mathbb{P}^2_K$ meet and since $E_1^{(0)}, E_2^{(0)}, E_3^{(0)}$ do not meet in one point, we must show that adding, respectively removing, a condition such as $\eta_1 C_0^{-1} + \eta_2 C_0^{-1} + \eta_4 C_0^{-D} = \mathcal{O}_K$ for $E_1^{(0)} \cap E_2^{(0)} \cap E_4^{(0)} = \emptyset$ makes no difference. The emptiness of this intersection is equivalent to $R'(0, 0, 1) \neq 0$, i.e. the term Y_2^D appears in R' with a non-zero coefficient. In fact, this coefficient is ± 1 since all coefficients in R are ± 1 by assumption, and this could fail after the substitution in (4.9) only if two terms of R differed only by powers of $\tilde{\eta}_5, \dots, \tilde{\eta}_{r+4}$, which is impossible because of (4.8) and the homogeneity of R. If there was a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ dividing $\eta_1 C_0^{-1}, \eta_2 C_0^{-1}, \eta_4 C_0^{-D}$, then the relation $\eta_4 = R'(\eta_1, \eta_2, \eta_3)$ would imply that $\mathfrak{p}$ divides $\eta_3 C_0^{-1}$, contradicting the coprimality of $\eta_1 C_0^{-1}, \eta_2 C_0^{-1}, \eta_3 C_0^{-1}$. □

Lemma 4.4. *Assume that Claim 4.2 holds for some $i - 1 \in \{0, \dots, r-1\}$. If ρ_i in (4.3) is the blow-up of a point on $\widetilde{S}_{i-1}$ lying on precisely two of $E_1^{(i-1)}, \dots, E_{i+3}^{(i-1)}$, if these two meet transversally in that point and meet nowhere else, and if the strict transform on $\widetilde{S}$ of at least one of these two is a negative curve, then Claim 4.2 holds for i.*

Remark 4.5. For most steps of the proof of Lemma 4.4, we consider the following more general situation for ρ_i. Let J_0 be the set of all $j \in \{1, \dots, i+3\}$ such that $E_j^{(i-1)}$ contains the blown-up point $p_i \in \widetilde{S}_{i-1}$. Assume that p_i has multiplicity 1 on each $E_j^{(i-1)}$ with $j \in J_0$, that we have $\bigcap_{j \in J_0} E_j^{(i)} = \emptyset$ for their strict transforms on $\widetilde{S}_i$, and that the strict transform E_j on $\widetilde{S}$ is a negative curve for some $j \in J_0$.

The additional assumption $|J_0| = 2$ in Lemma 4.4 is used only for parts of one direction of the coprimality conditions, see (4.13) below. Without this assumption, we expect that we must use the torsor equation to derive the coprimality conditions for $J \subset J_0 \cup \{i+4\}$ of Claim 4.2 for i.

Proof of Lemma 4.4. Except in the paragraph containing (4.13), we work in the situation of Remark 4.5.

We write $E'_j := E_j^{(i-1)}$ for divisors on $\widetilde{S}_{i-1}$ and $E''_j := E_j^{(i)}$ for their strict transforms on $\widetilde{S}_i$. The exceptional divisor of ρ_i is $E''_{i+4} := E_{i+4}^{(i)}$.

Let M', respectively M'', be the disjoint union in step $i-1$, respectively step i, of Claim 4.2. We construct a bijection between the $(\mathcal{O}_K^\times)^i$-orbits in M' and the $(\mathcal{O}_K^\times)^{i+1}$-orbits in M''. We use

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η'_j for coordinates of points in M' and η''_j for coordinates in M'', and similarly $\mathcal{O}'_j := \mathcal{O}_j^{(i-1)}$, $\mathcal{O}''_j := \mathcal{O}_j^{(i)}$ and $I'_j := I_j^{(i-1)}$, $I''_j := I_j^{(i)}$ for their corresponding (fractional) ideals.

Given $\boldsymbol{\eta}' = (\eta'_1, \dots, \eta'_{i+3}) \in M'$, we have corresponding $C_0, \dots, C_{i-1} \in \mathcal{C}$ and $\mathcal{O}'_j$ with $\eta'_j \in \mathcal{O}'_{j*}$, and $I'_j = \eta'_j \mathcal{O}'^{-1}_j$. Since E_j is a negative curve on $\widetilde{S}$ for some $j \in J_0$, at least one of the η'_j with $j \in J_0$ is non-zero. Therefore, there is a unique $C_i \in \mathcal{C}$ such that $[\sum_{j \in J_0} I'_j] = [C_i^{-1}]$, giving $\mathcal{O}''_j$ and I''_j for $j = 1, \dots, i+3$. Choose $\eta''_{i+4} \in C_i = \mathcal{O}''_{i+4}$ such that $I''_{i+4} = \sum_{j \in J_0} I'_j$, which is unique up to multiplication by $\mathcal{O}_K^\times$. Then we define $\eta''_j := \eta'_j / \eta''_{i+4}$ for $j \in J_0$ and $\eta''_j := \eta'_j$ for all $j \in \{1, \dots, i+3\} \smallsetminus J_0$, giving $\boldsymbol{\eta}'' = (\eta''_1, \dots, \eta''_{i+4}) \in \mathcal{O}''_{1*} \times \dots \times \mathcal{O}''_{i+4*}$, uniquely up to the action of $\lambda_i \in \mathcal{O}_K^\times$ by $\eta''_{i+4} \mapsto \lambda_i \eta''_{i+4}$ and $\eta''_j \mapsto \lambda_i^{-1} \eta''_j$ for all $j \in J_0$ and $\eta''_j \mapsto \eta''_j$ for all $j \in \{1, \dots, i+3\} \smallsetminus J_0$.

We check that these $\boldsymbol{\eta}''$ satisfy the coprimality conditions on M''. For $J \subset \{1, \dots, i+4\}$ with $J \not\subset J_0 \cup \{i+4\}$, assume first that $i+4 \notin J$. Since blowing up p_i only separates divisors meeting in p_i and since $J \not\subset J_0$, we have $\bigcap_{j \in J} E''_j = \emptyset$ only for $\bigcap_{j \in J} E'_j = \emptyset$, hence $\sum_{j \in J} I'_j = \mathcal{O}_K$ and hence $\sum_{j \in J} I''_j = \mathcal{O}_K$, as desired, because each I''_j divides I'_j. Assume next that $i+4 \in J$. Then only the case $J = \{k, i+4\}$ with $k \notin J_0$ is relevant because of the minimality assumption on J, so $E''_k \cap E''_{i+4} = \emptyset$; by the assumption $\bigcap_{j \in J_0} E''_j = \emptyset$, we have $\bigcap_{j \in J_0} E'_j = \{p_i\}$, hence $E'_k \cap (\bigcap_{j \in J_0} E'_j) = \emptyset$; hence $I'_k + \sum_{j \in J_0} I'_j = \mathcal{O}_K$ and since I''_{i+4} divides all I'_j with $j \in J_0$, we conclude $I''_k + I''_{i+4} = \mathcal{O}_K$.

It remains to check the coprimality conditions for

$$J \subset J_0 \cup \{i+4\}. \tag{4.13}$$

Here we use the additional assumption $|J_0| = 2$, say $J_0 = \{a, b\}$. Then our other assumptions imply $([E'_a], [E'_b]) = 1$, hence $([E''_a], [E''_b]) = 0$ and $([E''_a], [E''_{i+4}]) = ([E''_b], [E''_{i+4}]) = 1$. Therefore, the only remaining coprimality condition is $I''_a + I''_b = \mathcal{O}_K$, and this is clearly fulfilled using $I''_a = I'_a / I''_{i+4}$ and $I''_b = I'_b / I''_{i+4}$ with $I''_{i+4} = I'_a + I'_b$.

To check that the $\boldsymbol{\eta}''$ constructed above satisfy the torsor equation on M'', we first discuss how the polynomial R behaves under analogous substitutions. Let $c_0\ell_0 + \dots + c_r\ell_r$ be the degree of the homogeneous relation R of the Cox ring. Then $R(T'_1, \dots, T'_{i+3}, 1, \dots, 1)$ is homogeneous of degree $c_0\ell_0 + \dots + c_{i-1}\ell_{i-1}$ if we give each T'_j the degree $[E'_j] = a_{j,0}\ell_0 + \dots + a_{j,i-1}\ell_{i-1}$ for the moment. Similarly, $R(T''_1, \dots, T''_{i+4}, 1, \dots, 1)$ is homogeneous of degree $c_0\ell_0 + \dots + c_i\ell_i$ if we give each T''_j the degree $[E''_j] = a_{j,0}\ell_0 + \dots + a_{j,i}\ell_i$. If we substitute T'_j in $R(T'_1, \dots, T'_{i+3}, 1, \dots, 1)$ by $T''_j T''_{i+4}$ for $j \in J_0$ and by T''_j for $j \in \{1, \dots, i+3\} \smallsetminus J_0$, then we obtain an expression in $T''_1, \dots, T''_{i+4}$ that is homogeneous of the same degree $c_0\ell_0 + \dots + c_{i-1}\ell_{i-1}$. Indeed, $T''_j T''_{i+4}$ has the same degree as T'_j for $j \in J_0$ since $[E''_j] = [E'_j] - \ell_i$ and $[E''_{i+4}] = \ell_i$, and similarly for $j \in \{1, \dots, i+3\} \smallsetminus J_0$. Furthermore, the result of the substitution clearly agrees with $R(T''_1, \dots, T''_{i+4}, 1, \dots, 1)$ up to powers of T''_{i+4} in each term. But both are homogeneous of degrees differing by $c_i\ell_i$, so the result of the substitution is $T''^{-c_i}_{i+4} R(T''_1, \dots, T''_{i+4}, 1, \dots, 1)$.

Since $\eta'_j = \eta''_j \eta''_{i+4}$ for $j \in J_0$ and $\eta'_j = \eta''_j$ for $j \in \{1, \dots, i+3\} \smallsetminus J_0$, this implies that

$$\eta''^{-c_i}_{i+4} R(\eta''_1, \dots, \eta''_{i+4}, 1, \dots, 1) = R(\eta'_1, \dots, \eta'_{i+3}, 1, \dots, 1).$$

Since $R(\eta'_1, \dots, \eta'_{i+3}, 1, \dots, 1) = 0$ and $\eta''_{i+4} \neq 0$, this implies that $\boldsymbol{\eta}''$ satisfies the torsor equation on M''. In total, we have constructed for $\boldsymbol{\eta}' \in M'$ an $\mathcal{O}_K^\times$-orbit of $\boldsymbol{\eta}'' \in M''$.

In the other direction, given $\boldsymbol{\eta}'' \in M''$ with corresponding $C_0, \dots, C_i \in \mathcal{C}$, we define $\eta'_j := \eta''_j \eta''_{i+4}$ for $j \in J_0$ and $\eta'_j := \eta''_j$ for $j \in \{1, \dots, i+3\} \smallsetminus J_0$, giving $\boldsymbol{\eta}' = (\eta'_1, \dots, \eta'_{i+3}) \in \mathcal{O}'_{1*} \times \dots \times \mathcal{O}'_{i+3*}$.

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If $\boldsymbol{\eta}'' \in M''$ satisfies the coprimality conditions, the same holds for $\boldsymbol{\eta}'$ that we just defined. Indeed, if $\bigcap_{j\in J} E'_j = \emptyset$, then $\bigcap_{j\in J} E''_j = \emptyset$ since blowing up only decreases intersection numbers, so $\sum_{j\in J} I''_j = \mathcal{O}_K$. Since $\bigcap_{j\in J} E'_j = \emptyset$ does not contain p_i, there is at least one $k \in J$ with $k \notin J_0$, so $([E''_k],[E''_{i+4}]) = 0$, hence $I''_k + I''_{i+4} = \mathcal{O}_K$. In particular, the factors η''_{i+4} in $\eta'_j = \eta''_j\eta''_{i+4}$ for all $j \in J \cap J_0$ do not contribute to the greatest common divisor, so we have $\sum_{j\in J} I'_j = \mathcal{O}_K$. Therefore, $\boldsymbol{\eta}'$ satisfies the coprimality conditions on M'. Similarly as above, $\boldsymbol{\eta}'$ satisfies the torsor equation. Clearly all $\boldsymbol{\eta}''$ in the same $\mathcal{O}_K^\times$-orbit give the same $\boldsymbol{\eta}'$.

Obviously, $\boldsymbol{\eta}' \mapsto \boldsymbol{\eta}'' \mapsto \boldsymbol{\eta}'$ is the identity on M' (for any choice of $\boldsymbol{\eta}''$ in the corresponding $\mathcal{O}_K^\times$-orbit). The assumption $\bigcap_{j\in J_0} E''_j = \emptyset$ gives the coprimality condition $\sum_{j\in J_0} I''_j = \mathcal{O}_K$ on M'', and this ensures that $\boldsymbol{\eta}'' \mapsto \boldsymbol{\eta}' \mapsto \boldsymbol{\eta}''$ yields an element of the same $\mathcal{O}_K^\times$-orbit as the original $\boldsymbol{\eta}''$. We have thus constructed a bijection between M' and $\mathcal{O}_K^\times$-orbits in M''.

Moreover, it is clear that the $\mathcal{O}_K^\times$-orbits in M'' are contained in the $(\mathcal{O}_K^\times)^{i+1}$-orbits from Claim 4.2, and that $\boldsymbol{\eta}'_1, \boldsymbol{\eta}'_2 \in M'$ are in the same $(\mathcal{O}_K^\times)^i$-orbit if and only if $\boldsymbol{\eta}''_1$ and $\boldsymbol{\eta}''_2$ are in the same $(\mathcal{O}_K^\times)^{i+1}$-orbit. Hence, our bijection induces the claimed bijection between orbits on M' and M''.

Using the coprimality condition $\sum_{j\in J_0} I''_j = \mathcal{O}_K$, we see that the union defining M'' is disjoint if the union defining M'' is disjoint.

To conclude our proof, it is enough to show that the map $M'' \to \mathbb{P}_K^{\deg(S)}(K)$ defined in Claim 4.2, step i, coincides with the composition $M'' \to U(K)$ of the map $M'' \to M'$ constructed above and the map $M' \to U(K)$ from step $i-1$. Using the same gradings and substitutions as in the discussion of R, we note that $\Psi_i(T'_1, \dots, T'_{i+3}, 1, \dots, 1)$ is homogeneous of degree $3\ell_0 - \ell_1 - \cdots - \ell_{i-1}$. Our substitution turns this into a monic monomial of the same degree that coincides up to powers of T''_{i+4} with the monic monomial $\Psi_i(T''_1, \dots, T''_{i+4}, 1, \dots, 1)$, which is homogeneous of degree $3\ell_0 - \ell_1 - \cdots - \ell_i$. Since T''_{i+4} has degree ℓ_i, the substitution gives $T''_{i+4}\Psi_i(T''_1, \dots, T''_{i+4}, 1, \dots, 1)$. Thus, both maps send $\boldsymbol{\eta}'' \in M''$ to K-rational points in projective space that differ by a factor of $\eta''_{i+4} \neq 0$ in each coordinate, hence are the same. $\square$

Remark 4.6. By our assumption, in the Cox ring relation $R(\tilde\eta_1, \dots, \tilde\eta_{r+4}) = \sum_{k=1}^t \lambda_k \tilde\eta_1^{b_{1,k}} \cdots \tilde\eta_{r+4}^{b_{r+4,k}}$ with exponents $b_{j,k} \in \mathbb{Z}_{\geqslant 0}$, all coefficients λ_k are ± 1. For $j = 1, \dots, r+4$, write $\mathcal{O}_j := \mathcal{O}_j^{(r)}$ for simplicity. Then the fractional ideals $\lambda_k \mathcal{O}_1^{b_{1,k}} \cdots \mathcal{O}_{r+4}^{b_{r+4,k}}$ coincide for all $k = 1, \dots, t$. Indeed, since R is homogeneous of some degree $c_0\ell_0 + \cdots + c_r\ell_r \in \operatorname{Pic}(\widetilde{S})$, each of them is $C_0^{c_0} \cdots C_r^{c_r}$.

5. The first summation

Let K be an imaginary quadratic field, which we regard as a subfield of $\mathbb{C}$. Given a parameterization as in Claim 4.1 of rational points on a del Pezzo surface S, we must estimate the cardinality of each $M_{\mathbf{C}}(B)$. As indicated in § 1.3, we start by estimating the number of η_{B_0}, η_{C_0} in the fractional ideals $\mathcal{O}_{B_0}, \mathcal{O}_{C_0}$, say, satisfying the torsor equation, with the remaining variables fixed. The details depend on the precise shape of the torsor equation and coprimality conditions, via the configuration of curves on $\widetilde{S}$ encoded in an extended Dynkin diagram. In this section, we assume that they are as in (5.1) and Figure 1. As discussed in [Der09, Remark 2.1], this is true for the majority of singular del Pezzo surfaces described in [Der14], and the additional assumptions for Proposition 5.3 are expected to follow from Claim 4.1.

We use the following notation, similar to [Der09, Section 2]. Let r, s, $t \in \mathbb{Z}_{\geqslant 0}$, $(a_0, \dots, a_r) \in \mathbb{Z}_{>0}^{r+1}$, $(b_0, \dots, b_s) \in \mathbb{Z}_{>0}^{s+1}$, $(c_1, \dots, c_t) \in \mathbb{Z}_{>0}^t$. Let $G = (V, E)$ be the graph given in Figure 1, and let $G' = (V', E')$ be the graph obtained from G by deleting the vertices B_0, C_0 (see Figure 2).

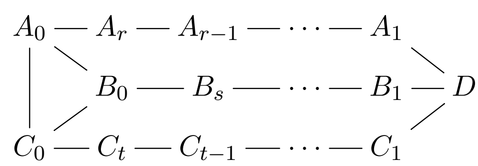

FIGURE 1. Graph of $G = (V, E)$.

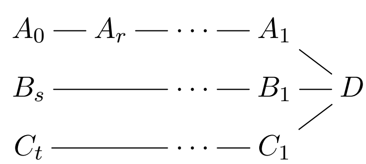

FIGURE 2. Graph of $G' = (V', E')$.

For $v \in V$, let $\mathcal{O}_v$ be a non-zero fractional ideal of K such that

$$\mathcal{O}_{A_0}^{a_0} \cdots \mathcal{O}_{A_r}^{a_r} = \mathcal{O}_{B_0}^{b_0} \cdots \mathcal{O}_{B_s}^{b_s} = \mathcal{O}_{C_0} \mathcal{O}_{C_1}^{c_1} \cdots \mathcal{O}_{C_t}^{c_t} =: \mathcal{O},$$

see Remark 4.6. We define

$$\mathcal{O}_{v*} := \begin{cases} \mathcal{O}_{A_0} \text{ or } \mathcal{O}_{A_0}^{\neq 0} & \text{if } v = A_0, \\ \mathcal{O}_v & \text{if } v \in \{B_0, C_0\}, \\ \mathcal{O}_v^{\neq 0} & \text{if } v \in V \smallsetminus \{A_0, B_0, C_0\}. \end{cases}$$

For $B > 0$, let $M(B)$ be the set of all $(\eta_v)_{v \in V} \in \prod_{v \in V} \mathcal{O}_{v*}$ with the following properties.

- The tuple $(\eta_v)_{v \in V \smallsetminus \{D\}}$ satisfies the *torsor equation*

$$\eta_{A_0}^{a_0} \cdots \eta_{A_r}^{a_r} + \eta_{B_0}^{b_0} \cdots \eta_{B_s}^{b_s} + \eta_{C_0} \eta_{C_1}^{c_1} \cdots \eta_{C_t}^{c_t} = 0. \tag{5.1}$$

- The tuple $(\eta_v)_{v \in V' \cup \{B_0\}}$ satisfies *height conditions* written as

$$((\eta_v)_{v \in V'}, \eta_{B_0}) \in \mathcal{R}(B), \tag{5.2}$$

  for a subset $\mathcal{R}(B) \subset \mathbb{C}^{V'} \times \mathbb{C}$. Moreover, we assume that for all $(\eta_v)_{v \in V'}$ and B, the set $\mathcal{R}((\eta_v)_{v \in V'}; B)$ of all $z \in \mathbb{C}$ with $((\eta_v)_{v \in V'}, z) \in \mathcal{R}(B)$ is of class m (see Definition 3.1) and contained in the union of k closed balls of radius $R((\eta_v)_{v \in V'}; B)$. Here, k, m are fixed constants.

- The ideals

$$I_v := \eta_v \mathcal{O}_v^{-1}, v \in V,$$

  of $\mathcal{O}_K$ satisfy the *coprimality conditions encoded by the graph* G, in the following sense: For any two non-adjacent vertices v and w in G, the corresponding ideals I_v and I_w are relatively prime. We impose the additional coprimality condition

  Each prime ideal $\mathfrak{p}$ dividing I_D may divide at most one of I_{A_0}, I_{B_0}, I_{C_0},

  which is only relevant if at least two of r, s, t are 0. Thus, $(I_{A_0}, I_{B_0}, I_{C_0})$ is the only triplet of ideals I_v allowed to have a nontrivial common divisor.

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In this section, we count, for fixed $(\eta_v)_{v\in V'}$, the number of all (η_{B_0},η_{C_0}) such that $(\eta_v)_{v\in V}$ satisfies the above conditions. This is analogous to [Der09, § 2], except that non-uniqueness of factorization in our case (if $h_K>1$) leads to technical difficulties. For ease of notation, we write $\boldsymbol{\eta}' := (\eta_v)_{v\in V'}$, $\mathbf{I}' := (I_v)_{v\in V'}$,

$$\begin{aligned} \boldsymbol{\eta_A} &:= (\eta_{A_1},\dots,\eta_{A_r}), & \boldsymbol{\eta_B} &:= (\eta_{B_1},\dots,\eta_{B_s}), & \boldsymbol{\eta_C} &:= (\eta_{C_1},\dots,\eta_{C_t}),\\ \mathbf{I_A} &:= (I_{A_1},\dots,I_{A_r}), & \mathbf{I_B} &:= (I_{B_1},\dots,I_{B_s}), & \mathbf{I_C} &:= (I_{C_1},\dots,I_{C_t}). \end{aligned}$$

Let

$$\Pi(\boldsymbol{\eta_A}) := \eta_{A_1}^{a_1}\cdots\eta_{A_r}^{a_r},\quad \Pi(\mathbf{I_A}) := I_{A_1}^{a_1}\cdots I_{A_r}^{a_r},$$

and

$$\Pi'(I_D,\mathbf{I_A}) := \begin{cases} I_D I_{A_1}\cdots I_{A_{r-1}} & \text{if } r\geqslant 1,\\ \mathcal{O}_K & \text{if } r=0.\end{cases}$$

Analogously, we define $\Pi(\boldsymbol{\eta_B})$, $\Pi(\mathbf{I_B})$, $\Pi'(I_D,\mathbf{I_B})$ and $\Pi(\boldsymbol{\eta_C})$, $\Pi(\mathbf{I_C})$, $\Pi'(I_D,\mathbf{I_C})$.

The following notation encoding coprimality conditions is similar to that in Definition 2.6. For any prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, let

$$J_{\mathfrak{p}}(\mathbf{I}') := \{v\in V' : \mathfrak{p}\mid I_v\}. \tag{5.3}$$

We define $\theta_0(\mathbf{I}') := \prod_{\mathfrak{p}}\theta_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$, where

$$\theta_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J=\emptyset, J=\{v\} \text{ with } v\in V' \text{ or } J=\{v,w\}\in E',\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 5.1. *If $(\eta_v)_{v\in V\smallsetminus\{D\}}$ satisfy the torsor equation (5.1), then the coprimality conditions encoded by G are equivalent to*

$$I_{B_0} + \Pi'(I_D,\mathbf{I_B})\Pi(\mathbf{I_A}) = \mathcal{O}_K \tag{5.4}$$
$$I_{C_0} + \Pi'(I_D,\mathbf{I_C}) = \mathcal{O}_K \tag{5.5}$$
$$\theta_0(\mathbf{I}') = 1. \tag{5.6}$$

Proof. This is analogous to [Der09, Lemma 2.3]. Condition (5.6) is equivalent to the coprimality conditions encoded by G for all I_v, $v\in V'$. Conditions (5.4), (5.5) are clearly implied by the coprimality conditions for I_{B_0}, I_{C_0}, respectively. Using the torsor equation (5.1), one can easily check that (5.4) and (5.6) imply $I_{B_0}+\Pi'(I_D,\mathbf{I_B})\Pi(\mathbf{I_A})\Pi(\mathbf{I_C})=\mathcal{O}_K$, and that (5.4), (5.5), (5.6) imply $I_{C_0}+\Pi'(I_D,\mathbf{I_C})\Pi(\mathbf{I_A})\Pi(\mathbf{I_B})=\mathcal{O}_K$. □

For given $\boldsymbol{\eta}'$, let $\mathfrak{A}=\mathfrak{A}(\boldsymbol{\eta}')$ be a non-zero ideal of $\mathcal{O}_K$ that is relatively prime to $\Pi'(I_D,\mathbf{I_C})\Pi(\mathbf{I_C})$, such that we can write

$$\eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta_A}) = \Pi_1\Pi_2^{b_0},$$

with $\Pi_2=\Pi_2(\boldsymbol{\eta}')\in\mathfrak{A}\mathcal{O}_{B_0}$ and $\Pi_1=\Pi_1(\boldsymbol{\eta}')\in\mathcal{O}(\mathfrak{A}\mathcal{O}_{B_0})^{-b_0}$.

Remark 5.2. For example, we can choose $\mathfrak{A}:=\mathfrak{p}$ to be a suitable prime ideal $\mathfrak{p}$ not dividing $\Pi'(I_D,\mathbf{I_C})\Pi(\mathbf{I_C})$, such that $\mathfrak{p}\mathcal{O}_{B_0}$ is a principal fractional ideal (t), and let $\Pi_2:=t$, $\Pi_1:=\eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta_A})/t^{b_0}$. However, in some applications it is desirable to use $\Pi_2^{b_0}$ to collect b_0-th powers of the variables η_{A_i} appearing in $\eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta_A})$.

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PROPOSITION 5.3. *With all the above definitions, we have*

$$|M(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \prod_{v \in V'} \mathcal{O}_{v*}} \theta_1(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B) + O\Bigg(\sum_{\boldsymbol{\eta}', (5.7)} 2^{\omega_K(\Pi'(I_D, \mathbf{I}_{\boldsymbol{C}})) + \omega_K(\Pi'(I_D, \mathbf{I}_{\boldsymbol{B}})\Pi(\mathbf{I}_{\boldsymbol{A}}))} b_0^{\omega_K(I_D \Pi(\mathbf{I}_{\boldsymbol{C}}))} \bigg(\frac{R(\boldsymbol{\eta}'; B)}{\mathfrak{N}\Pi(\mathbf{I}_{\boldsymbol{C}})^{1/2}} + 1 \bigg) \Bigg),$$

where the sum in the error term runs over all $\boldsymbol{\eta}' \in \prod_{v \in V'} \mathcal{O}_{v*}$ *such that*

$$\mathcal{R}(\boldsymbol{\eta}'; B) \neq \emptyset, \tag{5.7}$$

and the implicit constant may depend on K, k, m, *and* $\mathcal{O}_{B_0}$. *In the main term,*

$$V_1(\boldsymbol{\eta}'; B) := \int_{z \in \mathcal{R}(\boldsymbol{\eta}'; B)} \frac{1}{\mathfrak{N}(\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}_{B_0})} \, dz,$$

and

$$\theta_1(\boldsymbol{\eta}') := \sum_{\substack{\mathfrak{k}_\mathfrak{c} | \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_\mathfrak{c})}{\mathfrak{N}\mathfrak{k}_\mathfrak{c}} \tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c}) \sum_{\substack{\rho \bmod \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) \\ \rho\mathcal{O}_K + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K \\ \rho^{b_0} \equiv_{\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})} -\Pi_1/\Pi(\boldsymbol{\eta}_{\boldsymbol{B}})}} 1.$$

Here,

$$\tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c}) := \theta_0(\mathbf{I}') \frac{\phi_K^*(\Pi'(I_D, \mathbf{I}_{\boldsymbol{B}})\Pi(\mathbf{I}_{\boldsymbol{A}}))}{\phi_K^*(\Pi'(I_D, \mathbf{I}_{\boldsymbol{B}}) + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}))},$$

and $\Pi_1/\Pi(\boldsymbol{\eta}_{\boldsymbol{B}})$ *is invertible modulo* $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ *whenever* $\theta_0(\mathbf{I}') \neq 0$. *In the inner sum,* ρ *runs through a system of representatives for the invertible residue classes modulo* $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ *whose* b_0*-th power is the class of* $-\Pi_1/\Pi(\boldsymbol{\eta}_{\boldsymbol{B}})$.

If $b_0 = 1$, then the sum over ρ in the definition of θ_1 is just 1 whenever $\theta_0(\mathbf{I}') \neq 0$, so $\theta_1(\boldsymbol{\eta}') = \theta_1'(\mathbf{I}')$, where

$$\theta_1'(\mathbf{I}') := \sum_{\substack{\mathfrak{k}_\mathfrak{c} | \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_\mathfrak{c})}{\mathfrak{N}\mathfrak{k}_\mathfrak{c}} \tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c}). \tag{5.8}$$

In our applications, the function $\theta_1'(\mathbf{I}')$ plays an important role in the computation of the main term in the second summation, regardless of whether $b_0 = 1$ or not. Thus, let us investigate θ_1', at least in the case where s, $t \geqslant 1$. Recall that the I_v, $v \in V' \smallsetminus \{A_0\}$, are always non-zero ideals of $\mathcal{O}_K$. In the following, we will assume that $I_{A_0} \neq \{0\}$ holds as well.

LEMMA 5.4. *Let* s, $t \geqslant 1$. *Then we have*

$$\theta_1'(\mathbf{I}') = \prod_{\mathfrak{p}} \theta_{1,\mathfrak{p}}'(J_\mathfrak{p}(\mathbf{I}')), \tag{5.9}$$

where $J_\mathfrak{p}$ *is defined in (5.3), and for any* $J \subset V'$,

$$\theta_{1,\mathfrak{p}}'(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{B_s\}, \{C_t\}, \{A_0\}, \\ 1 - \dfrac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{D\}, \\ 1 - \dfrac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{v\}, \text{ with } v \in V' \smallsetminus \{B_s, C_t, A_0, D\}, \\ & \text{or } J = \{v, w\} \in E', \\ 0 & \text{otherwise.} \end{cases}$$

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In particular, $\theta_1' \in \Theta'_{r+s+t+2}(2)$ *and, with* $\rho := r+s+t+1$*, we have*

$$\mathcal{A}(\theta_1'(\mathbf{I}'), \mathbf{I}') = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{\rho} \left(1 + \frac{\rho}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right). \tag{5.10}$$

Moreover, let $v \in V' \smallsetminus \{A_1, B_1, C_1, D\}$ *and let* $\mathfrak{b}$ *be the product of all prime ideals of* $\mathcal{O}_K$ *dividing at least one* I_w *with* $w \in V' \smallsetminus \{v\}$ *not adjacent to* v*. Then, considered as a function of* I_v*, we have* $\theta_1'(\mathbf{I}') \in \Theta(\mathfrak{b}, 1, 1, 1)$*.*

Proof. We write $\theta_1'(\mathbf{I}')$ as

$$\theta_0(\mathbf{I}') \frac{\phi_K^*(\Pi'(I_D, \mathbf{I}_{\boldsymbol{B}})\Pi(\mathbf{I}_{\boldsymbol{A}}))}{\phi_K^*(\Pi'(I_D, \mathbf{I}_{\boldsymbol{B}}) + \Pi(\mathbf{I}_{\boldsymbol{C}}))} \sum_{\substack{\mathfrak{k}_{\mathfrak{c}} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_{\mathfrak{c}} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_{\mathfrak{c}})}{\mathfrak{N}\mathfrak{k}_{\mathfrak{c}}} \prod_{\substack{\mathfrak{p} \mid (\mathfrak{k}_{\mathfrak{c}} + \Pi'(I_D, \mathbf{I}_{\boldsymbol{B}})) \\ \mathfrak{p} \nmid \Pi(\mathbf{I}_{\boldsymbol{C}})}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^{-1}.$$

The first factor is defined as a product of local factors which depend only on the set $J_{\mathfrak{p}}(\mathbf{I}')$. It is obvious how to write the second factor as such a product. Recall that we assumed s, $t \geqslant 1$. Whenever $\theta_0(\mathbf{I}') \neq 0$, we can write the third factor as

$$\prod_{\substack{\mathfrak{p} \mid I_D \\ \mathfrak{p} \nmid I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}})\Pi(\mathbf{I}_{\boldsymbol{C}})}} \frac{\mathfrak{N}\mathfrak{p} - 2}{\mathfrak{N}\mathfrak{p} - 1} \prod_{\substack{\mathfrak{p} \mid (\Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) + \Pi(\mathbf{I}_{\boldsymbol{C}})) \\ \mathfrak{p} \nmid I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}})}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right).$$

Now (5.9) can be proved by a straightforward inspection of the local factors. To prove (5.10), we use (2.2) in Lemma 2.8. Then (5.9) and counting the vertices and edges in G' show that the local factor at each prime ideal $\mathfrak{p}$ is indeed as in (5.10).

The last assertion in the lemma is again an immediate consequence of (5.9). □

An analogous version of the last assertion in Lemma 5.4 holds for $\tilde{\theta}_1$.

Lemma 5.5. *Let* $v \in V'$ *and let* $\mathfrak{b}$ *be the product of all prime ideals of* $\mathcal{O}_K$ *dividing at least one* I_w *with* $w \in V' \smallsetminus \{v\}$ *not adjacent to* v*. Then, considered as a function of* I_v*, we have* $\tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_{\mathfrak{c}}) \in \Theta(\mathfrak{b}, 1, 1, 1)$*.*

Proof. This follows immediately from the definition of $\tilde{\theta}_1$. □

5.1 Proof of Proposition 5.3

The proof is mostly analogous to [Der09, Proposition 2.4], but the lack of unique factorization in $\mathcal{O}_K$ leads to some technical difficulties. We use two simple lemmas.

Lemma 5.6. *Let* $\mathfrak{a}$ *be an ideal and* $\mathfrak{f}$ *a non-zero fractional ideal of* $\mathcal{O}_K$*. Let* y_1, $y_2 \in \mathfrak{f}$ *such that* $(y_1\mathfrak{f}^{-1}, \mathfrak{a}) = (y_2\mathfrak{f}^{-1}, \mathfrak{a}) = \mathcal{O}_K$*. Then* y_2/y_1 *is invertible modulo* $\mathfrak{a}$ *and, for* $x \in \mathcal{O}_K$*, we have*

$$xy_1 - y_2 \in \mathfrak{a}\mathfrak{f} \quad \textit{if and only if} \quad x \equiv_{\mathfrak{a}} y_2/y_1.$$

Proof. For every prime ideal $\mathfrak{p} \mid \mathfrak{a}$, we have $v_{\mathfrak{p}}(y_1) = v_{\mathfrak{p}}(\mathfrak{f}) = v_{\mathfrak{p}}(y_2)$, so y_2/y_1 is invertible modulo $\mathfrak{a}$. Moreover, $xy_1 - y_2 \in \mathfrak{a}\mathfrak{f}$ holds if and only if $v_{\mathfrak{p}}(x - y_2/y_1) \geqslant v_{\mathfrak{p}}(\mathfrak{a}) - v_{\mathfrak{p}}(y_1\mathfrak{f}^{-1})$ for all prime ideals $\mathfrak{p}$. Given our assumptions, this is equivalent to $x \equiv_{\mathfrak{a}} y_2/y_1$. □

Lemma 5.7. *Let* $\mathfrak{a}_1$, $\mathfrak{a}_2$ *be fractional ideals of* $\mathcal{O}_K$ *and let* $x, y \in \mathfrak{a}_2$ *such that* $x - y \in \mathfrak{a}_1\mathfrak{a}_2$*. Then, for any positive integer* n*, we have* $x^n - y^n \in \mathfrak{a}_1\mathfrak{a}_2^n$*.*

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Proof. Clearly, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}) \in \mathfrak{a}_1\mathfrak{a}_2 \cdot \mathfrak{a}_2^{n-1}$. $\square$

For fixed $B > 0$ and $\boldsymbol{\eta}' \in \prod_{v \in V'} \mathcal{O}_{v*}$ subject to (5.6), let $N_1 = N_1(\boldsymbol{\eta}'; B)$ be the number of all $(\eta_{B_0}, \eta_{C_0}) \in \mathcal{O}_{B_0} \times \mathcal{O}_{C_0}$ such that the torsor equation (5.1), the coprimality conditions (5.4), (5.5), and the height conditions (5.2) are satisfied. Then

$$|M(B)| = \sum_{\substack{\boldsymbol{\eta}' \in \prod_{v \in V'} \mathcal{O}_{v*} \\ (5.6)}} N_1(\boldsymbol{\eta}'; B).$$

By Möbius inversion for (5.5), we obtain

$$N_1 = \sum_{\mathfrak{k}_\mathfrak{c} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}})} \mu(\mathfrak{k}_\mathfrak{c}) |\{(\eta_{B_0}, \eta_{C_0}) \in \mathcal{O}_{B_0} \times \mathfrak{k}_\mathfrak{c}\mathcal{O}_{C_0} \mid (5.1), (5.2), (5.4)\}|.$$

We note that, given $\eta_{B_0} \in \mathcal{O}_{B_0}$, there is a (unique) $\eta_{C_0} \in \mathfrak{k}_\mathfrak{c}\mathcal{O}_{C_0}$ with (5.1) if and only if

$$\eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta_A}) + \eta_{B_0}^{b_0}\Pi(\boldsymbol{\eta_B}) \in \Pi(\boldsymbol{\eta_C})\mathfrak{k}_\mathfrak{c}\mathcal{O}_{C_0} = \Pi(\mathbf{I}_{\boldsymbol{C}})\mathfrak{k}_\mathfrak{c}\mathcal{O}. \tag{5.11}$$

Similarly as in the proof of [Der09, Proposition 2.4], we see that (5.4) and (5.6) can only hold if $\mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K$, so

$$N_1 = \sum_{\substack{\mathfrak{k}_\mathfrak{c} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \mu(\mathfrak{k}_\mathfrak{c}) |\{\eta_{B_0} \in \mathcal{O}_{B_0} \mid (5.2), (5.4), (5.11)\}|.$$

Let us consider condition (5.11). Recall the definition of Π_1 and Π_2 before Proposition 5.3. We note that

$$\Pi_1(\mathcal{O}(\mathfrak{A}\mathcal{O}_{B_0})^{-b_0})^{-1} \cdot (\Pi_2(\mathfrak{A}\mathcal{O}_{B_0})^{-1})^{b_0} = \eta_{A_0}^{a_0}\Pi(\boldsymbol{\eta_A})\mathcal{O}^{-1} = I_{A_0}^{a_0}\Pi(\mathbf{I}_{\boldsymbol{A}}),$$

so $\Pi_1(\mathcal{O}(\mathfrak{A}\mathcal{O}_{B_0})^{-b_0})^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ and $\Pi_2(\mathfrak{A}\mathcal{O}_{B_0})^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ are $\mathcal{O}_K$.

LEMMA 5.8. *For all $\eta_{B_0} \in \mathcal{O}_{B_0}$ satisfying (5.11) there exists ρ in $\mathcal{O}_K$, unique modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$, such that*

$$\eta_{B_0} - \rho\Pi_2 \in \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}_{B_0}. \tag{5.12}$$

This ρ satisfies

$$\rho^{b_0} \equiv_{\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})} -\Pi_1/\Pi(\boldsymbol{\eta_B}). \tag{5.13}$$

Here, $\Pi_1/\Pi(\boldsymbol{\eta_B})$ is invertible modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$, so ρ is invertible modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ as well.

Conversely, if $\eta_{B_0} \in \mathcal{O}_{B_0}$ satisfies (5.12) for some ρ with (5.13) then it satisfies (5.11).

Proof. We write (5.11) as

$$\eta_{B_0}^{b_0}\Pi(\boldsymbol{\eta_B}) + \Pi_1\Pi_2^{b_0} \in \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}. \tag{5.14}$$

Since $\Pi_1\Pi_2^{b_0}\mathcal{O}^{-1} = I_{A_0}^{a_0}\Pi(\mathbf{I}_{\boldsymbol{A}})$ is coprime to $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$, we see that $\eta_{B_0}^{b_0}\Pi(\boldsymbol{\eta_B})\mathcal{O}^{-1} = I_{B_0}^{b_0}\Pi(\mathbf{I}_{\boldsymbol{B}})$ is coprime to $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ as well.

Therefore, $\eta_{B_0}\mathcal{O}_{B_0}^{-1} = I_{B_0}$ is relatively prime with $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$. Moreover, $\Pi_2 \in \mathcal{O}_{B_0}$ and, by our choice of $\mathfrak{A}$, we have $\Pi_2\mathcal{O}_{B_0}^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$. Therefore, we can apply Lemma 5.6 with $x := \rho$, $y_1 := \Pi_2$, $y_2 := \eta_{B_0}$, $\mathfrak{a} := \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$, and $\mathfrak{f} := \mathcal{O}_{B_0}$ to see that there is a unique ρ modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ with (5.12).

By Lemma 5.7, this ρ satisfies

$$\eta_{B_0}^{b_0} - (\rho\Pi_2)^{b_0} \in \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}_{B_0}^{b_0}. \tag{5.15}$$

Clearly, $\Pi(\boldsymbol{\eta_B})\mathcal{O}_K = \Pi(\mathbf{I}_{\boldsymbol{B}})\mathcal{O}\mathcal{O}_{B_0}^{-b_0} \subset \mathcal{O}\mathcal{O}_{B_0}^{-b_0}$, so (5.14) and (5.15) imply

$$\rho^{b_0}\Pi(\boldsymbol{\eta_B})\Pi_2^{b_0} + \Pi_1\Pi_2^{b_0} \in \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}. \tag{5.16}$$

Now $\Pi(\boldsymbol{\eta_B})\Pi_2^{b_0}\mathcal{O}^{-1} = \Pi(\mathbf{I}_{\boldsymbol{B}})\Pi_2^{b_0}\mathcal{O}_{B_0}^{-b_0}$, so $\Pi(\boldsymbol{\eta_B})\Pi_2^{b_0} \in \mathcal{O}$ and $\Pi(\boldsymbol{\eta_B})\Pi_2^{b_0}\mathcal{O}^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$. We have already seen that $\Pi_1\Pi_2^{b_0} \in \mathcal{O}$ and $\Pi_1\Pi_2^{b_0}\mathcal{O}^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$. By Lemma 5.6, $\Pi_1/\Pi(\boldsymbol{\eta_B})$ is invertible modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ and (5.13) holds.

Now assume that we are given $\eta_{B_0} \in \mathcal{O}_{B_0}$ such that (5.12) and (5.13) hold for some ρ. By the same argument as in the above paragraph, using the reverse implication in Lemma 5.6, (5.13) implies (5.16). By Lemma 5.7, (5.12) implies that $\eta_{B_0}^{b_0} - (\rho\Pi_2)^{b_0} \in \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}_{B_0}^{b_0}$, which, together with (5.16), yields (5.11). $\square$

By the lemma,

$$N_1 = \sum_{\substack{\mathfrak{k}_\mathfrak{c} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \mu(\mathfrak{k}_\mathfrak{c}) \sum_{\substack{\rho \bmod \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) \\ \rho\mathcal{O}_K + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K \\ (5.13)}} |\{\eta_{B_0} \in \mathcal{O}_{B_0} \mid (5.2), (5.4), (5.12)\}|.$$

After Möbius inversion for the coprimality condition (5.4), we have

$$N_1 = \sum_{\substack{\mathfrak{k}_\mathfrak{c} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_0}\Pi(\mathbf{I}_{\boldsymbol{A}})\Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K}} \mu(\mathfrak{k}_\mathfrak{c}) \sum_{\substack{\rho \bmod \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) \\ \rho\mathcal{O}_K + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K \\ (5.13)}} \sum_{\mathfrak{k}_\mathfrak{b} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{B}})\Pi(\mathbf{I}_{\boldsymbol{A}})} N_2(\mathfrak{k}_\mathfrak{c}, \mathfrak{k}_\mathfrak{b}, \rho),$$

where

$$N_2(\mathfrak{k}_\mathfrak{c}, \mathfrak{k}_\mathfrak{b}, \rho) := |\{\eta_{B_0} \in \mathfrak{k}_\mathfrak{b}\mathcal{O}_{B_0} \mid (5.2), (5.12)\}|.$$

Since $\rho\Pi_2\mathcal{O}_{B_0}^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$, congruence (5.12) implies that $\eta_{B_0}\mathcal{O}_{B_0}^{-1} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$. Therefore, we can add the condition $\mathfrak{k}_\mathfrak{b} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K$ to the sum over $\mathfrak{k}_\mathfrak{b}$.

Let $\delta \in \mathcal{O}_K \smallsetminus \{0\}$ such that $\delta\mathcal{O}_{B_0}$ is an integral ideal of $\mathcal{O}_K$. The conditions $\eta_{B_0} \in \mathfrak{k}_\mathfrak{b}\mathcal{O}_{B_0}$ and (5.12) can be written as a system of congruences

$$\begin{aligned} \delta\eta_{B_0} &\equiv 0 \bmod \mathfrak{k}_\mathfrak{b}(\delta\mathcal{O}_{B_0}) \\ \delta\eta_{B_0} &\equiv \delta\rho\Pi_2 \bmod \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})(\delta\mathcal{O}_{B_0}). \end{aligned}$$

Since $\mathfrak{k}_\mathfrak{b}\delta\mathcal{O}_{B_0} + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\delta\mathcal{O}_{B_0} = \delta\mathcal{O}_{B_0}$ and $\delta\rho\Pi_2 \equiv 0 \bmod \delta\mathcal{O}_{B_0}$, we can apply the Chinese remainder theorem. Thus, there is an element $x \in \mathcal{O}_K$ such that these congruences are equivalent to

$$\delta\eta_{B_0} \equiv x \bmod \mathfrak{k}_\mathfrak{b}\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})(\delta\mathcal{O}_{B_0}).$$

Hence,

$$N_2(\mathfrak{k}_\mathfrak{c}, \mathfrak{k}_\mathfrak{b}, \rho) = |(x/\delta + \mathfrak{k}_\mathfrak{b}\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})\mathcal{O}_{B_0}) \cap \mathcal{R}(\boldsymbol{\eta}'; B)|.$$

With our assumptions on $\mathcal{R}(\boldsymbol{\eta}'; B)$, Lemma 3.3 yields

$$N_2(\mathfrak{k}_\mathfrak{c}, \mathfrak{k}_\mathfrak{b}, \rho) = \frac{2}{\sqrt{|\Delta_K|}} \frac{V_1(\boldsymbol{\eta}'; B)}{\mathfrak{N}(\mathfrak{k}_\mathfrak{b}\mathfrak{k}_\mathfrak{c})} + O\biggl(\frac{R(\boldsymbol{\eta}'; B)}{\mathfrak{N}\Pi(\mathbf{I}_{\boldsymbol{C}})^{1/2}} + 1\biggr).$$

Now a simple computation shows that the main term in the proposition is the correct one. For the error term, we note that the number of ρ modulo $\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_{\boldsymbol{C}})$ with (5.13) is $\ll b_0^{\omega_K(I_D\Pi(\mathbf{I}_{\boldsymbol{C}}))}$ by Hensel's lemma.

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6. The second summation

As in the previous section, K denotes an imaginary quadratic field. We provide tools to sum the main term resulting from Proposition 5.3 over a further variable.

First, we fix some notation: let $\mathcal{O}$ be a non-zero fractional ideal of K, let $\mathfrak{q} \in \mathcal{I}_K$, and $n \in \mathbb{Z}_{>0}$. Let $A \in K$ such that $v_{\mathfrak{p}}(A\mathcal{O}) = 0$ for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ dividing $\mathfrak{q}$. In particular, Az is defined modulo $\mathfrak{q}$ for all $z \in \mathcal{O}$.

We consider a function $\vartheta : \mathcal{I}_K \to \mathbb{R}$ such that, with constants $c_\vartheta > 0$ and $C \geqslant 0$,

$$\sum_{\substack{\mathfrak{a} \in \mathcal{I}_K \\ \mathfrak{N}\mathfrak{a} \leqslant t}} |(\vartheta * \mu_K)(\mathfrak{a})| \cdot \mathfrak{N}\mathfrak{a} \ll c_\vartheta t (\log(t+2))^C \tag{6.1}$$

holds for all $t > 0$. We write

$$\mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{q}) := \sum_{\substack{\mathfrak{a} \in \mathcal{I}_K \\ \mathfrak{a} + \mathfrak{q} = \mathcal{O}_K}} \frac{(\vartheta * \mu_K)(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}}.$$

(For $\vartheta \in \Theta(\mathfrak{b}, C_1, C_2, C_3)$, this is consistent with the definition given in § 2.)

For $1 \leqslant t_1 \leqslant t_2$, let $g : [t_1, t_2] \to \mathbb{R}$ be a function such that there exists a partition of $[t_1, t_2]$ into at most $R(g)$ intervals on whose interior g is continuously differentiable and monotonic. Moreover, with constants $c_g > 0$ and $a \leqslant 0$, we assume that

$$|g(t)| \ll c_g t^a \quad \text{on } [t_1, t_2]. \tag{6.2}$$

We find an asymptotic formula for the sum

$$S(t_1, t_2) := \sum_{\substack{z \in \mathcal{O}^{\neq 0} \\ t_1 < \mathfrak{N}(z\mathcal{O}^{-1}) \leqslant t_2}} \vartheta(z\mathcal{O}^{-1}) \sum_{\substack{\rho \bmod \mathfrak{q} \\ \rho\mathcal{O}_K + \mathfrak{q} = \mathcal{O}_K \\ \rho^n \equiv_{\mathfrak{q}} Az}} g(\mathfrak{N}(z\mathcal{O}^{-1})).$$

Proposition 6.1. *With the above definitions, we have*

$$S(t_1, t_2) = \frac{2\pi}{\sqrt{|\Delta_K|}} \phi_K^*(\mathfrak{q}) \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathfrak{q}) \int_{t_1}^{t_2} g(t)\, dt + O(c_\vartheta c_g (\sqrt{\mathfrak{N}\mathfrak{q}} \mathcal{E}_1 + \mathfrak{N}\mathfrak{q} \mathcal{E}_2)),$$

where

$$\mathcal{E}_1 \ll_{a,C} R(g) \begin{cases} \sup_{t_1 \leqslant t \leqslant t_2} (t^{a+1/2}) & \text{if } a \neq -1/2, \\ \log(t_2 + 2) & \text{if } a = -1/2, \end{cases} \tag{6.3}$$

and

$$\mathcal{E}_2 \ll_{a,C} R(g) \begin{cases} t_1^a \log(t_1 + 2)^{C+1} & \text{if } a < 0, \\ \log(t_2 + 2)^{C+1} & \text{if } a = 0. \end{cases} \tag{6.4}$$

Moreover, the same formula holds if, in the definition of $S(t_1, t_2)$, the range $t_1 < \mathfrak{N}(z\mathcal{O}^{-1}) \leqslant t_2$ is replaced by $t_1 \leqslant \mathfrak{N}(z\mathcal{O}^{-1}) \leqslant t_2$.

Remark 6.2. In particular, we can apply Proposition 6.1 with $\mathfrak{q} = \mathcal{O}_K$, $n = A = 1$ to handle sums of the form

$$S(t_1, t_2) = \sum_{\substack{z \in \mathcal{O}^{\neq 0} \\ t_1 < \mathfrak{N}(z\mathcal{O}^{-1}) \leqslant t_2}} \vartheta(z\mathcal{O}^{-1}) g(\mathfrak{N}(z\mathcal{O}^{-1})).$$

In this case, the error term is $\ll_{a,C} R(g) c_\vartheta c_g \sup_{t_1 \leqslant t \leqslant t_2} (t^{a+1/2})$ if $a \neq -1/2$ and $\ll_C R(g) c_\vartheta c_g \log(t_2 + 2)$ if $a = -1/2$. (Note that $t_2 \geqslant t_1 \geqslant 1$.)

Recall the notation of § 5, in particular Proposition 5.3. In a typical application, we have r, s, $t \geqslant 1$, $b_0 \in \{1, 2\}$, and

$$V_1(\boldsymbol{\eta}'; B) =: \tilde{V}_1((\mathfrak{N} I_v)_{v \in V'}; B)$$

depends only on B and the absolute norms of the ideals I_v, and not on the η_v. Let us describe how we apply Proposition 6.1 to sum the main term in the result of Proposition 5.3 over a further variable, say, η_w. We write $V'' := V' \smallsetminus \{w\}$, $\boldsymbol{\eta}'' := (\eta_v)_{v \in V''}$ and assume that $g(t) := \tilde{V}_1((\mathfrak{N} I_v)_{v \in V''}, t; B)$ satisfies the hypotheses from the beginning of this section. We define

$$V_2((\mathfrak{N} I_v)_{v \in V''}; B) := \pi \int_{t \geqslant 1} g(t)\, dt$$

and distinguish between two cases.

In the first case, let $b_0 = 1$. As mentioned after Proposition 5.3, $\theta_1(\boldsymbol{\eta}') = \theta_1'(\mathbf{I}')$. Let $\vartheta(I_w) := \theta_1'(\mathbf{I}')$, considered as a function of I_w. By the last assertion of Lemma 5.4 and Lemma 2.2(2), ϑ satisfies (6.1) with $c_\theta = 2^{\omega(\mathfrak{b})}$ and $C = 0$. Up to a possible contribution of $\eta_w = 0$ (if $w = A_0$), we can use Remark 6.2 to estimate

$$\frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \prod_{v \in V''} \mathcal{O}_{v*}} \sum_{\eta_w \in \mathcal{O}_{w*}} \vartheta(\eta_w \mathcal{O}_w^{-1}) g(\mathfrak{N}(\eta_w \mathcal{O}_w^{-1})).$$

We obtain a main term

$$\left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \prod_{v \in V''} \mathcal{O}_{v*}} \mathcal{A}(\theta_1'(\mathbf{I}'), I_v) V_2((\mathfrak{N} I_v)_{v \in V''}; B). \tag{6.5}$$

It remains to bound the sum over $\boldsymbol{\eta}''$ of the error term from Remark 6.2.

In the second case $b_0 \geqslant 2$, the sum over ρ in the definition of θ_1 is not just 1. However, we note that, if r, $s \geqslant 1$, the condition $\mathfrak{k}_\mathfrak{c} + I_{A_0} \Pi(\mathbf{I}_{\boldsymbol{A}}) \Pi(\mathbf{I}_{\boldsymbol{B}}) = \mathcal{O}_K$ can be replaced by $\mathfrak{k}_\mathfrak{c} + I_{A_1} I_{B_1} = \mathcal{O}_K$, since the remaining coprimality conditions follow from $\theta_0(\mathbf{I}') = 1$. We additionally assume that

$$w \in \{A_0, A_2, \ldots, A_r\}$$

and that $-\Pi_1/\Pi(\boldsymbol{\eta_B})$ has the form $A\eta_w$, where A does not depend on η_w. Then $v_\mathfrak{p}(A\mathcal{O}_w) = 0$ for all $\mathfrak{p} \mid \mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}})$. We apply Proposition 6.1 once for every summand in the sum over $\mathfrak{k}_\mathfrak{c}$, to sum the expression

$$\tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c}) \sum_{\substack{\rho \bmod \mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}}) \\ \rho \mathcal{O}_K + \mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}}) = \mathcal{O}_K \\ \rho^{b_0} \equiv_{\mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}})} A\eta_w}} V_1(\boldsymbol{\eta}'; B)$$

over $\eta_w \in \mathcal{O}_{w*}$. Let $\vartheta(I_w) := \tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c})$, considered as a function of I_w. By Lemmas 5.5 and 2.2(2), ϑ satisfies (6.1) with $c_\vartheta = 2^{\omega_K(\mathfrak{b})}$ and $C = 0$. After applying Proposition 6.1 and summing the result over $\mathfrak{k}_\mathfrak{c}$, we obtain a main term

$$\left(\frac{2}{\sqrt{\Delta_K}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \prod_{v \in V''} \mathcal{O}_{v*}} \theta_2(\mathbf{I}'') V_2((\mathfrak{N} I_v)_{v \in V''}; B),$$

where

$$\theta_2(\mathbf{I}'') := \sum_{\substack{\mathfrak{k}_\mathfrak{c} \mid \Pi'(I_D, \mathbf{I}_{\boldsymbol{C}}) \\ \mathfrak{k}_\mathfrak{c} + I_{A_1} I_{B_1} = \mathcal{O}_K}} \frac{\mu_K(\mathfrak{k}_\mathfrak{c})}{\mathfrak{N}\mathfrak{k}_\mathfrak{c}} \phi_K^*(\mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}})) \mathcal{A}(\vartheta(I_w), I_w, \mathfrak{k}_\mathfrak{c} \Pi(\mathbf{I}_{\boldsymbol{C}})). \tag{6.6}$$

The following lemma shows that the main term is the same as in the case $b_0 = 0$. It remains to bound the sum over η'' and $\mathfrak{k}_\mathfrak{c}$ of the error term multiplied by $\mu_K(\mathfrak{k}_\mathfrak{c})/\mathfrak{N}\mathfrak{k}_\mathfrak{c}$.

LEMMA 6.3. *Assume that $r, s \geqslant 1$, choose $w \in \{A_0, A_2, \ldots, A_r, B_2, \ldots, B_s\}$, and let $\vartheta(I_w) := \tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c})$, considered as a function of I_w. Then $\vartheta(I_w) \in \Theta(\mathfrak{b}, 1, 1, 1)$, where $\mathfrak{b}$ is given in Lemma 5.5. Define $\theta_2(\mathbf{I}'')$ as in (6.6) and $\theta_1'(\mathbf{I}')$ as in (5.8). Then we have*

$$\theta_2(\mathbf{I}'') = \mathcal{A}(\theta_1'(\mathbf{I}'), I_w).$$

Proof. It is enough to prove that $\phi_K^*(\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C))\mathcal{A}(\vartheta(I_w), I_w, \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C)) = \mathcal{A}(\vartheta(I_w), I_w)$ holds whenever $\mathfrak{k}_\mathfrak{c}$ satisfies the conditions under the sum. This is clearly true if ϑ is the zero function. If not, write $\vartheta(I_w) = \prod_\mathfrak{p} A_\mathfrak{p}(v_\mathfrak{p}(I_w))$ with $A_\mathfrak{p}(n) = A_\mathfrak{p}(1)$ for all prime ideals $\mathfrak{p}$ and all $n \geqslant 1$. By Lemma 2.2(3), $\phi_K^*(\mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C))\mathcal{A}(\vartheta(I_w), I_w, \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C))$ is given by

$$\prod_{\mathfrak{p} \nmid \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C)} \left(\left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)A_\mathfrak{p}(0) + \frac{1}{\mathfrak{N}\mathfrak{p}}A_\mathfrak{p}(1)\right) \prod_{\mathfrak{p} \mid \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C)} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)A_\mathfrak{p}(0),$$

and

$$\mathcal{A}(\vartheta(I_w), I_w) = \prod_\mathfrak{p}\left(\left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)A_\mathfrak{p}(0) + \frac{1}{\mathfrak{N}\mathfrak{p}}A_\mathfrak{p}(1)\right).$$

By our choice of w, we have $\vartheta(I_w) = \tilde{\theta}_1(\mathbf{I}', \mathfrak{k}_\mathfrak{c}) = 0$ whenever $\mathfrak{p} \mid (I_w + \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C))$. Since ϑ is not identically zero, this implies $A_\mathfrak{p}(1) = 0$ for all $\mathfrak{p} \mid \mathfrak{k}_\mathfrak{c}\Pi(\mathbf{I}_C)$. □

6.1 Proof of Proposition 6.1

First, we prove a version of Lemma 2.5 that counts elements in a given residue class instead of ideals.

LEMMA 6.4. *Let $\mathfrak{a}$ be an ideal of K and let $\beta \in \mathcal{O}_K$ such that $\mathfrak{a} + \mathfrak{q} = \beta\mathcal{O}_K + \mathfrak{q} = \mathcal{O}_K$. Moreover, let $\vartheta : \mathcal{I}_K \to \mathbb{R}$ satisfy (6.1). Then, for $t \geqslant 0$,*

$$\sum_{\substack{z \in \mathfrak{a} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \mathfrak{N}(z\mathfrak{a}^{-1}) \leqslant t}} \vartheta(z\mathfrak{a}^{-1}) = \frac{2\pi}{\sqrt{|\Delta_K|}\mathfrak{N}\mathfrak{q}}\mathcal{A}(\vartheta(\mathfrak{b}), \mathfrak{b}, \mathfrak{q})t + O_C\left(c_\vartheta\left(\sqrt{\frac{t}{\mathfrak{N}\mathfrak{q}}} + \log(t+2)^{C+1}\right)\right).$$

Proof. The case $t < 1$ can be handled as in Lemma 2.5, so let $t \geqslant 1$. Using $\vartheta = (\vartheta * \mu_K) * 1$, we see that

$$\sum_{\substack{z \in \mathfrak{a} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \mathfrak{N}(z\mathfrak{a}^{-1}) \leqslant t}} \vartheta(z\mathfrak{a}^{-1}) = \sum_{\mathfrak{N}\mathfrak{b} \leqslant t} (\vartheta * \mu_K)(\mathfrak{b}) \sum_{\substack{z \in \mathfrak{a}\mathfrak{b} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \mathfrak{N}(z\mathfrak{a}^{-1}) \leqslant t}} 1 = \sum_{\substack{\mathfrak{N}\mathfrak{b} \leqslant t \\ \mathfrak{b} + \mathfrak{q} = \mathcal{O}_K}} (\vartheta * \mu_K)(\mathfrak{b}) \sum_{\substack{z \in \mathfrak{a}\mathfrak{b} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \|z\|_\infty \leqslant t\mathfrak{N}\mathfrak{a}}} 1.$$

For the second equality, note that the inner sum is 0 whenever $\mathfrak{b} + \mathfrak{q} \neq \mathcal{O}_K$. Now $\mathfrak{a}\mathfrak{b} + \mathfrak{q} = \mathcal{O}_K$, so the Chinese remainder theorem yields an $x \in \mathcal{O}_K$ such that

$$\sum_{\substack{z \in \mathfrak{a} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \mathfrak{N}(z\mathfrak{a}^{-1}) \leqslant t}} \vartheta(z\mathfrak{a}^{-1}) = \sum_{\substack{\mathfrak{N}\mathfrak{b} \leqslant t \\ \mathfrak{b} + \mathfrak{q} = \mathcal{O}_K}} (\vartheta * \mu_K)(\mathfrak{b}) \sum_{\substack{z \in \mathcal{O}_K^{\neq 0} \\ z \equiv x \bmod \mathfrak{a}\mathfrak{b}\mathfrak{q} \\ \|z\|_\infty \leqslant t\mathfrak{N}\mathfrak{a}}} 1.$$

We use Lemma 3.3 to estimate the inner sum and obtain

$$\sum_{\substack{z \in \mathfrak{a} \smallsetminus \{0\} \\ z \equiv \beta \bmod \mathfrak{q} \\ \mathfrak{N}(z\mathfrak{a}^{-1}) \leqslant t}} \vartheta(z\mathfrak{a}^{-1}) = \sum_{\substack{\mathfrak{N}\mathfrak{b} \leqslant t \\ \mathfrak{b} + \mathfrak{q} = \mathcal{O}_K}} (\vartheta * \mu_K)(\mathfrak{b})\left(\frac{2\pi t}{\sqrt{|\Delta_K|}\mathfrak{N}\mathfrak{b}\mathfrak{N}\mathfrak{q}} + O\left(\sqrt{\frac{t}{\mathfrak{N}\mathfrak{b}\mathfrak{N}\mathfrak{q}}} + 1\right)\right),$$

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which we expand to the main term in the lemma plus an error term

$$\ll \frac{t}{\mathfrak{N}\mathfrak{q}} \sum_{\mathfrak{N}\mathfrak{b}>t} \frac{|(\vartheta * \mu_K)(\mathfrak{b})|}{\mathfrak{N}\mathfrak{b}} + \sqrt{\frac{t}{\mathfrak{N}\mathfrak{q}}} \sum_{\mathfrak{N}\mathfrak{b}\leqslant t} \frac{|(\vartheta * \mu_K)(\mathfrak{b})|}{\sqrt{\mathfrak{N}\mathfrak{b}}} + \sum_{\mathfrak{N}\mathfrak{b}\leqslant t} |(\vartheta * \mu_K)(\mathfrak{b})|.$$

By (6.1) and Lemma 2.4, the first part of the error term is $\ll_C c_\vartheta \mathfrak{N}\mathfrak{q}^{-1}\log(t+2)^C$, the second part is $\ll_C c_\vartheta\sqrt{t/\mathfrak{N}\mathfrak{q}}$, and the third part is $\ll_C c_\vartheta \log(t+2)^{C+1}$. □

Lemma 6.5. *Using the notation from the beginning of this section, we have*

$$\sum_{\substack{z\in\mathcal{O}^{\neq 0}\\ \mathfrak{N}(z\mathcal{O}^{-1})\leqslant t}} \vartheta(z\mathcal{O}^{-1}) \sum_{\substack{\rho \bmod \mathfrak{q}\\ \rho\mathcal{O}_K+\mathfrak{q}=\mathcal{O}_K\\ \rho^n\equiv_{\mathfrak{q}} Az}} 1 = \frac{2\pi}{\sqrt{|\Delta_K|}}\phi_K^*(\mathfrak{q})\mathcal{A}(\vartheta(\mathfrak{a}),\mathfrak{a},\mathfrak{q})t + O(c_\vartheta(\sqrt{\mathfrak{N}\mathfrak{q}t} + \mathfrak{N}\mathfrak{q}\log(t+2)^{C+1})). \tag{6.7}$$

Proof. Denote the expression on the left-hand side of (6.7) by L. Since $v_{\mathfrak{p}}(A\mathcal{O}) = 0$ for all $\mathfrak{p}\mid\mathfrak{q}$, we can, by weak approximation, find $A_1\in\mathcal{O}^{-1}$, $A_2\in\mathcal{O}_K$ such that $A = A_1/A_2$ and $A_1\mathcal{O}+\mathfrak{q} = A_2\mathcal{O}_K+\mathfrak{q} = \mathcal{O}_K$. Changing the order of summation, we obtain

$$L = \sum_{\substack{\rho \bmod \mathfrak{q}\\ \rho\mathcal{O}_K+\mathfrak{q}=\mathcal{O}_K}} \sum_{\substack{z\in\mathcal{O}^{\neq 0}\\ A_1z\equiv A_2\rho^n \bmod \mathfrak{q}\\ \mathfrak{N}(z\mathcal{O}^{-1})\leqslant t}} \vartheta(z\mathcal{O}^{-1}) = \sum_{\substack{\rho \bmod \mathfrak{q}\\ \rho\mathcal{O}_K+\mathfrak{q}=\mathcal{O}_K}} \sum_{\substack{A_1z\in A_1\mathcal{O}^{\neq 0}\\ A_1z\equiv A_2\rho^n \bmod \mathfrak{q}\\ \mathfrak{N}(A_1z(A_1\mathcal{O})^{-1})\leqslant t}} \vartheta(A_1z(A_1\mathcal{O})^{-1}).$$

The lemma now follows from Lemma 6.4 and the trivial estimate $\phi_K(\mathfrak{q})\leqslant\mathfrak{N}\mathfrak{q}$. □

Define $\tilde\vartheta : \mathcal{I}_K\to\mathbb{R}$ by

$$\tilde\vartheta(\mathfrak{a}) := \vartheta(\mathfrak{a}) \sum_{\substack{z\in\mathcal{O}^{\neq 0}\\ z\mathcal{O}^{-1}=\mathfrak{a}}} \sum_{\substack{\rho \bmod \mathfrak{q}\\ \rho\mathcal{O}_K+\mathfrak{q}=\mathcal{O}_K\\ \rho^n\equiv_{\mathfrak{q}} Az}} 1.$$

The first sum is finite, since $|\mathcal{O}_K^\times|<\infty$. Then

$$S(t_1,t_2) = \sum_{\substack{\mathfrak{a}\in[\mathcal{O}^{-1}]\cap\mathcal{I}_K\\ t_1<\mathfrak{N}\mathfrak{a}\leqslant t_2}} \tilde\vartheta(\mathfrak{a})g(\mathfrak{N}\mathfrak{a}),$$

and by Lemma 6.5 we have

$$\sum_{\substack{a\in[\mathcal{O}^{-1}]\cap\mathcal{I}_K\\ \mathfrak{N}\mathfrak{a}\leqslant t}} \tilde\vartheta(\mathfrak{a}) = \frac{2\pi}{\sqrt{|\Delta_K|}}\phi_K^*(\mathfrak{q})\mathcal{A}(\vartheta(\mathfrak{a}),\mathfrak{a},\mathfrak{q})t + O(c_\vartheta(\sqrt{\mathfrak{N}\mathfrak{q}t} + \mathfrak{N}\mathfrak{q}\log(t+2)^{C+1})).$$

With (6.2) and simple calculations, the proposition now follows from Lemma 2.10.

7. Further summations

Here, we show how to evaluate the main term of the second summation as in (6.5), once the sums over $\mathbf{C}\in\mathcal{C}^{r+1}$ from Claim 4.1 and over elements $\boldsymbol{\eta}''\in\mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{r+1*}$ have been transformed into a sum over ideals $(\mathfrak{a}_1,\ldots,\mathfrak{a}_{r+1})\in\mathcal{I}_K^{r+1}$ (see Lemma 9.4, for example).

In this section, K can be an arbitrary number field of degree $d\geqslant 2$. For $K=\mathbb{Q}$, we refer to [Der09]. Let $r\in\mathbb{Z}_{>0}$, $s\in\{0,1\}$. We consider functions $V:\mathbb{R}_{\geqslant 1}^{r+s}\times\mathbb{R}_{\geqslant 3}\to\mathbb{R}_{\geqslant 0}$ similar to the ones in [Der09, Propositions 3.9 and 3.10]. That is, we consider three cases.

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(a) We have $s = 0$ and
$$V(t_1, \dots, t_r; B) \ll \frac{B}{t_1 \cdots t_r}.$$
(b) We have $s = 1$ and there exist $k_2, \dots, k_r \in \mathbb{R}$, k_1, $k_{r+1} \in \mathbb{R}_{\neq 0}$, $a \in \mathbb{R}_{>0}$ with
$$V(t_1, \dots, t_{r+1}; B) \ll \frac{B}{t_1 \cdots t_{r+1}} \cdot \left(\frac{B}{t_1^{k_1} \cdots t_{r+1}^{k_{r+1}}}\right)^{-a}.$$
Moreover, $V(t_1, \dots, t_{r+1}; B) = 0$ unless $t_1^{k_1} \cdots t_{r+1}^{k_{r+1}} \leqslant B$.

(c) We have $s = 1$ and there exist $k_2, \dots, k_r \in \mathbb{R}$, k_1, $k_{r+1} \in \mathbb{R}_{\neq 0}$, a, $b \in \mathbb{R}_{>0}$ with
$$V(t_1, \dots, t_{r+1}; B) \ll \frac{B}{t_1 \cdots t_{r+1}} \cdot \min\left\{\left(\frac{B}{t_1^{k_1} \cdots t_{r+1}^{k_{r+1}}}\right)^{-a}, \left(\frac{B}{t_1^{k_1} \cdots t_{r+1}^{k_{r+1}}}\right)^{b}\right\}.$$

In addition, we assume that $V(t_1, \dots, t_{r+s}) = 0$ unless $t_1, \dots, t_{r+s} \leqslant B$, and that there is a constant $R(V)$ such that for all fixed $t_1, \dots, t_{r+s-1}$, B, there is a partition of $[1, B]$ into at most $R(V)$ intervals on whose interior $V(t_1, \dots, t_{r+s}; B)$, considered as a function of t_{r+s}, is continuously differentiable and monotonic. We note that case (b) implies case (c) for any $b > 0$.

Lemma 7.1. *Let $V(t_1, \dots, t_{r+s}; B)$ be as above, $t_{r+s} \geqslant 1$, $B \geqslant 3$. Then*
$$\sum_{\mathfrak{a}_1, \dots, \mathfrak{a}_{r+s-1} \in \mathcal{I}_K} V(\mathfrak{N}\mathfrak{a}_1, \dots, \mathfrak{N}\mathfrak{a}_{r+s-1}, t_{r+s}; B) \ll \frac{B(\log B)^{r-1}}{t_{r+s}}.$$

Proof. In case (a) this follows immediately from Lemma 2.4 with $C = 0$, $\kappa = 1$ applied r times. In case (b), we apply Lemma 2.4 with $C = 0$, $\kappa = 1 - ak_1$ to the sum over $\mathfrak{a}_1$ and then proceed as in case (a).

In case (c), we split the sum over $\mathfrak{a}_1$ into two sums: one over all $\mathfrak{a}_1$ with $\mathfrak{N}\mathfrak{a}_1^{k_1} \cdots \mathfrak{N}\mathfrak{a}_{r+1}^{k_{r+1}} \leqslant B$ and one where the opposite inequality holds. For the first, we use
$$V(\mathfrak{N}\mathfrak{a}_1, \dots, \mathfrak{N}\mathfrak{a}_r, t_{r+1}; B) \ll B/(\mathfrak{N}\mathfrak{a}_1 \cdots \mathfrak{N}\mathfrak{a}_r t_{r+1})(B/(\mathfrak{N}\mathfrak{a}_1^{k_1} \cdots \mathfrak{N}\mathfrak{a}_r^{k_r} t_{r+1}^{k_{r+1}}))^{-a}$$
and proceed as in case (b). For the second sum, we use
$$V(\mathfrak{N}\mathfrak{a}_1, \dots, \mathfrak{N}\mathfrak{a}_r, t_{r+1}; B) \ll B/(\mathfrak{N}\mathfrak{a}_1 \cdots \mathfrak{N}\mathfrak{a}_r t_{r+1})(B/(\mathfrak{N}\mathfrak{a}_1^{k_1} \cdots \mathfrak{N}\mathfrak{a}_r^{k_r} t_{r+1}^{k_{r+1}}))^{b}$$
and apply Lemma 2.4 with $C = 0$, $\kappa = 1 + bk_1$. The remaining summations over $\mathfrak{a}_2, \dots, \mathfrak{a}_r$ are again handled as in case (a). $\square$

Proposition 7.2. *Let V be as above and $\theta \in \Theta'_{r+s}(C)$ for some $C \in \mathbb{Z}_{>0}$. Then*
$$\begin{aligned}\sum_{\mathfrak{a}_1, \dots, \mathfrak{a}_{r+s}} &\theta(\mathfrak{a}_1, \dots, \mathfrak{a}_{r+s}) V(\mathfrak{N}\mathfrak{a}_1, \dots, \mathfrak{N}\mathfrak{a}_{r+s}; B) \\ &= \rho_K h_K \sum_{\mathfrak{a}_1, \dots, \mathfrak{a}_{r+s-1}} \mathcal{A}(\theta(\mathfrak{a}_1, \dots, \mathfrak{a}_{r+s}), \mathfrak{a}_{r+s}) \int_1^\infty V(\mathfrak{N}\mathfrak{a}_1, \dots, \mathfrak{N}\mathfrak{a}_{r+s-1}, t_{r+s}; B)\, dt_{r+s} \\ &\quad + O_{V,C}(B(\log B)^{r-1} \log\log B).\end{aligned}$$

Proof. This is mostly analogous to a special case of [Der09, Propositions 3.9 and 3.10], but we could simplify the third step significantly.

We define $T := (\log B)^{d((2C-1)(r+s-1)+s)}$ and proceed in three steps:

(1) bound
$$\sum_{\substack{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}\\ \mathfrak{N}\mathfrak{a}_{r+s}<T}} \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}) V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s};B);$$

(2) bound the sum over $\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1}$ of
$$\sum_{\mathfrak{N}\mathfrak{a}_{r+s}\geqslant T} \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}) V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s};B)$$
$$- \rho_K h_K \mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}),\mathfrak{a}_{r+s}) \int_T^\infty V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s-1},t_{r+s};B)\,dt_{r+s};$$

(3) bound
$$\sum_{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1}} \mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}),\mathfrak{a}_{r+s}) \int_1^T V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s-1},t_{r+s};B)\,dt_{r+s}.$$

Using $0 \leqslant \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}) \leqslant 1$ and Lemma 7.1, we see that the expression in step (1) is indeed bounded by
$$\sum_{\substack{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}\\ \mathfrak{N}\mathfrak{a}_{r+s}<T}} V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s};B) \ll \sum_{\mathfrak{N}\mathfrak{a}_{r+s}<T} \frac{B(\log B)^{r-1}}{\mathfrak{N}\mathfrak{a}_{r+s}} \ll B(\log B)^{r-1}\log\log B.$$

Analogously, since $\mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s}),\mathfrak{a}_{r+s}) \in \Theta'_{r+s-1}(2C)$, the expression in step (3) is bounded by
$$\begin{aligned}\sum_{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1}} \int_1^T V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+s-1},t_{r+s};B)\,dt_{r+s} &\ll \int_1^T \frac{B(\log B)^{r-1}}{t_{r+s}}\,dt_{r+s}\\ &\ll B(\log B)^{r-1}\log\log B.\end{aligned}$$

For step (2), we note that in all three cases (a), (b) and (c) we have
$$V(t_1,\dots,t_{r+s};B) \ll \frac{B}{t_1\cdots t_{r+s}}.$$

By Corollary 2.7, we may apply Lemma 2.10 with $m=1$, $c_1 = (2C)^{\omega(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1})}$, $b_1 = 1-1/d$, $k_1 = 0$, $c_g = B/(\mathfrak{N}\mathfrak{a}_1\cdots\mathfrak{N}\mathfrak{a}_{r+s-1})$, $a=-1$ for the sum over $\mathfrak{a}_{r+s}$. We obtain an error term of order
$$\begin{aligned}\ll_{V,C} \sum_{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1}} \frac{(2C)^{\omega(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+s-1})}B}{\mathfrak{N}\mathfrak{a}_1\cdots\mathfrak{N}\mathfrak{a}_{r+s-1}} T^{-1/d} &\ll_C B(\log B)^{(2C)(r+s-1)}T^{-1/d}\\ &\ll B(\log B)^{r-1}.\end{aligned}$$ □

Let $V_{r+1} := V$ be as in cases (b) and (c) at the start of this section. For all $l \in \{0,\dots,r\}$, we define
$$V_l(t_1,\dots,t_l;B) := \int_{t_{l+1},\dots,t_{r+1}\geqslant 1} V(t_1,\dots,t_{r+1};B)\,dt_{l+1}\cdots dt_{r+1}.$$

For $l \geqslant 1$, and fixed $t_1,\dots,t_{l-1}$, B, we additionally require that there is a partition of $[1,B]$ into at most $R(V)$ intervals on which $V_l(t_1,\dots,t_l;B)$, as a function of t_l, is continuously differentiable and monotonic. For $\theta \in \Theta'_{r+1}(C)$, let
$$\theta_l(\mathfrak{a}_1,\dots,\mathfrak{a}_l) := \mathcal{A}(\theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+1}),\mathfrak{a}_{r+1},\dots,\mathfrak{a}_{l+1}) \in \Theta'_l(2^{r-l+1}C).$$

The following proposition is analogous to [Der09, Proposition 4.3 and Remark 4.4].

Proposition 7.3. *Let V be as above and $\theta \in \Theta'_{r+1}(C)$. Then*

$$\sum_{\mathfrak{a}_1,\dots,\mathfrak{a}_{r+1}} \theta(\mathfrak{a}_1,\dots,\mathfrak{a}_{r+1}) V(\mathfrak{N}\mathfrak{a}_1,\dots,\mathfrak{N}\mathfrak{a}_{r+1};B)$$
$$= (\rho_K h_K)^{r+1}\theta_0 V_0(B) + O_{V,C}(B(\log B)^{r-1}\log\log B).$$

Proof. By a similar argument as in Lemma 7.1, we see that, for $l \in \{1,\dots,r\}$,

$$V_l(t_1,\dots,t_l;B) \ll \frac{B(\log B)^{r-l}}{t_1\cdots t_l}.$$

Since $\theta_l(\mathfrak{a}_1,\dots,\mathfrak{a}_l) \in \Theta'_l(2^{r-l+1}C)$, we can apply Proposition 7.2 inductively to V_{r+1}, V_r, $V_{r-1}/\log B,\dots,V_1/(\log B)^{r-1}$. □

Note that θ_0 can be computed by Lemma 2.8.

8. The factor α

Let K be an imaginary quadratic field. Let S be a split singular del Pezzo surface of degree $d = 9-r$ over K, with minimal desingularization $\widetilde{S}$. The final result of our summation process is typically provided by Proposition 7.3. To derive Manin's conjecture as in Theorem 1.1 from this, it remains to compare the integral $V_0(B)$ with $\alpha(\widetilde{S})\pi^{r+1}\omega_\infty B(\log B)^r$. Here, $\alpha(\widetilde{S})$ is a constant defined in [Pey95, Définition 2.4] and [BT95, Definition 2.4.6] that is expected to be a factor of the leading constant $c_{S,H}$ in Manin's conjecture (1.2).

For a split singular del Pezzo surface S of degree $d \leqslant 7$, its value can be computed by [DJT08, Theorem 1.3] as

$$\alpha(\widetilde{S}) = \frac{\alpha(S_0)}{|W|} \tag{8.1}$$

where S_0 is a split ordinary del Pezzo surface of the same degree and $|W|$ is the order of the Weyl group W associated to the singularities of S. For example, $|W| = (n+1)!$ if S has precisely one singularity whose type is $\mathbf{A}_n$. The value of $\alpha(S_0)$ can be computed by [Der07a, Theorem 4], with $\alpha(S_0) = 1/180$ in degree 4.

To rewrite $\alpha(\widetilde{S})$ as an integral, it is most convenient to work with [DEJ14, Definition 1.1], giving

$$\alpha(\widetilde{S}) := (r+1)\cdot \mathrm{vol}\{x \in \Lambda^\vee_{\mathrm{eff}}(\widetilde{S}) \mid (x, -K_{\widetilde{S}}) \leqslant 1\},$$

since $\mathrm{Pic}(\widetilde{S})$ has rank $r+1$, where $\Lambda^\vee_{\mathrm{eff}}(\widetilde{S}) \subset (\mathrm{Pic}(\widetilde{S}) \otimes_{\mathbb{Z}} \mathbb{R})^\vee$ is the dual of the effective cone of $\widetilde{S}$ (which is generated by the classes of the negative curves since $d \leqslant 7$), $(\cdot,\cdot)$ is the natural pairing between $\mathrm{Pic}(\widetilde{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ and its dual space, and the volume is normalized such that $\mathrm{Pic}(\widetilde{S})^\vee$ has covolume 1.

Suppose that the negative curves on $\widetilde{S}$ are $E_1,\dots,E_{r+1+s}$, for some $s \geqslant 0$, where $E_1,\dots,E_{r+1}$ are a basis of $\mathrm{Pic}(\widetilde{S})$; for example, this holds in the ordering chosen in § 4. Expressing $-K_{\widetilde{S}}$ and $E_{r+2},\dots,E_{r+1+s}$ in terms of this basis, we have

$$[-K_{\widetilde{S}}] = \sum_{j=1}^{r+1} c_j [E_j]$$

and, for $i = 1, \dots, s$,

$$[E_{r+1+i}] = \sum_{j=1}^{r+1} b_{i,j}[E_j]$$

for some $b_{i,j}, c_j \in \mathbb{Z}$.

Lemma 8.1. *With the above notation, assume that $c_{r+1} > 0$. Define, for $j = 1, \dots, r$ and $i = 1, \dots, s$,*

$$a_{0,j} := c_j, \quad a_{i,j} := b_{i,r+1}c_j - b_{i,j}c_{r+1}, \quad A_0 := 1, \quad A_i := b_{i,r+1}.$$

Then

$$\alpha(\widetilde{S})(\log B)^r = \frac{1}{c_{r+1}\pi^r}\int_{R_1(B)} \frac{1}{\|\eta_1\cdots\eta_r\|_\infty}\, d\eta_1\cdots d\eta_r$$

with a domain of integration

$$R_1(B) := \left\{ (\eta_1, \dots, \eta_r) \in \mathbb{C}^r \;\middle|\; \begin{array}{l} \|\eta_1\|_\infty, \dots, \|\eta_r\|_\infty \geqslant 1, \\ \prod_{j=1}^{r} \|\eta_j\|_\infty^{a_{i,j}} \leqslant B^{A_i} \text{ for all } i \in \{0, \dots, s\} \end{array} \right\}.$$

Proof. Since $[E_1], \dots, [E_{r+1+s}]$ generate the effective cone of $\widetilde{S}$, the value of $\alpha(\widetilde{S})$ is

$$(r+1)\cdot \mathrm{vol}\left\{ (t_1', \dots, t_{r+1}') \in \mathbb{R}_{\geqslant 0}^{r+1} \;\middle|\; \sum_{j=1}^{r+1} b_{i,j}t_j' \geqslant 0 \ (i = 1, \dots, s), \sum_{j=1}^{r+1} c_j t_j' \leqslant 1 \right\}.$$

We make a linear change of variables $(t_1, \dots, t_r, t_{r+1}) = (t_1', \dots, t_r', c_1t_1' + \cdots + c_{r+1}t_{r+1}')$, with Jacobian c_{r+1}. This transforms the polytope in the previous formula into a pyramid whose base is $R_0 \times \{1\}$ in the hyperplace $\{t_{r+1} = 1\}$ in $\mathbb{R}^{r+1}$, and whose apex is the origin, where

$$R_0 := \left\{ (t_1, \dots, t_r) \in \mathbb{R}_{\geqslant 0}^r \;\middle|\; \sum_{j=1}^{r} a_{i,j}t_j \leqslant A_i \text{ for all } i \in \{0, \dots, s\} \right\}.$$

This pyramid has volume $(r+1)^{-1}\,\mathrm{vol}\, R_0$ since its height is 1 and its dimension is $r+1$. Writing $\mathrm{vol}\, R_0$ as an integral, we get

$$\alpha(\widetilde{S}) = \frac{1}{c_{r+1}} \int_{(t_1,\dots,t_r)\in R_0} dt_r \cdots dt_1,$$

where the factor c_{r+1}^{-1} appears because of our change of coordinates. Now the change of coordinates $\eta_i = B^{t_i}$ for $i \in \{1, \dots, r\}$ gives a real integral with the factor $(\log B)^r$. The final complex integral with the factor π^r is obtained via polar coordinates. $\square$

9. The quartic del Pezzo surface of type $\mathbf{A}_3$ with five lines

Let $S \subset \mathbb{P}_K^4$ be the anticanonically embedded del Pezzo surface defined by (1.3). In this section, we apply our general techniques to prove Manin's conjecture for S (Theorem 1.1).

Our surface S contains precisely one singularity $(0:0:0:0:1)$ (of type $\mathbf{A}_3$) and the five lines $\{x_0 = x_1 = x_2 = 0\}$, $\{x_0 = x_2 = x_3 = 0\}$, $\{x_0 = x_3 = x_4 = 0\}$, $\{x_1 = x_2 = x_3 = 0\}$, $\{x_1 = x_3 = x_4 = 0\}$. Let U be the complement of these lines in S.

By [DL10, DL13], S is not an equivariant compactification of an algebraic group, so that Manin's conjecture does not follow from [BT98a, CLT02, TT12].

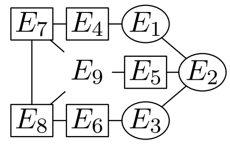

Figure 3. Configuration of curves on $\widetilde{S}$.

9.1 Passage to a universal torsor

To parameterize the rational points on $U \subset S$ by integral points on an affine hypersurface, we apply the strategy described in § 4, based on the description of the Cox ring of its minimal desingularization $\widetilde{S}$ in [Der14]. In particular, we will refer to the extended Dynkin diagram in Figure 3 encoding the configuration of curves $E_1, \dots, E_9$ corresponding to generators of $\mathrm{Cox}(\widetilde{S})$. Here, a vertex marked by a circle (respectively a box) corresponds to a (-2)-curve (respectively (-1)-curve), and there are $([E_j],[E_k])$ edges between the vertices corresponding to E_j and E_k.

For any given $\mathbf{C} = (C_0, \dots, C_5) \in \mathcal{C}^6$, we define $u_{\mathbf{C}} := \mathfrak{N}(C_0^3 C_1^{-1} \cdots C_5^{-1})$ and

$$\begin{aligned} &\mathcal{O}_1 := C_1 C_4^{-1}, && \mathcal{O}_2 := C_0 C_1^{-1} C_2^{-1} C_3^{-1}, && \mathcal{O}_3 := C_2 C_5^{-1},\\ &\mathcal{O}_4 := C_4, && \mathcal{O}_5 := C_3, && \mathcal{O}_6 := C_5,\\ &\mathcal{O}_7 := C_0 C_1^{-1} C_4^{-1}, && \mathcal{O}_8 := C_0 C_2^{-1} C_5^{-1}, && \mathcal{O}_9 := C_0 C_3^{-1}. \end{aligned} \tag{9.1}$$

Let

$$\mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0}, & j \in \{1, \dots, 8\},\\ \mathcal{O}_j, & j = 9. \end{cases}$$

For $\eta_j \in \mathcal{O}_j$, we define

$$I_j := \eta_j \mathcal{O}_j^{-1}.$$

For $B \geqslant 0$, let $\mathcal{R}(B)$ be the set of all $(\eta_1, \dots, \eta_8) \in \mathbb{C}^8$ with $\eta_5 \neq 0$ and

$$\|\eta_1^2 \eta_2^2 \eta_3 \eta_4^2 \eta_5 \eta_7\|_\infty \leqslant B, \tag{9.2}$$

$$\|\eta_1 \eta_2^2 \eta_3^2 \eta_5 \eta_6^2 \eta_8\|_\infty \leqslant B, \tag{9.3}$$

$$\|\eta_1^2 \eta_2^3 \eta_3^2 \eta_4 \eta_5^2 \eta_6\|_\infty \leqslant B, \tag{9.4}$$

$$\|\eta_1 \eta_2 \eta_3 \eta_4 \eta_6 \eta_7 \eta_8\|_\infty \leqslant B, \tag{9.5}$$

$$\left\| \frac{\eta_1 \eta_4^2 \eta_7^2 \eta_8 + \eta_3 \eta_6^2 \eta_7 \eta_8^2}{\eta_5} \right\|_\infty \leqslant B. \tag{9.6}$$

Moreover, let $M_{\mathbf{C}}(B)$ be the set of all

$$(\eta_1, \dots, \eta_9) \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$$

that satisfy the *height conditions*

$$(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}} B), \tag{9.7}$$

the *torsor equation*

$$\eta_1 \eta_4^2 \eta_7 + \eta_3 \eta_6^2 \eta_8 + \eta_5 \eta_9 = 0, \tag{9.8}$$

and the *coprimality conditions*

$$I_j + I_k = \mathcal{O}_K \quad \text{for all distinct non-adjacent vertices } E_j,\ E_k \text{ in Figure 3.} \tag{9.9}$$

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Lemma 9.1. *Let K be a imaginary quadratic field. Then*

$$N_{U,H}(B) = \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}(B)|.$$

Proof. We apply the strategy from § 4. We work with the data in [Der14]. For our surface S, Claim 4.1 specializes precisely to the statement of our lemma (where (9.6) is $\|\eta_7\eta_8\eta_9\|_\infty \leqslant B$ with η_9 eliminated using (9.8)).

We prove it via the induction process described in Claim 4.2. It is based on the construction of the minimal desingularization $\pi : \widetilde{S} \to S$ by the following sequence of blow-ups

$$\rho = \rho_1 \circ \cdots \circ \rho_5 : \widetilde{S} \to \mathbb{P}_K^2.$$

Starting with the curves

$$E_7^{(0)} := \{y_0 = 0\}, \quad E_8^{(0)} := \{y_1 = 0\}, \quad E_2^{(0)} := \{y_2 = 0\}, \quad E_9^{(0)} := \{-y_0 - y_1 = 0\}$$

in $\mathbb{P}_K^2$:

(1) blow up $E_2^{(0)} \cap E_7^{(0)}$, giving $E_1^{(1)}$;
(2) blow up $E_2^{(1)} \cap E_8^{(1)}$, giving $E_3^{(2)}$;
(3) blow up $E_2^{(2)} \cap E_9^{(2)}$, giving $E_5^{(3)}$;
(4) blow up $E_1^{(3)} \cap E_7^{(3)}$, giving $E_4^{(4)}$;
(5) blow up $E_3^{(4)} \cap E_8^{(4)}$, giving $E_6^{(5)}$.

The inverse $\pi \circ \rho^{-1} : \mathbb{P}_K^2 \dashrightarrow S$ of the projection

$$\phi = \rho \circ \pi^{-1} : S \dashrightarrow \mathbb{P}_K^2, \quad (x_0 : \cdots : x_4) \mapsto (x_0 : x_1 : x_2)$$

is given by

$$(y_0 : y_1 : y_2) \mapsto (y_0y_2^2 : y_1y_2^2 : y_2^3 : y_0y_1y_2 : -y_0y_1(y_0 + y_1)). \tag{9.10}$$

In our case, the map Ψ appearing in Claim 4.2 sends $(\eta_1, \ldots, \eta_9)$ to

$$(\eta_1^2\eta_2^2\eta_3\eta_4^2\eta_5\eta_7, \eta_1\eta_2^2\eta_3^2\eta_5\eta_6^2\eta_8, \eta_1^2\eta_2^3\eta_3^2\eta_4\eta_5^2\eta_6, \eta_1\eta_2\eta_3\eta_4\eta_6\eta_7\eta_8, \eta_7\eta_8\eta_9).$$

Claim 4.2 holds for $i = 0$ by Lemma 4.3 since the map $\psi : \mathbb{P}_K^2 \dashrightarrow S$ obtained from Ψ by the substitution $(\eta_1, \ldots, \eta_8) \mapsto (1, y_2, 1, 1, 1, 1, y_0, y_1, -y_0 - y_1)$ as in (4.11), (4.12) agrees with $\pi \circ \rho^{-1}$ on $\mathbb{P}_K^2 \smallsetminus \{(1 : 0 : 0), (0 : 1 : 0), (1 : -1 : 0)\}$.

Since the five blow-ups described above satisfy the assumptions of Lemma 4.4, Claim 4.2 follows by induction for $i = 1, \ldots, 5$.

Hence Ψ induces a ω_K^6-to-1 map from the set of all $(\eta_1, \ldots, \eta_9) \in \bigcup_{\mathbf{C} \in \mathcal{C}^6} \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}$ satisfying (9.8), (9.9), $H(\Psi(\eta_1, \ldots, \eta_9)) \leqslant B$ to the set of K-rational points on U of height bounded by B. One easily sees that (9.9) implies that

$$\eta_1^2\eta_2^2\eta_3\eta_4^2\eta_5\eta_7\mathcal{O}_K + \cdots + \eta_7\eta_8\eta_9\mathcal{O}_K = C_0^3C_1^{-1} \cdots C_5^{-1}.$$

As discussed after Claim 4.2, this completes the proof of Claim 4.1. □

9.2 Summations

In a direct application of Proposition 5.3 to $M_{\mathbf{C}}(B)$, our height conditions would not yield sufficiently good estimates for the sum over the error terms, so we consider two cases: let $M_{\mathbf{C}}^{(8)}(B)$ be the set of all $(\eta_1, \ldots, \eta_9) \in M_{\mathbf{C}}(B)$ with $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$, and let $M_{\mathbf{C}}^{(7)}(B)$ be the set of all

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$(\eta_1, \dots, \eta_9) \in M_{\mathbf{C}}(B)$ with $\mathfrak{N}I_7 > \mathfrak{N}I_8$. Moreover, let

$$N_8(B) := \frac{1}{\omega_K^6} \sum_{\mathbf{C} \in \mathcal{C}^6} |M_{\mathbf{C}}^{(8)}(B)|,$$

and define $N_7(B)$ analogously. Then clearly $N_{U,H}(B) = N_8(B) + N_7(B)$.

9.2.1 *The first summation over η_8 in $M_{\mathbf{C}}^{(8)}(B)$ with dependent η_9.*

Lemma 9.2. *Write $\boldsymbol{\eta}' := (\eta_1, \dots, \eta_7)$ and $\mathbf{I}' := (I_1, \dots, I_7)$. For $B > 0$, $\mathbf{C} \in \mathcal{C}^6$, we have*

$$|M_{\mathbf{C}}^{(8)}(B)| = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{7*}} \theta_8(\mathbf{I}') V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_7; B) + O_{\mathbf{C}}(B(\log B)^3),$$

where

$$V_8(t_1, \dots, t_7; B) := \frac{1}{t_5} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ \|\eta_8\|_\infty \geqslant t_7}} d\eta_8$$

with a complex variable η_8, and where

$$\theta_8(\mathbf{I}') := \prod_{\mathfrak{p}} \theta_{8,\mathfrak{p}}(J_{\mathfrak{p}}(\mathbf{I}'))$$

with $J_{\mathfrak{p}}(\mathbf{I}') := \{j \in \{1, \dots, 7\} : \mathfrak{p} \mid I_j\}$ and

$$\theta_{8,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \emptyset, \{5\}, \{6\}, \{7\}, \\ 1 - \frac{1}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{1\}, \{3\}, \{4\}, \{1,2\}, \{1,4\}, \{2,3\}, \{2,5\}, \{3,6\}, \{4,7\}, \\ 1 - \frac{2}{\mathfrak{N}\mathfrak{p}} & \text{if } J = \{2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the same asymptotic formula holds if we replace the condition $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$ in the definition of $M_{\mathbf{C}}^{(8)}(B)$ by $\mathfrak{N}I_8 > \mathfrak{N}I_7$.

Proof. We express the condition $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$ as

$$\|\sqrt{\mathfrak{N}I_7}\|_\infty \leqslant \|\sqrt{\mathfrak{N}\mathcal{O}_8^{-1}}\eta_8\|_\infty.$$

Let $\boldsymbol{\eta}' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{7*}$. By Lemma 3.2, the subset $\mathcal{R}(\boldsymbol{\eta}'; u_{\mathbf{C}}B) \subset \mathbb{C}$ of all η_8 with $(\eta_1, \dots, \eta_8) \in \mathcal{R}(u_{\mathbf{C}}B)$ and $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$ is of class m, where m is an absolute constant. Moreover, by Lemma 3.4(1), applied to (9.6) with $u_{\mathbf{C}}B$ instead of B, we see that $\mathcal{R}(\boldsymbol{\eta}'; u_{\mathbf{C}}B)$ is contained in the union of two balls of radius

$$R(\boldsymbol{\eta}'; u_{\mathbf{C}}B) := (u_{\mathbf{C}}B\|\eta_3^{-1}\eta_5\eta_6^{-2}\eta_7^{-1}\|_\infty)^{1/4} \ll_{\mathbf{C}} (B\mathfrak{N}I_3^{-1}\mathfrak{N}I_5\mathfrak{N}I_6^{-2}\mathfrak{N}I_7^{-1})^{1/4}.$$

We may sum over all $\eta_8 \in \mathcal{O}_8$ instead of $\eta_8 \in \mathcal{O}_{8*}$, since $0 \notin \mathcal{R}(\boldsymbol{\eta}'; u_{\mathbf{C}}B)$. We apply Proposition 5.3 with $(A_1, A_2, A_0) := (1, 4, 7)$, $(B_1, B_2, B_0) := (3, 6, 8)$, $(C_1, C_0) := (5, 9)$, $D := 2$, and $u_{\mathbf{C}}B$ instead of B. We choose Π_1 and Π_2 as suggested by Remark 5.2. Then

$$V_1(\boldsymbol{\eta}'; u_{\mathbf{C}}B) = \frac{1}{\mathfrak{N}(I_5\mathcal{O}_8)} \int_{\eta_8 \in \mathcal{R}(\boldsymbol{\eta}'; u_{\mathbf{C}}B)} d\eta_8.$$

A straightforward computation shows that $\eta_8 \in \mathcal{R}(\boldsymbol{\eta}'; u_{\mathbf{C}}B)$ if and only if

$$(\sqrt{\mathfrak{N}I_1}, \ldots, \sqrt{\mathfrak{N}I_7}, \varphi(\eta_8)) \in \mathcal{R}(B) \quad \text{and} \quad \|\varphi(\eta_8)\|_\infty \geqslant \mathfrak{N}I_7,$$

where $\varphi : \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto e^{i\arg(\eta_3\eta_6^2/(\eta_1\eta_4^2\eta_7))}/\sqrt{\mathfrak{N}\mathcal{O}_8} \cdot z$. Therefore, $V_1(\boldsymbol{\eta}'; u_{\mathbf{C}}B) = V_8(\mathfrak{N}I_1, \ldots, \mathfrak{N}I_7; B)$. Moreover, since $b_0 = 1$, Lemma 5.4 shows that $\theta_1(\boldsymbol{\eta}') = \theta_1'(\mathbf{I}') = \theta_8(\mathbf{I}')$, so the main term is as desired.

The error term from Proposition 5.3 is

$$\ll \sum_{\boldsymbol{\eta}', (5.7)} 2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}\left(\frac{R(\boldsymbol{\eta}'; u_{\mathbf{C}}B)}{\mathfrak{N}(I_5)^{1/2}} + 1\right).$$

Using (9.2), (9.3), the definitions of $u_{\mathbf{C}}$ and $\mathcal{O}_j$, and our assumption $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$, we see that (5.7) (with $u_{\mathbf{C}}B$ instead of B) implies

$$\mathfrak{N}I_1^2\mathfrak{N}I_2^2\mathfrak{N}I_3\mathfrak{N}I_4^2\mathfrak{N}I_5\mathfrak{N}I_7 \leqslant B \quad \text{and} \tag{9.11}$$

$$\mathfrak{N}I_1\mathfrak{N}I_2^2\mathfrak{N}I_3^2\mathfrak{N}I_5\mathfrak{N}I_6^2\mathfrak{N}I_7 \leqslant B. \tag{9.12}$$

Let $\mathfrak{a} \in \mathcal{I}_K$. Since there are at most $|\mathcal{O}_K^\times| < \infty$ elements $\eta_j \in \mathcal{O}_j$ with $I_j = \mathfrak{a}$, we can sum over the ideals $I_j \in \mathcal{I}_K$ instead of the $\eta_j \in \mathcal{O}_j$. Moreover, we can replace (5.7) by (9.11) and (9.12), and estimate the error term by

$$\begin{aligned}
&\ll_{\mathbf{C}} \sum_{\substack{I_1\ldots,I_7\\(9.11),(9.12)}} 2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}\left(\frac{B^{1/4}}{\mathfrak{N}I_3^{1/4}\mathfrak{N}I_5^{1/4}\mathfrak{N}I_6^{1/2}\mathfrak{N}I_7^{1/4}} + 1\right)\\
&\ll \sum_{\substack{I_1,\ldots,I_5,I_7\\(9.11)}} \left(\frac{2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}B^{1/2}}{\mathfrak{N}I_1^{1/4}\mathfrak{N}I_2^{1/2}\mathfrak{N}I_3^{3/4}\mathfrak{N}I_5^{1/2}\mathfrak{N}I_7^{1/2}} + \frac{2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}B^{1/2}}{\mathfrak{N}I_1^{1/2}\mathfrak{N}I_2\mathfrak{N}I_3\mathfrak{N}I_5^{1/2}\mathfrak{N}I_7^{1/2}}\right)\\
&\ll \sum_{\substack{I_1,\ldots,I_5\\\mathfrak{N}I_j\leqslant B}} \left(\frac{2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}B}{\mathfrak{N}I_1^{5/4}\mathfrak{N}I_2^{3/2}\mathfrak{N}I_3^{5/4}\mathfrak{N}I_4\mathfrak{N}I_5} + \frac{2^{\omega(I_2)+\omega(I_1I_2I_3I_4)}B}{\mathfrak{N}I_1^{3/2}\mathfrak{N}I_2^2\mathfrak{N}I_3^{3/2}\mathfrak{N}I_4\mathfrak{N}I_5}\right)\\
&\ll B(\log B)^3.
\end{aligned}$$

For the last estimation, we used Lemmas 2.9 and 2.4.

Let $M_{\mathbf{C}}^{(8)'}(B)$ be defined as $M_{\mathbf{C}}^{(8)}(B)$, except that the condition $\mathfrak{N}I_8 \geqslant \mathfrak{N}I_7$ is replaced by $\mathfrak{N}I_8 = \mathfrak{N}I_7$. We apply Proposition 5.3 in an analogous way as above. Since then $V_1(\boldsymbol{\eta}'; u_{\mathbf{C}}B) = 0$, we obtain $|M_{\mathbf{C}}^{(8)'}(B)| \ll B(\log B)^3$. This shows the last assertion of the lemma. □

For the second summation, we need another dichotomy: let $M_{\mathbf{C}}^{(87)}(B)$ be the main term in the expression for $|M_{\mathbf{C}}^{(8)}(B)|$ given in Lemma 9.2 with the additional condition $\mathfrak{N}I_7 > \mathfrak{N}I_4$ in the sum, and let $M_{\mathbf{C}}^{(84)}(B)$ be the same main term with the additional condition $\mathfrak{N}I_4 \geqslant \mathfrak{N}I_7$ in the sum, so that $|M_{\mathbf{C}}^{(8)}(B)| = M_{\mathbf{C}}^{(87)}(B) + M_{\mathbf{C}}^{(84)}(B) + O(B(\log B)^3)$. Moreover, let

$$N_{87}(B) := \frac{1}{\omega_K^6}\sum_{\mathbf{C}\in\mathcal{C}^6} M_{\mathbf{C}}^{(87)}(B)$$

and define $N_{84}(B)$ analogously. Then $N_8(B) = N_{87}(B) + N_{84}(B) + O(B(\log B)^3)$.

9.2.2 *The second summation over η_7 in $M_{\mathbf{C}}^{(87)}(B)$.*

LEMMA 9.3. *Write $\boldsymbol{\eta}'' := (\eta_1, \dots, \eta_6)$. For $B \geqslant 3$, $\mathbf{C} \in \mathcal{C}^6$, we have*

$$M_{\mathbf{C}}^{(87)}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \mathcal{A}(\theta_8(\mathbf{I}'), I_7) V_{87}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B) + O_{\mathbf{C}}(B(\log B)^3).$$

For $t_1, \dots, t_6 \geqslant 1$,

$$V_{87}(t_1, \dots, t_6; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1}, \dots, \sqrt{t_7}, \eta_8) \in \mathcal{R}(B) \\ t_4 < t_7 \leqslant \|\eta_8\|_\infty}} dt_7 \, d\eta_8,$$

with a real variable t_7 and a complex variable η_8.

Proof. We use the strategy described in § 6 in the case $b_0 = 1$. For $\mathfrak{a} \in \mathcal{I}_K$, $t \geqslant 1$, let $\vartheta(\mathfrak{a}) := \theta_8(I_1, \dots, I_6, \mathfrak{a})$ and $g(t) := V_8(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6, t; B)$. Then

$$M_{\mathbf{C}}^{(87)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}'' \in \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{6*}} \sum_{\substack{\eta_7 \in \mathcal{O}_{7*} \\ \mathfrak{N}I_7 > \mathfrak{N}I_4}} \vartheta(I_7) g(\mathfrak{N}I_7). \tag{9.13}$$

By Lemmas 5.4 and 2.2, ϑ satisfies (6.1) with $C = 0$ and $c_\vartheta = 2^{\omega_K(I_1 I_2 I_3 I_5 I_6)}$.

The first height condition (9.2) implies that $g(t) = 0$ whenever $t > t_2 := B/(\mathfrak{N}I_1^2 \mathfrak{N}I_2^2 \mathfrak{N}I_3 \mathfrak{N}I_4^2 \mathfrak{N}I_5)$. Moreover, applying Lemma 3.4(2) to the fifth height condition (9.6), we see that

$$g(t) \ll \frac{1}{\mathfrak{N}I_5} \cdot \frac{(B\mathfrak{N}I_5)^{1/2}}{(\mathfrak{N}I_3 \mathfrak{N}I_6^2 t)^{1/2}} = \frac{B^{1/2}}{\mathfrak{N}I_3^{1/2} \mathfrak{N}I_5^{1/2} \mathfrak{N}I_6} t^{-1/2}.$$

We may assume that $\mathfrak{N}I_4 \leqslant t_2$. By Lemma 3.6, g is piecewise continuously differentiable and monotonic on $[\mathfrak{N}I_4, t_2]$, and the number of pieces can be bounded by an absolute constant. Using the notation from § 6 (with $a = -1/2$), we see that the sum over η_7 in (9.13) is just $S(\mathfrak{N}I_4, t_2)$, and Proposition 6.1, applied as suggested by Remark 6.2, yields

$$S(\mathfrak{N}I_4, t_2) = \frac{2\pi}{\sqrt{|\Delta_K|}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K) \int_{t \geqslant \mathfrak{N}I_4} g(t) \, dt + O\left(\frac{2^{\omega(I_1 I_2 I_3 I_5 I_6)} B^{1/2} (\log B)}{\mathfrak{N}I_3^{1/2} \mathfrak{N}I_5^{1/2} \mathfrak{N}I_6}\right). \tag{9.14}$$

Clearly, $\pi \int_{t \geqslant \mathfrak{N}I_4} g(t) \, dt = V_{87}(\mathfrak{N}I_1, \dots, \mathfrak{N}I_6; B)$, so we obtain the correct main term.

Let us consider the error term. Taking the product of (9.2) and (9.4) together with $\mathfrak{N}I_7 > \mathfrak{N}I_4$ (respectively $t > \mathfrak{N}I_4$), we see that both the sum and the integral in (9.14) are zero unless

$$\mathfrak{N}I_1^4 \mathfrak{N}I_2^5 \mathfrak{N}I_3^3 \mathfrak{N}I_4^4 \mathfrak{N}I_5^3 \mathfrak{N}I_6 \leqslant B^2. \tag{9.15}$$

Since $|\mathcal{O}_K^\times| < \infty$, we may sum over the $\mathbf{I}'' := (I_1, \dots, I_6)$ satisfying (9.15) instead of the $\boldsymbol{\eta}''$, so the total error term is

$$\ll \sum_{\substack{I_1,\dots,I_6 \in \mathcal{I}_K \\ (9.15)}} \frac{2^{\omega(I_1 I_2 I_3 I_5 I_6)} B^{1/2} (\log B)}{\mathfrak{N} I_3^{1/2} \mathfrak{N} I_5^{1/2} \mathfrak{N} I_6}$$

$$\ll \sum_{\substack{I_2,\dots,I_6 \in \mathcal{I}_K \\ \mathfrak{N} I_j \leqslant B}} \frac{2^{\omega(I_2 I_3 I_5 I_6)} B (\log B)^2}{\mathfrak{N} I_2^{5/4} \mathfrak{N} I_3^{5/4} \mathfrak{N} I_4 \mathfrak{N} I_5^{5/4} \mathfrak{N} I_6^{5/4}} \ll B (\log B)^3.$$

In the summations, we used (9.15), Lemmas 2.9 and 2.4. □

LEMMA 9.4. *If $\mathbf{I}''$ runs over all six-tuples $(I_1, \dots, I_6)$ of non-zero ideals of $\mathcal{O}_K$, then we have*

$$N_{87}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8(\mathbf{I}'', I_7), I_7) V_{87}(\mathfrak{N} I_1, \dots, \mathfrak{N} I_6; B) + O(B (\log B)^3).$$

Proof. It follows directly from (9.1) that $([\mathcal{O}_1^{-1}], \dots, [\mathcal{O}_6^{-1}])$ runs through all six-tuples of ideal classes whenever $\mathbf{C}$ runs through $\mathcal{C}^6$. If $\mathcal{O}_j^{-1}$ runs through a set of representatives for the ideal classes and η_j runs through all non-zero elements in $\mathcal{O}_j$, then $I_j = \eta_j \mathcal{O}_j^{-j}$ runs through all non-zero integral ideals of $\mathcal{O}_K$, each one occurring $|\mathcal{O}_K^\times| = \omega_K$ times. This proves the lemma. □

9.2.3 *The remaining summations in $N_{87}(B)$.*

LEMMA 9.5. *We have*

$$N_{87}(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}} \right)^8 \left(\frac{h_K}{\omega_K} \right)^6 \theta_0 V_{870}(B) + O(B (\log B)^4 \log \log B),$$

where

$$V_{870}(B) := \int_{t_1,\dots,t_6 \geqslant 1} V_{87}(t_1, \dots, t_6; B) \, dt_1 \cdots dt_6$$

and

$$\theta_0 := \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}} \right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2} \right). \tag{9.16}$$

Proof. We start from Lemma 9.4. Applying Lemma 3.4(6) to (9.6), we see that

$$V_{87}(t_1, \dots, t_6; B) \ll \frac{1}{t_5} \cdot \frac{B^{2/3} t_5^{2/3}}{t_1^{1/3} t_3^{1/3} t_4^{2/3} t_6^{2/3}} = \frac{B}{t_1 \cdots t_6} \left(\frac{B}{t_1^2 t_2^3 t_3^2 t_4 t_5^2 t_6} \right)^{-1/3}.$$

We apply Proposition 7.3 with $r = 5$ (the assumptions on $V = V_{87}$ are satisfied by Lemma 3.6). By Lemma 5.4, $\theta_0 = \mathcal{A}(\theta_8(\mathbf{I}'), \mathbf{I}')$ has the desired form. □

9.2.4 *The second summation over η_4 in $M_{\mathbf{C}}^{(84)}(B)$.*

LEMMA 9.6. *Let $\boldsymbol{\eta}'' := (\eta_1, \eta_2, \eta_3, \eta_5, \eta_6, \eta_7)$ and $\boldsymbol{\mathcal{O}}'' := \mathcal{O}_{1*} \times \mathcal{O}_{2*} \times \mathcal{O}_{3*} \times \mathcal{O}_{5*} \times \mathcal{O}_{6*} \times \mathcal{O}_{7*}$. We have*

$$M_{\mathbf{C}}^{(84)}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}} \right)^2 \sum_{\boldsymbol{\eta}'' \in \boldsymbol{\mathcal{O}}''} \mathcal{A}(\theta_8(\mathbf{I}'), I_4) V_{84}(\mathfrak{N} I_1, \mathfrak{N} I_2, \mathfrak{N} I_3, \mathfrak{N} I_5, \mathfrak{N} I_6, \mathfrak{N} I_7; B)$$
$$+ O_{\mathbf{C}}(B (\log B)^3),$$

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For $t_1,\, t_2,\, t_3,\, t_5,\, t_6,\, t_7 \geqslant 1$,

$$V_{84}(t_1, t_2, t_3, t_5, t_6, t_7; B) := \frac{\pi}{t_5} \int_{\substack{(\sqrt{t_1},\dots,\sqrt{t_7},\eta_8)\in\mathcal{R}(B)\\ t_4\geqslant t_7, \|\eta_8\|_\infty \geqslant t_7}} dt_4\, d\eta_8,$$

with a real variable t_4 and a complex variable η_8.

Proof. This is similar to Lemma 9.3. Let

$$\vartheta(\mathfrak{a}) := \theta_8(I_1, I_2, I_3, \mathfrak{a}, I_5, I_6, I_7) \quad \text{and} \quad g(t) := V_8(\mathfrak{N}I_1, \mathfrak{N}I_2, \mathfrak{N}I_3, t, \mathfrak{N}I_5, \mathfrak{N}I_6, \mathfrak{N}I_7; B).$$

Then

$$M_{\mathbf{C}}^{(84)}(B) = \frac{2}{\sqrt{|\Delta_K|}} \sum_{\boldsymbol{\eta}''\in\boldsymbol{\mathcal{O}}''} \sum_{\substack{\eta_4\in\mathcal{O}_{4*}\\ \mathfrak{N}I_4\geqslant\mathfrak{N}I_7}} \vartheta(I_4) g(\mathfrak{N}I_4) + O(B(\log B)^3). \tag{9.17}$$

By Lemmas 5.4 and 2.2, ϑ satisfies (6.1) with $C = 0$ and $c_\theta \ll 2^{\omega(I_2I_3I_5I_6)}$. By (9.2), $g(t) = 0$ whenever $t > t_2 := B^{1/2}/(\mathfrak{N}I_1\mathfrak{N}I_2\mathfrak{N}I_3^{1/2}\mathfrak{N}I_5^{1/2}\mathfrak{N}I_7^{1/2})$. Moreover, applying Lemma 3.4(2) to (9.6), we see that

$$g(t) \ll \frac{1}{\mathfrak{N}I_5} \cdot \frac{(B\mathfrak{N}I_5)^{1/2}}{(\mathfrak{N}I_3\mathfrak{N}I_6^2\mathfrak{N}I_7)^{1/2}} = \frac{B^{1/2}}{\mathfrak{N}I_3^{1/2}\mathfrak{N}I_5^{1/2}\mathfrak{N}I_6\mathfrak{N}I_7^{1/2}} =: c_g.$$

Clearly, we may assume that $\mathfrak{N}I_7 \leqslant t_2$. Using the notation from § 6 (with $a = 0$), the sum over η_4 in (9.17) is just $S(\mathfrak{N}I_7, t_2)$, and Proposition 6.1 yields

$$S(\mathfrak{N}I_7, t_2) = \frac{2\pi}{\sqrt{|\Delta_K|}} \mathcal{A}(\vartheta(\mathfrak{a}), \mathfrak{a}, \mathcal{O}_K) \int_{t\geqslant\mathfrak{N}I_7} g(t)\, dt + O\left(\frac{2^{\omega(I_2I_3I_5I_6)} B^{1/2}}{\mathfrak{N}I_3^{1/2}\mathfrak{N}I_5^{1/2}\mathfrak{N}I_6\mathfrak{N}I_7^{1/2}} \cdot \frac{B^{1/4}}{\mathfrak{N}I_1^{1/2}\mathfrak{N}I_2^{1/2}\mathfrak{N}I_3^{1/4}\mathfrak{N}I_5^{1/4}\mathfrak{N}I_7^{1/4}}\right).$$

Now $\pi \int_{t\geqslant\mathfrak{N}I_7} g(t)\, dt = V_{84}(\mathfrak{N}I_1, \mathfrak{N}I_2, \mathfrak{N}I_3, \mathfrak{N}I_5, \mathfrak{N}I_6, \mathfrak{N}I_7; B)$, so we obtain the correct main term. Let us consider the error term. Height condition (9.4) and $\mathfrak{N}I_4 \geqslant \mathfrak{N}I_7$ imply that both the sum and the integral are zero unless

$$\mathfrak{N}I_1^2\mathfrak{N}I_2^3\mathfrak{N}I_3^2\mathfrak{N}I_5^2\mathfrak{N}I_6\mathfrak{N}I_7 \leqslant B. \tag{9.18}$$

Since $|\mathcal{O}_K^\times| < \infty$, we may sum over the $\mathbf{I}'' := (I_1, I_2, I_3, I_5, I_6, I_7)$ satisfying (9.18) instead of the $\boldsymbol{\eta}''$, so the error term is

$$\begin{aligned}
&\ll \sum_{\substack{I_1,I_2,I_3,I_5,I_6,I_7\in\mathcal{I}_K\\ (9.18)}} \frac{2^{\omega(I_2I_3I_5I_6)} B^{3/4}}{\mathfrak{N}I_1^{1/2}\mathfrak{N}I_2^{1/2}\mathfrak{N}I_3^{3/4}\mathfrak{N}I_5^{3/4}\mathfrak{N}I_6\mathfrak{N}I_7^{3/4}}\\
&\ll \sum_{\substack{I_1,I_2,I_3,I_5,I_6\in\mathcal{I}_K\\ \mathfrak{N}I_j\leqslant B}} \frac{2^{\omega(I_2I_3I_5I_6)} B}{\mathfrak{N}I_1\mathfrak{N}I_2^{5/4}\mathfrak{N}I_3^{5/4}\mathfrak{N}I_5^{5/4}\mathfrak{N}I_6^{5/4}}\\
&\ll B(\log B).
\end{aligned}$$

□

Lemma 9.7. *If $\mathbf{I}''$ runs over all six-tuples $(I_1, I_2, I_3, I_5, I_6, I_7)$ of non-zero ideals of $\mathcal{O}_K$, then we have*

$$N_{84}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^2 \sum_{\mathbf{I}''} \mathcal{A}(\theta_8(\mathbf{I}'), I_4) V_{84}(\mathfrak{N}I_1, \mathfrak{N}I_2, \mathfrak{N}I_3, \mathfrak{N}I_5, \mathfrak{N}I_6, \mathfrak{N}I_7; B) + O(B(\log B)^3).$$

Proof. This is entirely analogous to the proof of Lemma 9.4. □

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9.2.5 *The remaining summations in $N_{84}(B)$.*

Lemma 9.8. *We have*

$$N_{84}(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 V_{840}(B) + O(B(\log B)^4 \log\log B),$$

where

$$V_{840}(B) := \int_{t_1,t_2,t_3,t_5,t_6,t_7 \geqslant 1} V_{84}(t_1,t_2,t_3,t_5,t_6,t_7;B)\, dt_1\, dt_2\, dt_3\, dt_5\, dt_6\, dt_7$$

and θ_0 is given in (9.16).

Proof. We start from Lemma 9.7. Using Lemma 3.4(5), applied to (9.6), we have

$$\begin{aligned} V_{84}(t_1,t_2,t_3,t_5,t_6,t_7;B) &\ll \frac{1}{t_5} \cdot \frac{B^{3/4} t_5^{3/4}}{t_1^{1/2} t_3^{1/4} t_6^{1/2} t_7^{5/4}} \\ &= \frac{B}{t_1 t_2 t_3 t_5 t_6 t_7} \left(\frac{B}{t_1^2 t_2^4 t_3^3 t_5^3 t_6^2 t_7^{-1}}\right)^{-1/4}. \end{aligned}$$

Moreover, using (9.2) to bound t_4 and (9.3) to bound η_8, we have

$$\begin{aligned} V_{84}(t_1,t_2,t_3,t_5,t_6,t_7;B) &\ll \frac{1}{t_5} \cdot \frac{B^{1/2}}{t_1 t_2 t_3^{1/2} t_5^{1/2} t_7^{1/2}} \cdot \frac{B}{t_1 t_2^2 t_3^2 t_5 t_6^2} \\ &= \frac{B}{t_1 t_2 t_3 t_5 t_6 t_7} \left(\frac{B}{t_1^2 t_2^4 t_3^3 t_5^3 t_6^2 t_7^{-1}}\right)^{1/2}. \end{aligned}$$

We apply Proposition 7.3 with $r = 5$. Again, we evaluate $\theta_0 = \mathcal{A}(\theta_8(\mathbf{I}'), \mathbf{I}')$ using Lemma 5.4. $\square$

9.2.6 *Combining the summations.*

Lemma 9.9. *We have*

$$N_{U,H}(B) = \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 V_0(B) + O(B(\log B)^4 \log\log B),$$

where θ_0 is given in (9.16) and

$$V_0(B) := \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1 \\ (\eta_1,\dots,\eta_8)\in\mathcal{R}(B)}} \frac{1}{\|\eta_5\|_\infty}\, d\eta_1 \cdots d\eta_8,$$

with complex variables η_i.

Proof. Similarly as in the proof of Lemma 9.2, we note that $(\eta_1,\dots,\eta_8) \in \mathcal{R}(B)$ holds if and only if $(|\eta_1|,\dots,|\eta_7|, e^{i\arg((\eta_3\eta_6^2)/(\eta_1\eta_4^2\eta_7))}\eta_8) \in \mathcal{R}(B)$. Using polar coordinates, we obtain

$$\begin{aligned} V_{870}(B) + V_{840}(B) &= \pi \int_{\substack{t_1,\dots,t_7 \geqslant 1, \|\eta_8\|_\infty \geqslant t_7 \\ (\sqrt{t_1},\dots,\sqrt{t_7},\eta_8)\in\mathcal{R}(B)}} \frac{1}{t_5}\, dt_1 \cdots dt_7\, d\eta_8 \\ &= \pi^{-6} \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1, \|\eta_8\|_\infty \geqslant \|\eta_7\|_\infty \\ (\eta_1,\dots,\eta_8)\in\mathcal{R}(B)}} \frac{1}{\|\eta_5\|_\infty}\, d\eta_1 \cdots d\eta_8 =: \tilde{V}_8(B). \end{aligned}$$

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Therefore,

$$N_8(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 \tilde{V}_8(B) + O(B(\log B)^4 \log\log B).$$

For the computation of $N_7(B)$, we note that our height and coprimality conditions are symmetric with respect to swapping the indices $(1,4,7)$ with $(3,6,8)$. This allows us to perform the first summation over η_7 analogously to Lemma 9.2, the second summation over η_8 (respectively η_6) analogously to Lemma 9.3 (respectively Lemma 9.6), and the remaining summations analogously to Lemma 9.5 (respectively Lemma 9.8). We obtain

$$N_7(B) = \pi^6 \left(\frac{2}{\sqrt{|\Delta_K|}}\right)^8 \left(\frac{h_K}{\omega_K}\right)^6 \theta_0 \tilde{V}_7(B) + O(B(\log B)^4 \log\log B),$$

where

$$\tilde{V}_7(B) := \pi^{-6} \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1, \|\eta_7\|_\infty \geqslant \|\eta_8\|_\infty \\ (\eta_1,\dots,\eta_8)\in\mathcal{R}(B)}} \frac{1}{\|\eta_5\|_\infty}\, d\eta_1 \cdots d\eta_8.$$

The lemma follows immediately. $\square$

9.3 Proof of Theorem 1.1

To compare the result of Lemma 9.9 with Theorem 1.1, we introduce the conditions

$$\|\eta_1^2\eta_3^2\eta_4\eta_5^2\eta_6\|_\infty \leqslant B, \tag{9.19}$$

$$\|\eta_1^2\eta_3^2\eta_4\eta_5^2\eta_6\|_\infty \leqslant B, \quad \|\eta_1^2\eta_3^{-1}\eta_4^4\eta_5^{-1}\eta_6^{-2}\|_\infty \leqslant B, \tag{9.20}$$

$$\|\eta_1^2\eta_3^2\eta_4\eta_5^2\eta_6\|_\infty \leqslant B, \quad \|\eta_1^2\eta_3^{-1}\eta_4^4\eta_5^{-1}\eta_6^{-2}\|_\infty \leqslant B, \quad \|\eta_1^{-1}\eta_3^2\eta_4^{-2}\eta_5^{-1}\eta_6^4\|_\infty \leqslant B. \tag{9.21}$$

Lemma 9.10. *Let ω_∞ be as in Theorem 1.1, $\mathcal{R}(B)$ as in (9.2)–(9.6), and*

$$V_0'(B) := \int_{\substack{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B) \\ \|\eta_1\|_\infty,\|\eta_3\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1 \\ (9.21)}} \frac{1}{\|\eta_5\|_\infty}\, d\eta_1 \cdots d\eta_8.$$

Then $\frac{1}{4320}\pi^6\omega_\infty B(\log B)^5 = 4V_0'(B)$.

Proof. Note that substituting $y_0 = \eta_1\eta_4^2\eta_7$, $y_1 = \eta_3\eta_6^2\eta_8$, $y_2 = \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6$, $-(y_0+y_1) = \eta_5\eta_9$ (which are obtained using the substitutions in § 4) in (9.10) and cancelling out $\eta_1\eta_3\eta_4^2\eta_5\eta_6^2$ gives $\Psi(\eta_1,\dots,\eta_9)$ as in the proof of Lemma 9.1. This motivates the following substitutions in ω_∞: let η_1, η_3, η_4, η_5, $\eta_6 \in \mathbb{C}\smallsetminus\{0\}$ and $B \in \mathbb{R}_{>0}$. Let η_2, η_7, η_8 be complex variables. With $l := (B\|\eta_1\eta_3\eta_4^2\eta_5\eta_6^2\|_\infty)^{1/2}$, we apply the coordinate transformation $y_0 = l^{-1/3}\eta_1\eta_4^2\cdot\eta_7$, $y_1 = l^{-1/3}\eta_3\eta_6^2\cdot\eta_8$, $y_2 = l^{-1/3}\eta_1\eta_3\eta_4\eta_5\eta_6\cdot\eta_2$ of Jacobi determinant

$$\frac{\|\eta_1\eta_3\eta_4\eta_5\eta_6\|_\infty}{B}\frac{1}{\|\eta_5\|_\infty} \tag{9.22}$$

and obtain

$$\omega_\infty = \frac{12}{\pi}\frac{\|\eta_1\eta_3\eta_4\eta_5\eta_6\|_\infty}{B}\int_{(\eta_1,\dots,\eta_8)\in\mathcal{R}(B)} \frac{1}{\|\eta_5\|_\infty}\, d\eta_2\, d\eta_7\, d\eta_8. \tag{9.23}$$

An application of Lemma 8.1 with exchanged roles of η_2 and η_6 gives

$$\alpha(\widetilde{S})(\log B)^5 = \frac{1}{3\pi^5}\int_{\substack{\|\eta_1\|_\infty,\|\eta_3\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1 \\ (9.21)}} \frac{d\eta_1\, d\eta_3 \cdots d\eta_6}{\|\eta_1\eta_3\cdots\eta_6\|_\infty},$$

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since $[-K_{\widetilde{S}}] = [2E_1 + 3E_2 + 2E_3 + E_4 + 2E_5 + E_6]$, $[E_7] = [E_2 + E_3 - E_4 + E_5 + E_6]$, and $[E_8] = [E_1 + E_2 + E_4 + E_5 - E_6]$. By (8.1), we have $\alpha(\widetilde{S}) = 1/4320$.

The lemma follows by substituting this and (9.23) in $\frac{1}{4320}\pi^6\omega_\infty B(\log B)^5$. □

To finish our proof, we compare $V_0(B)$ from Lemma 9.9 with $V_0'(B)$ from Lemma 9.10. Let

$$\begin{aligned}
\mathcal{D}_0(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1\},\\
\mathcal{D}_1(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1,\ (9.19)\},\\
\mathcal{D}_2(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1,\ (9.20)\},\\
\mathcal{D}_3(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_8\|_\infty \geqslant 1,\ (9.21)\},\\
\mathcal{D}_4(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty, \|\eta_8\|_\infty \geqslant 1,\ (9.21)\},\\
\mathcal{D}_5(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1,\ (9.21)\},\\
\mathcal{D}_6(B) &:= \{(\eta_1,\dots,\eta_8) \in \mathcal{R}(B) \mid \|\eta_1\|_\infty, \|\eta_3\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1,\ (9.21)\}.
\end{aligned}$$

Moreover, let

$$V_i(B) := \int_{\mathcal{D}_i(B)} \frac{1}{\|\eta_5\|_\infty}\, d\eta_1 \cdots d\eta_8.$$

Then clearly $V_0(B)$ is as in Lemma 9.9 and $V_6(B) = V_0'(B)$. We show that, for $i = 1,\dots,6$, $V_i(B) - V_{i-1}(B) = O(B(\log B)^4)$. This is clear for $i = 1$, since, by (9.4) and $t_2 \geqslant 1$, we have $\mathcal{D}_1 = \mathcal{D}_0$. Moreover, using Lemma 3.4(4) and (9.6) to bound the integral over η_7 and η_8, we have

$$V_2(B) - V_1(B) \ll \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1\\ \|\eta_1^2\eta_2^2\eta_3\eta_4^2\eta_5\|_\infty \leqslant B\\ \|\eta_1^2\eta_3^{-1}\eta_4^4\eta_5^{-1}\eta_6^{-2}\|_\infty > B}} \frac{B^{2/3}}{\|\eta_1\eta_3\eta_4^2\eta_5\eta_6^2\|_\infty^{1/3}}\, d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

An entirely symmetric argument shows that $V_3(B) - V_2(B) \ll B(\log B)^4$. Using Lemma 3.4(2) and (9.6) to bound the integral over η_8, we obtain

$$V_4(B) - V_3(B) \ll \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1\\ \|\eta_7\|_\infty < 1,\ (9.21)\\ \|\eta_1^2\eta_2^3\eta_3^2\eta_4\eta_5^2\eta_6\|_\infty \leqslant B}} \frac{B^{1/2}}{\|\eta_3\eta_5\eta_6^2\eta_7\|_\infty^{1/2}}\, d\eta_1 \cdots d\eta_7 \ll B(\log B)^4.$$

Here, we first integrate over η_7 and t_2. Again, an analogous argument shows that $V_5(B) - V_4(B) \ll B(\log B)^4$. Finally, using Lemma 3.4(4) and (9.6) to bound the integral over η_7 and η_8, we have

$$V_6(B) - V_5(B) \ll \int_{\substack{\|\eta_1\|_\infty,\dots,\|\eta_6\|_\infty \geqslant 1\\ 0 < t_2 < 1,\ (9.19)}} \frac{B^{2/3}}{\|\eta_1\eta_3\eta_4^2\eta_5\eta_6\|_\infty^{1/3}}\, d\eta_1 \cdots d\eta_6 \ll B(\log B)^4.$$

Thus, $V_0(B) = V_0'(B) + O(B(\log B)^4)$. Using Lemmas 9.9 and 9.10, this implies Theorem 1.1.

9.4 Over $\mathbb{Q}$

The following result is the analog over $\mathbb{Q}$ of Theorem 1.1.

THEOREM 9.11. *For the number of $\mathbb{Q}$-rational points of bounded height on the subset U obtained by removing the lines of $S \subset \mathbb{P}^4_{\mathbb{Q}}$, defined over $\mathbb{Q}$ by (1.3), and $B \geqslant 3$, we have*

$$N_{U,H}(B) = c_{S,H} B(\log B)^5 + O(B(\log B)^4 \log\log B),$$

where

$$c_{S,H} = \frac{1}{4320} \cdot \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right) \cdot \omega_\infty$$

with

$$\omega_\infty = \frac{3}{2} \int_{\max\{|y_0 y_2^2|, |y_1 y_2^2|, |y_2^3|, |y_0 y_1 y_2|, |y_0 y_1 (y_0+y_1)|\} \leqslant 1} dy_0\, dy_1\, dy_2.$$

Proof. This is similar to the case of imaginary quadratic K above, so we shall be very brief.

The parameterization of rational points by integral points on the universal torsor is as in Lemma 9.1, here and everywhere below with $\omega_{\mathbb{Q}} = 2$, $h_K = 1$ so that $\mathcal{C}$ contains only the trivial ideal class, with $\mathcal{O}_j = \mathbb{Z}$ for $j = 1, \dots, 9$, $\mathcal{O}_{1*} = \cdots = \mathcal{O}_{8*} = \mathbb{Z}_{\neq 0}$ and $\mathcal{O}_{9*} = \mathbb{Z}$, and with $\|\cdot\|_\infty$ replaced by the ordinary absolute value $|\cdot|$ on $\mathbb{R}$ in (9.7).

The proof of the asymptotic formula proceeds as in the imaginary quadratic case, but using the original techniques over $\mathbb{Q}$ from [Der09]. In the statements of the intermediate results, we must always replace $2/\sqrt{|\Delta_K|}$ by 1, complex by real integration, π by 2, and $\sqrt{t_i}$ by t_i. The computation of the main terms is always analogous, but less technical. The estimation of the error terms is often analogous and sometimes easier.

The main changes are as follows. For the first summation, we apply [Der09, Proposition 2.4]. The error term $2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3\eta_4)}$ can be estimated as the second summand of the error term in Lemma 9.2.

For the second summation over η_7, we can apply [Der09, Lemma 3.1, Corollary 6.9]. The error term is

$$\sum_{\eta_1,\dots,\eta_6} 2^{\omega(\eta_1\eta_2\eta_3\eta_5\eta_6)} \sup_{|\eta_7|>|\eta_4|} \tilde{V}_8(\eta_1,\dots,\eta_7;B) \ll \sum_{\eta_1,\dots,\eta_6} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_5\eta_6)} B^{1/2}\log B}{|\eta_3|^{1/2}|\eta_4|^{1/2}|\eta_5|^{1/2}|\eta_6|}$$
$$\ll \sum_{\eta_2,\dots,\eta_6} \frac{2^{\omega(\eta_2\eta_3\eta_5\eta_6)} B(\log B)^2}{|\eta_2|^{5/4}|\eta_3|^{5/4}|\eta_4|^{3/2}|\eta_5|^{5/4}|\eta_6|^{5/4}} \ll B(\log B)^2$$

where (using $|\eta_4| < |\eta_7|$)

$$|\eta_1| \leqslant \left(\frac{B}{|\eta_2^2\eta_3\eta_4^3\eta_5|}\right)^{1/4}\left(\frac{B}{|\eta_2^3\eta_3^2\eta_4\eta_5^2\eta_6|}\right)^{1/4}.$$

For the second summation over η_4, the computation is very similar.

The remaining summations and the completion of the proof of Theorem 9.11 remain essentially unchanged. □

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