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Reducibility of some induced representations of split classical p-adic groups

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Abstract. In this paper we study reducibility of representations of split classical p-adic groups induced from self-contragredient supercuspidal representations of general linear groups. For a supercuspidal representation associated via Howe's construction to an admissible character, we show that in many cases Shahidi's criterion for reducibility of the induced representation reduces to a simple condition on the admissible character.

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1. Introduction

Let F be a p-adic field of characteristic zero and odd residue characteristic. Suppose that $G' = \mathrm{SO}_{4n}(F)$, $\mathrm{Sp}_{4n}(F)$, or $\mathrm{SO}_{4n+1}(F)$. Then G' has a maximal parabolic subgroup $P_{\max}$ with Levi factor isomorphic to $G = \mathrm{GL}_{2n}(F)$. Let π be an irreducible unitary supercuspidal representation of G. Assume that π is self-contragredient. In [Sh], Shahidi derives a criterion for reducibility of the representation $\mathrm{I}(\pi)$ induced from the representation $\pi\otimes\mathbf{1}$ of $P_{\max}$. The criterion is expressed in terms of the values of a particular twisted orbital integral $\mathcal{I}$ at functions f in $C_c^\infty(G)$ which represent matrix coefficients of π. If $G' = \mathrm{SO}_{4n}(F)$, $\mathrm{Sp}_{4n}(F)$, resp. $\mathrm{SO}_{4n+1}(F)$, then $\mathrm{I}(\pi)$ is irreducible, resp. reducible, if and only if $\mathcal{I}(f)$ is nonzero for some such function f.

Suppose that π arises via the construction of Howe ([H2]) from an admissible character θ of the multiplicative group of a tamely ramified degree $2n$ extension E of F. As π is unitary and self-contragredient, θ is unitary and satisfies $\theta\circ\sigma = \theta^{-1}$ for some involutive automorphism σ of E/F. In this paper, we prove that, for many such π, Shahidi's criterion reduces to a simple condition on θ. If L is the fixed field of σ, then there are only two possibilities for the restriction of θ to $L^\times$. If this restriction is non-trivial, then it is the quadratic character of $L^\times$ associated to E by class field theory. In the case where E is ramified over L and $\theta\,|\,L^\times$ is trivial, we show that the integral in Shahidi's criterion is nonzero for a particular choice of

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function which represents a (sum of) matrix coefficient(s) of π. If E is unramified over L, we get the same type of result under some additional assumptions on θ; sometimes the integral is nonzero when $\theta \mid L^{\times}$ is non-trivial. The contents of the paper are described in more detail below.

Properties of the Howe factorization of θ relative to the automorphism σ are discussed in Section 2.

The twisted orbital integral $\mathcal{I}$ can be expressed as an integral over the fixed points in G of a certain involutive anti-automorphism φ of $\mathfrak{g}l_{2n}(F)$. The third section contains results describing the action of φ on filtrations of the parahoric subalgebra attached to the extension E, and on related subgroups of G.

The representation π is induced from an irreducible representation κ of an open, compact-mod-centre subgroup H. In Section 4 we state Shahidi's reducibility criterion and define a particular function $\mathcal{F}$ which represents a finite sum of matrix coefficients of π. The function $\mathcal{F}$ is chosen in such a way that the corresponding twisted orbital integral $\mathcal{I}(\mathcal{F})$ reduces to an integral of the character of κ over a certain set of φ-invariant points in H.

In Section 5, we establish some properties of θ and κ. We prove that if κ is one-dimensional and $\theta \mid L^{\times}$ is trivial, then $\mathcal{I}(\mathcal{F})$ is non-zero (Proposition 5.3).

The inducing representation κ is a tensor product of finitely many representations κ_i, $i = 1, \ldots, r$, corresponding to the Howe factors of θ. In Section 6, we show that if a Heisenberg representation is used in the construction of κ_i, then the character of κ_i is real-valued on the set of φ-invariant points in H. Then Section 7 is devoted to computing the sign of the character on certain φ-invariant points.

In Section 8, we state a particular case of a result of Digne and Michel ([DM]) which gives a character formula for Deligne–Lusztig characters of a non-connected finite reductive group.

Next we consider the case when κ_r has level one; that is, κ_r is an inflation of an irreducible cuspidal representation of a general linear group over a finite field. In the first part of Section 9, we determine the map which φ induces on the finite general linear group. Certain sums of values of Deligne–Lusztig characters of finite general linear groups occur in the integral $\mathcal{I}(\mathcal{F})$. Using properties of φ and κ_r, we express these sums in terms of values of a Deligne–Lusztig character of a non-connected finite reductive group (whose identity component is a general linear group). In the main result of Section 9 (Proposition 9.9), we determine the signs of the sums using the character formula from Section 8.

In Section 10, assuming that κ_r has level one and κ_i is one-dimensional for $1 \leqslant i \leqslant r \Leftrightarrow 1$, we derive an expression for $\mathcal{I}(\mathcal{F})$ in terms of values of θ and the sums considered in Section 9. Results of Section 9 are then applied to obtain our main result (Theorem 10.7) in this case.

In Theorem 11.1, we show that under certain assumptions on $\theta \mid L^{\times}$, the integral $\mathcal{I}(\mathcal{F})$ is nonzero (subject to the additional condition mentioned above if κ_r has level one and $r > 1$). When E is ramified over L, it suffices to assume that $\theta \mid L^{\times}$ is trivial. When E is unramified over L, there exists an intermediate extension

$F \subset E_1 \subset E$ (appearing in the Howe factorization of θ) such that $\theta \mid L^\times$ must be assumed to be trivial, resp. non-trivial, when $[E\colon E_1]$ is odd, resp. even. When θ satisfies the conditions of Theorem 11.1, the non-vanishing of $\mathcal{I}(\mathcal{F})$ translates into results concerning reducibility of the induced representation $\mathrm{I}(\pi)$ (Theorem 11.4). Reducibility of the non-unitary representation $\mathrm{I}(\pi \otimes |\det(\cdot)|^\alpha)$, for α a nonzero real number, is discussed in Corollary 11.5.

In the second part of Section 11, we formulate a conjecture giving necessary and sufficient conditions for reducibility of $\mathrm{I}(\pi)$ in terms of the values of $\theta \mid L^\times$. The conjecture is based on our results and on the expected relations between properties of θ and the conjectural representation of the Weil group parametrizing the L-packet $\{\pi\}$ of G. Shahidi ([Sh]) interpreted the reducibility of $\mathrm{I}(\pi)$ in terms of the conjectural theory of twisted endoscopy ([KS1], [KS2]). In particular, if θ satisfies the conditions of Theorem 11.1, then the L-packet $\{\pi\}$ of G should come via twisted endoscopy from an L-packet of representations of $\mathrm{SO}_{2n+1}(F)$. Our conjecture can be restated as a criterion which uses $\theta \mid L^\times$ to determine whether $\{\pi\}$ comes via twisted endoscopy from an L-packet of $\mathrm{SO}_{2n+1}(F)$ or of a quasi-split $\mathrm{SO}_{2n}(F)$.

Goldberg ([Go]) has expressed reducibility of certain induced representations of unitary groups in terms of non-vanishing of sums of twisted orbital integrals of matrix coefficients of supercuspidal representations of general linear groups. In a forthcoming paper ([MR]), the results of this paper are adapted to obtain reducibility results for unitary groups.

In an earlier version of this paper, in order to obtain some of our results in the case where κ_r has level one, we evaluated particular sums of Green polynomials of general linear groups. We have since found a more direct way to obtain these results via a character formula of Digne and Michel.

2. Self-contragredient Supercuspidal Representations

Let F be a p-adic field of characteristic zero and odd residue characteristic, and let $G = \mathrm{GL}_m(F)$. Let E be a tamely ramified extension of F of degree m, and let θ be an admissible character of $E^\times$ over F.

The character θ has a Howe factorization (see [H2], [Mo2])

$$\theta = (\chi \circ N_{E/F})\theta_r(\theta_{r-1} \circ N_{E/E_{r-1}}) \cdots (\theta_2 \circ N_{E/E_2})(\theta_1 \circ N_{E/E_1}). \tag{2.1}$$

Here θ uniquely determines the tower of fields $F = E_0 \subset E_1 \subset \cdots \subset E_r = E$ and χ, $\theta_1, \ldots, \theta_r$ are quasi-characters of $F^\times$, $E_1^\times, \ldots, E_r^\times$, respectively. Each quasi-character θ_i is generic over E_{i-1} ([H2]). The conductoral exponents are unique and satisfy

$$f_E(\theta_1 \circ N_{E/E_1}) > \cdots > f_E(\theta_r) > 0. \tag{2.2}$$

For each i, if $f_{E_i}(\theta_i) > 1$, choose an element $c_i \in E_i$ that 'represents' θ_i in the sense that

$$\theta_i(1+x) = \psi(\mathrm{tr}_{E_i/F}(c_i x)), \quad \text{for } x \in \mathfrak{p}_{E_i}^{[(f_{E_i}(\theta_i)+1)/2]}, \tag{2.3}$$

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where ψ is a character of the additive group F with conductor $\mathfrak{p}_F$; we must have $c_i \in \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+1} \setminus \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+2}$ (see [H2], [Mo2]). Note that the genericity of θ_i implies that c_i generates E_i over E_{i-1}.

The construction of Howe ([H2], [Mo2]) associates to each equivalence class of admissible characters θ an equivalence class of irreducible supercuspidal representations π of G. We will henceforth assume that θ (and hence the corresponding π) is unitary (see Corollary 11.6 for some results in the non-unitary case).

Suppose that the supercuspidal representation π of G attached to θ is self-contragredient (that is, π is equivalent to its contragredient). Since the contragredient of π is attached to the character $\bar{\theta}$ (see [Mo2]), it follows that there is a $\sigma \in \mathrm{Aut}(E/F)$ such that

$$\theta \circ \sigma = \bar{\theta} = \theta^{-1}. \tag{2.4}$$

Here the notation $\mathrm{Aut}(E/F)$ denotes the automorphisms of E fixing F pointwise (we are not assuming that E/F is Galois). Using the admissibility of θ, it is not hard to see that σ must have order two ([A]). In particular, m must be even. Henceforth we will let $m = 2n$, and consider $G = \mathrm{GL}_{2n}(F)$. Adler ([A]) has shown that given any tamely ramified degree $2n$ extension E of F such that E/L is quadratic for some intermediate field L, there exist unitary admissible characters θ of $E^\times$ satisfying (2.4) with σ the non-trivial element of $\mathrm{Gal}(E/L)$; hence there are self-contragredient supercuspidal representations of G associated to every such extension.

By comparing Howe factorizations of θ and $\theta \circ \sigma$, we also observe that $\sigma(E_i) = E_i$ for each i, although we shall see that σ does not fix E_i pointwise.

We claim that $f_E(\chi \circ N_{E/F}) \leqslant f_E(\theta_1 \circ N_{E/E_1})$. If not, then using (2.2), (2.4) and the fact that $\chi \circ N_{E/F}$ is invariant under σ, we see that $\chi \circ N_{E/F}$ must be a non-trivial real character on

$$(1 + \mathfrak{p}_E^{f_E(\theta_1 \circ N_{E/E_1})})/(1 + \mathfrak{p}_E^{f_E(\chi \circ N_{E/F})}).$$

Since p is odd, this group has odd order, which is impossible, proving the claim.

Next, we replace θ_1 with $\theta_1(\chi \circ N_{E_1/F})$ and drop χ from the notation. Note that $\chi \circ N_{E_1/F}$ is invariant under automorphisms of E_1/F, so any element representing $\chi \circ N_{E_1/F}$ can be chosen to be an element of F. Because of the claim just proved, this shows $\theta_1(\chi \circ N_{E_1/F})$ is still a generic character of E_1.

LEMMA 2.5. *The characters θ_i and the elements c_i can be chosen so that*

(i) θ_i *is unitary,*
(ii) $\theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$,
(iii) $\sigma(c_i) = -c_i$, *if* $f_E(\theta_i) > 1$.

Proof (i) We know that θ is a character (i.e., is unitary). In particular, if we write it as the product of a character of $\mathcal{O}_E^\times$ and a power of $|\cdot|_E$, then (2.4) shows that

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the power of $|\cdot|_E$ must take values in $\{\pm 1\}$. We can adjust the power of $|\cdot|_{E_i}$ occurring in θ_i so that each θ_i is unitary; this does not affect the genericity of θ_i.

(ii) From (2.4), we have that $\theta_1 \circ N_{E/E_1} \circ \sigma = (\theta_1 \circ N_{E/E_1})^{-1}$ on

$$(1+\mathfrak{p}_E^{f_E(\theta_2 \circ N_{E/E_2})})/(1+\mathfrak{p}_E^{f_E(\theta_1 \circ N_{E/E_1})}).$$

Since $(1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^{f_E(\theta_1 \circ N_{E/E_1})})$ is a p-group, it is possible to adjust θ_1 so that $\theta_1 \circ N_{E/E_1} \circ \sigma = (\theta_1 \circ N_{E/E_1})^{-1}$ on all of $1+\mathfrak{p}_E$. Using the Chinese Remainder Theorem, it can further be adjusted so that the same relation holds on all of $\mathcal{O}_E^{\times}$, and therefore on all of $E^{\times}$. Then, by an inductive argument, we can assume that for each i, $\theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$ on $E^{\times}$. This proves (ii).

(iii) Note that for any k,

$$\mathfrak{p}_E^k \cap E_i \subseteq \mathfrak{p}_{E_i}^{[(k-1)/\mathsf{e}(E/E_i)]+1},$$

so

$$f_E(\theta_i \circ N_{E/E_i}) = 1 + \mathsf{e}(E/E_i)(f_{E_i}(\theta_i) \Leftrightarrow 1).$$

Let $m_i = [(f_E(\theta_i \circ N_{E/E_i}) + 1)/2]$. To finish the proof, we will need the following technical result.

LEMMA 2.6. *Suppose* $1 \leqslant i \leqslant r$*; if* $i = r$*, then assume* $f_E(\theta_r) > 1$*. If* $x \in \mathfrak{p}_E^{m_i}$*, then*

$$\begin{aligned} \theta_i \circ N_{E/E_i}(1+x) &= \theta_i(1+\mathrm{tr}_{E/E_i}(x)) \\ &= \psi(\mathrm{tr}_{E_i/F}(c_i\,\mathrm{tr}_{E/E_i}(x))) = \psi(\mathrm{tr}_{E/F}(c_i x)). \end{aligned}$$

Proof. Note that

$$\begin{aligned} &\mathfrak{p}_E^{2m_i} \cap E_i \subseteq \mathfrak{p}_{E_i}^{[\mathsf{e}(E/E_i)(f_{E_i}(\theta_i)-1)/\mathsf{e}(E/E_i)]+1} = \mathfrak{p}_{E_i}^{f_{E_i}(\theta_i)} \\ &\quad \Longrightarrow N_{E/E_i}(1+x) \in 1 + \mathrm{tr}_{E/E_i}(x) + \mathfrak{p}_{E_i}^{f_{E_i}(\theta_i)}, \quad \text{for } x \in \mathfrak{p}_E^{m_i}. \end{aligned}$$

Also

$$\mathrm{tr}_{E/E_i}(x) \in \mathfrak{p}_E^{m_i} \cap E_i \subseteq \mathfrak{p}_{E_i}^{[(m_i-1)/\mathsf{e}(E/E_i)]+1} \subseteq \mathfrak{p}_{E_i}^{[(f_{E_i}(\theta_i)+1)/2]},$$

from which the result follows. □

Comparing with $\theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$, we see that $\sigma(c_i) \in \Leftrightarrow c_i + (\mathfrak{p}_E^{-m_i+1} \cap E_i)$. But c_i is only defined up to addition of elements of $\mathfrak{p}_{E_i}^{-[(f_{E_i}(\theta_i)+1)/2]+1}$. By adding something in $\mathfrak{p}_{E_i}^{-[(f_{E_i}(\theta_i)+1)/2]+1}$, we can assume that the c_i's satisfy $\sigma(c_i) = \Leftrightarrow c_i$. □

From now on we assume that θ_i and c_i are as in Lemma 2.5.

3. Filtrations and the map φ

Let the notation be as in Section 2. The representation π is of the form $\pi = \mathrm{Ind}_H^G \kappa$ where κ is an irreducible representation of an open compact-mod-centre subgroup H of $G = \mathrm{GL}_{2n}(F)$. We will define an anti-automorphism φ of $\mathfrak{g}l_{2n}(F)$ so that the integral we need to consider can be expressed as an integral over certain φ-invariant points in the inducing subgroup H. The field E will be embedded in $\mathfrak{g}l_{2n}(F)$ in such a way that the action of φ on E is given by σ; up to conjugation by a fixed matrix, the map $X \mapsto \Leftrightarrow\varphi(X)$ is the Lie algebra analogue of inverse transpose. We will also need to consider matrix algebras over φ-invariant fields intermediate between F and E. We will consider the action of φ on such algebras, and in particular will show that the standard filtrations and parahoric subgroups are φ-invariant.

Let L be the fixed field of σ. We choose a basis $\{\xi_1, \ldots, \xi_n\}$ of L/F, and let $\{\xi_1^*, \ldots, \xi_n^*\}$ be the dual basis with respect to the trace form. Let s be the matrix of the identity transformation from the basis $\{\xi_j\}$ to the basis $\{\xi_j^*\}$, i.e., the transition matrix; note that s is symmetric. Then for any $\eta \in L$,

$$s^{-1}\,{}^t\eta s = \eta.$$

If $E = L(\tau)$, with $\sigma(\tau) = \Leftrightarrow\tau$, we form a basis $\{\xi_1, \ldots, \xi_n, \tau\xi_1, \ldots, \tau\xi_n\}$ of E/F, and use it to embed E in $\mathfrak{g}l_{2n}(F)$. If we let

$$w = \begin{pmatrix} 0 & s \\ \Leftrightarrow s & 0 \end{pmatrix},$$

then w has a similar property for $\gamma \in E$, namely

$$w^{-1}\,{}^t\gamma w = \sigma(\gamma).$$

The relationship between left multiplication by w or w^{-1} and the standard basis above and its dual will be needed in Corollary 3.5 to show that the anti-automorphism φ preserves certain lattices in $\mathfrak{g}l_{2n}(F)$.

LEMMA 3.1. *Suppose $\gamma \in E$. Write $[\gamma]$ for the coefficients of γ with respect to the standard basis of E/F and $[\gamma]^*$ for the coefficients of γ with respect to the dual basis. Then*

(i) $w[\gamma] = [\Leftrightarrow(\sigma(\gamma)/2\tau)]^*$.
(ii) $w^{-1}[\gamma]^* = [\Leftrightarrow 2\tau\sigma(\gamma)]$.

Proof. We write $[\eta]_L$ (resp. $[\eta]_L^*$) for the coefficients of $\eta \in L$ with respect to the standard (resp. dual) basis of L/F. In particular, $s[\eta]_L = [\eta]_L^*$, for $\eta \in L$.

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Note that the basis of E/F dual to the standard basis $\{\xi_1,\ldots,\xi_n,\tau\xi_1,\ldots,\tau\xi_n\}$ is $\{\frac{1}{2}\xi_1^*,\ldots,\frac{1}{2}\xi_n^*,(1/2\tau)\xi_1^*,\ldots,(1/2\tau)\xi_n^*\}$.

For (i), we write $\gamma=\gamma_1+\tau\gamma_2$, with $\gamma_1,\gamma_2\in L$, so

$$[\gamma]=\begin{pmatrix}[\gamma_1]_L\\ [\gamma_2]_L\end{pmatrix}.$$

Then

$$w[\gamma]=\begin{pmatrix}s[\gamma_2]_L\\ \Leftrightarrow s[\gamma_1]_L\end{pmatrix}=\begin{pmatrix}[\gamma_2]_L^*\\ \Leftrightarrow[\gamma_1]_L^*\end{pmatrix}=\left[\frac{\gamma_2}{2}\Leftrightarrow\frac{\gamma_1}{2\tau}\right]^*=\left[\Leftrightarrow\frac{\sigma(\gamma)}{2\tau}\right]^*.$$

This proves (i). Part (ii) is obtained by inverting (i). $\square$

We define the map φ: $\mathfrak{g}l_{2n}(F)\to\mathfrak{g}l_{2n}(F)$ as follows: For $X\in\mathfrak{g}l_{2n}(F)$,

$$\varphi(X)=w^{-1t}Xw. \tag{3.2}$$

If $S\subset\mathfrak{g}l_{2n}(F)$ and $c=\pm1$, then $S^{c\varphi}$ will denote the $c\varphi$-invariant points in S.

Next we discuss how φ acts on matrices over different fields. We will want to apply this idea in different contexts, especially to intermediate fields $F\subset E_i\subset E$, but also to the map induced by φ on matrices over residue fields. Consider fields $F'\subset N'\subset E'$, $F'\subset L'\subset E'$, with $[E'\colon L']=2$, and σ the non-trivial automorphism of E'/L'. The results will often be applied to the fields $F\subset E_i\subset E$, $F\subset L\subset E$ or the corresponding residue fields. We let $n'=[L'\colon F']$. Our goal is to find a simple expression for the action of φ on matrices over N'.

LEMMA 3.3. *Suppose $F'\subset N'\subset L'$ and $m'=[N'\colon F']$. Any anti-automorphism of $\mathfrak{g}l_{n'/m'}(N')$ that fixes the scalars (i.e., the N'-scalars) can be written as the composition of the transpose in $\mathfrak{g}l_{n'/m'}(N')$ with an inner automorphism of $\mathfrak{g}l_{n'/m'}(N')$.*

Proof. Composition with the transpose gives an automorphism of $\mathfrak{g}l_{n'/m'}(N')$ that fixes the scalars. Composing with an inner automorphism, we can assume it preserves the diagonal matrices and fixes the scalars. Any such automorphism is inner. $\square$

Suppose $s'\in\mathrm{GL}_{n'}(F')$ is symmetric such that for any $\eta\in L'$, $s'^{-1}\,{}^t\eta s'=\eta$, and define

$$w'=\begin{pmatrix}0&s'\\ \Leftrightarrow s'&0\end{pmatrix},\qquad \varphi'(X)=w'^{-1}\,{}^tXw',\quad \text{for}\ \ X\in\mathfrak{g}l_{2n'}(F').$$

LEMMA 3.4. (i) *If $N'\subset L'$, let $m'=[N'\colon F']$. Then there exists a symmetric matrix $\mathcal{S}\in\mathrm{GL}_{n'/m'}(N')$ such that*

$$\varphi'(X)=\mathcal{S}^{-1T}X\mathcal{S},\quad X\in\mathfrak{g}l_{n'/m'}(N').$$

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Here ${}^{T}X$ *means the transpose over* N'.

(ii) *Let* $F' \subset N' \subset E'$ *be such that* $\sigma(N') = N'$ *and* $\sigma \,|\, N'$ *is non-trivial.* (*We will often apply this with* $N' = E_i$ *for some* i.) *Let* σ *be the non-trivial automorphism of* E' *over* L' *and let* $\tau_{N'}$ *be a generator of* N' *over* $N'_0 = N' \cap L'$ *such that* $\sigma(\tau_{N'}) = \Leftrightarrow\tau_{N'}$; *let* $m' = [N'_0 : F']$. *Let* $\sigma_{N'}$ *be the action on a matrix with entries in* N' *given by applying* $\sigma \,|\, N'$ *to each entry of the matrix. Then there exists a symmetric matrix* $\mathcal{S}_0 \in \mathrm{GL}_{n'/m'}(N'_0)$ *such that*

$$\varphi'(X + Y\tau_{N'}) = \mathcal{S}_0^{-1\,T}(\sigma_{N'}(X + Y\tau_{N'}))\mathcal{S}_0, \quad X + Y\tau_{N'} \in \mathfrak{gl}_{n'/m'}(N').$$

Proof. (i) Suppose $N' \subset L'$. Consider the map $\psi\colon \mathfrak{gl}_{n'}(F') \to \mathfrak{gl}_{n'}(F')$ defined by $\psi(X) = s'^{-1}\,{}^{t}Xs'$. Because of the form of s', $\psi \,|\, L'$ is the identity map. As $N' \subset L'$, it follows that $\psi(\mathfrak{gl}_{n'/m'}(N')) = \mathfrak{gl}_{n'/m'}(N')$. In fact we can say more than that: the restriction of ψ is an anti-automorphism of $\mathfrak{gl}_{n'/m'}(N')$ that fixes the scalars (i.e., the N'-scalars). Using Lemma 3.3, we see that there exists a matrix $\mathcal{S} \in \mathrm{GL}_{n'/m'}(N')$ such that

$$\psi(X) = \mathcal{S}^{-1\,T}X\mathcal{S}, \quad X \in \mathfrak{gl}_{n'/m'}(N') \subset \mathfrak{gl}_{n'}(F').$$

Since ψ^2 is the identity map, we know that $\mathcal{S}^{-1\,T}\mathcal{S}$ is a scalar, i.e., ${}^{T}\mathcal{S} = c\mathcal{S}$, for some $c \in N'$. But $\mathcal{S} = {}^{TT}\mathcal{S} = c^2\mathcal{S}$ and we find that $\mathcal{S}$ is symmetric or skew-symmetric. Let α be a generator of L' over N'. Let $\lambda_1, \ldots, \lambda_{n'/m'}$ be the eigenvalues of α (in some extension K of N'). We know that these eigenvalues are distinct and nonzero as α is regular 'elliptic'. There exists $x \in \mathrm{GL}_{n'/m'}(K)$ such that

$$x\alpha x^{-1} = \delta = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n'/m'}).$$

From $\alpha = \mathcal{S}^{-1\,T}\alpha\mathcal{S}$ it follows that $\delta = \mathcal{A}^{-1}\delta\mathcal{A}$, where $\mathcal{A} = x\mathcal{S}^{-1\,T}x$. Thus $\mathcal{A}$ must centralize δ. Therefore $\mathcal{A}$ is diagonal. If $\mathcal{S}$ were skew-symmetric, then $\mathcal{A}$ would also be. Clearly this is impossible. Therefore $\mathcal{S}$ is symmetric, proving (i).

(ii) Suppose $E' = L'(\tau)$ for some τ such that $\sigma(\tau) = \Leftrightarrow\tau$. With these choices we have $\tau_{N'} = \omega\tau$ for some $\omega \in L'^{\times}$. If a matrix commutes with all of N', then it commutes with N'_0 and therefore has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathfrak{gl}_{n'/m'}(N'_0).$$

The above matrix must also commute with

$$\tau_{N'} = \begin{pmatrix} 0 & \omega\tau^2 \\ \omega & 0 \end{pmatrix},$$

so we have $\omega\tau^2 C = B\omega$ and $D\omega = \omega A$. That is, the matrix has the form

$$\begin{pmatrix} \omega^{-1}X\omega & (\omega^{-1}Y\omega)\omega\tau^2 \\ Y\omega & X \end{pmatrix},$$

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with $X, Y \in \mathfrak{gl}_{n'/m'}(N'_0)$. We can easily check that mapping this matrix to $X + Y\tau_{N'}$ is a Lie algebra isomorphism between the centralizer of N' in $\mathfrak{gl}_{2n'}(F')$ and $\mathfrak{gl}_{n'/m'}(N') = \mathfrak{gl}_{(2n')/(2m')}(N')$.

A simple calculation gives

$$\varphi'(X + Y\tau_{N'}) = \begin{pmatrix} s'^{-1t}Xs' & \Leftrightarrow s'^{-1t}(\omega^{-1}Y\omega^2\tau^2)s' \\ \Leftrightarrow s'^{-1t}(Y\omega)s' & s'^{-1t}(\omega^{-1}X\omega)s' \end{pmatrix}.$$

Now, applying part (i) to matrices over N'_0, we find there is a symmetric $\mathcal{S} \in \mathrm{GL}_{n'/m'}(N'_0)$ so that

$$\varphi'(X + Y\tau_{N'}) = \begin{pmatrix} \mathcal{S}^{-1T}X\mathcal{S} & \Leftrightarrow \omega^2\tau^2\mathcal{S}^{-1T}Y\mathcal{S}\omega^{-1} \\ \Leftrightarrow \omega\mathcal{S}^{-1T}Y\mathcal{S} & \omega\mathcal{S}^{-1T}X\mathcal{S}\omega^{-1} \end{pmatrix}.$$

Using $\mathcal{S}^{-1\,T}\omega\mathcal{S} = \omega$, we see that

$$\begin{aligned} \varphi'(X + Y\tau_{N'}) &= (\omega\mathcal{S}^{-1T}X\mathcal{S}\omega^{-1}) \Leftrightarrow (\omega\mathcal{S}^{-1T}Y\mathcal{S}\omega^{-1})\tau_{N'}, \\ &\qquad X, Y \in \mathfrak{gl}_{n'/m'}(N'_0). \end{aligned}$$

Set $\mathcal{S}_0 = \mathcal{S}\omega^{-1}$. Then $\mathcal{S}_0$ is symmetric. We have shown that there exists a symmetric matrix $\mathcal{S}_0 \in \mathrm{GL}_{n'/m'}(N'_0)$ such that for any $X + Y\tau_{N'} \in \mathfrak{gl}_{n'/m'}(N')$,

$$\varphi'(X + Y\tau_{N'}) = \mathcal{S}_0^{-1T}(\sigma_{N'}(X + Y\tau_{N'}))\mathcal{S}_0,$$

as required. □

Now we define various subalgebras and subgroups. The parahoric 'subalgebra' $\mathcal{B} \subset \mathfrak{g} \Leftrightarrow \mathfrak{gl}_{2n}(F)$ attached to the embedding $E \hookrightarrow \mathfrak{g}$ is defined by

$$\mathcal{B} = \{X \in \mathfrak{g} \mid X\mathfrak{p}_E^k \subset \mathfrak{p}_E^k, \quad \text{for all } k\}.$$

The parahoric subgroup $P \subset G = \mathrm{GL}_{2n}(F)$ is the units

$$P = \mathcal{B}^\times.$$

For any integer j, we also define

$$\mathcal{B}_j = \{X \in \mathfrak{g} \mid X\mathfrak{p}_E^k \subset \mathfrak{p}_E^{k+j}, \quad \text{for all } k\}$$

and

$$P_0 = P, \qquad P_j = 1 + \mathcal{B}_j, \quad \text{for } j \geqslant 1.$$

We define a function ν on $\mathfrak{g}$ by $\nu(X) = j$, where j is the unique integer such that $X \in \mathcal{B}_j \backslash \mathcal{B}_{j+1}$. Note that if $X \in E$, then $\nu(X) = \mathrm{ord}_E(X)$.

At times it will be necessary to consider one of the intermediate fields E_i occurring in the Howe factorization of θ. It is possible to embed $\mathfrak{gl}_{[E:E_i]}(E_i)$ in $\mathfrak{gl}_{2n}(F)$ as the set of all elements of $\mathfrak{gl}_{2n}(F)$ that centralize $E_i \subset E \subset \mathfrak{gl}_{2n}(F)$. We will refer to this realization of $\mathfrak{gl}_{[E:E_i]}(E_i)$ as M_i.

In this situation, we will define

$$\begin{aligned} \mathcal{B}_j(i) &= \{X \in M_i \mid X\mathfrak{p}_{E_i}^k \subset \mathfrak{p}_{E_i}^{k+j}, \quad \text{for all } k\} = \mathcal{B}_j \cap M_i, \\ P_j(i) &= P_j \cap M_i \end{aligned}$$

and

$$\mathcal{B}(i) = \mathcal{B}_0(i), \qquad P(i) = P_0(i) = \mathcal{B}(i) \cap G.$$

Let φ be as in (3.2). Using Lemmas 3.1 and 3.4, we are now able to show that the filtrations and parahoric subgroups defined above are φ-invariant.

COROLLARY 3.5.
(i) $\varphi(\mathcal{B}_j) = \mathcal{B}_j$.
(ii) $\varphi(M_i) = M_i$.
(iii) $\varphi(\mathcal{B}_j(i)) = \mathcal{B}_j(i)$, $j \in \mathbb{Z}$.
(iv) $\varphi(P_j(i)) = P_j(i)$, $j \geqslant 0$.

Proof. Suppose $X \in \mathcal{B}_j$; this means $X\mathfrak{p}_E^k \subset \mathfrak{p}_E^{k+j}$, for all k.

If $\gamma \in \mathfrak{p}_E^k$, we write it as a column vector as discussed at the beginning of this section. Then by Lemma 3.1, $w[\gamma] = [\Leftrightarrow(\sigma(\gamma)/2\tau)]^*$.

Now ${}^tX$ is the matrix of X relative to the dual basis, so ${}^tX[\Leftrightarrow(\sigma(\gamma)/2\tau)]^* = [\Leftrightarrow X(\sigma(\gamma)/2\tau)]^*$. We find that

$$\varphi(X)[\gamma] = w^{-1}\,{}^tXw[\gamma] = w^{-1}\,{}^tX\left[\Leftrightarrow\frac{\sigma(\gamma)}{2\tau}\right]^* = w^{-1}\left[\Leftrightarrow X\frac{\sigma(\gamma)}{2\tau}\right]^*.$$

Now $\Leftrightarrow(\sigma(\gamma)/2\tau) \in (1/\tau)\mathfrak{p}_E^k$; since $X \in \mathcal{B}_j$, $\Leftrightarrow X(\sigma(\gamma)/2\tau)$ must be in $(1/\tau)\mathfrak{p}_E^{k+j}$. So

$$w^{-1}\left[\Leftrightarrow X\frac{\sigma(\gamma)}{2\tau}\right]^* \in w^{-1}\left[\Leftrightarrow\frac{1}{\tau}\mathfrak{p}_E^{k+j}\right]^* \subset [\mathfrak{p}_E^{k+j}].$$

This means $\varphi(X)(\mathfrak{p}_E^k) \subset \mathfrak{p}_E^{k+j}$, which means $\varphi(X) \in \mathcal{B}_j$, proving (i).

Part (ii) follows from Lemma 3.4, and part (iii) follows from parts (i) and (ii) and the fact that $\mathcal{B}_j(i) = \mathcal{B}_j \cap M_i$. Part (iv) follows immediately from (iii). □

We finish this section with some technical results that will be used in the Heisenberg construction of Section 6.

LEMMA 3.6. *Let* $1 \leqslant i \leqslant r$ *and* $j \geqslant 1$. *Suppose that* K_i *is a subgroup of* $P(i)$ *satisfying*

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(i) $K_i \cap P_j(i \Leftrightarrow 1) = P_j(i)$.
(ii) K_i *normalizes* $P_j(i \Leftrightarrow 1)$.
(iii) $\varphi(K_i) = K_i$.
(iv) $E^\times$ *normalizes* K_i.

Then any $x \in (E^\times K_i P_j(i \Leftrightarrow 1))^\varphi$ *can be written in the form* $x = yz$, *where* $y \in (E^\times K_i)^\varphi$, $z \in P_j(i \Leftrightarrow 1)$.

Proof. Write $x = uv$, $u \in E^\times K_i$ and $v \in P_j(i \Leftrightarrow 1)$. Note that $\varphi(E^\times K_i) = E^\times K_i$. This follows from (iii) and (iv) and $\varphi(E^\times) = E^\times$. Then, as $E^\times K_i$ normalizes $P_j(i \Leftrightarrow 1)$, we have

$$\varphi(x) = \varphi(v)\varphi(u) = \varphi(u)(\varphi(u)^{-1}\varphi(v)\varphi(u)) \in \varphi(u)P_j(i \Leftrightarrow 1).$$

Therefore, using $\varphi(x) = x$, we get

$$\varphi(u)^{-1}u \in P_j(i \Leftrightarrow 1) \cap \mathrm{GL}_{2n/[E_i:F]}(E_i) = P_j(i).$$

Write $u = \varphi(u)(1 + X)$, $X \in \mathcal{B}_j(i)$. By definition of X,

$$\varphi(u)X = u \Leftrightarrow \varphi(u).$$

Applying φ to this equality results in

$$\varphi(\varphi(u)X) = \varphi(u) \Leftrightarrow u = \Leftrightarrow (u \Leftrightarrow \varphi(u)) = \Leftrightarrow \varphi(u)X. \tag{3.7}$$

Now we write $u = \varpi_E^m \alpha k$, $\alpha \in \mathcal{O}_E^\times$, $k \in K_i$. We have

$$\varphi(u)X \in uP_j(i \Leftrightarrow 1)X \in \varpi_E^m \mathcal{O}_E^\times K_i \mathcal{B}_j(i) = \mathcal{B}_{j+\mathrm{em}}(i), \quad \mathrm{e} = \mathrm{e}(E/F).$$

Set $\mathcal{B}_\ell(i)^\pm = \{Y \in \mathcal{B}_\ell(i) \mid \varphi(Y) = \pm Y\}$, $\ell \in \mathbb{Z}$. It is easy to see that

$$\mathcal{B}_\ell(i) = \mathcal{B}_\ell(i)^+ \oplus \mathcal{B}_\ell(i)^-,$$

$$Y = \frac{Y + \varphi(Y)}{2} + \frac{Y \Leftrightarrow \varphi(Y)}{2}.$$

Therefore, by (3.7) we may write $\varphi(u)X = \varphi(X_1) \Leftrightarrow X_1$ for some $X_1 \in \mathcal{B}_{j+\mathrm{em}}(i)$. Set $X_2 = u^{-1}X_1$. From (iii)

$$X_2 \in k^{-1}\alpha^{-1}\varpi_E^{-m}\mathcal{B}_{j+\mathrm{em}}(i) \subset k^{-1}\mathcal{B}_j(i) = \mathcal{B}_j(i).$$

So we have

$$\varphi(u)X = \varphi(uX_2) \Leftrightarrow uX_2,$$

which implies

$$u = \varphi(u) + \varphi(uX_2) \Leftrightarrow uX_2,$$

or

$$u(1+X_2) = \varphi(u(1+X_2)).$$

Set $y = u(1+X_2)$ and $z = (1+X_2)^{-1}v$. Then $\varphi(y) = y$ and since $1+X_2 \in P_j(i)$, (i) implies $y \in E^\times K_i$. □

For $1 \leqslant i \leqslant r$, write $\ell_i = [(f_E(\theta_i \circ N_{E/E_i}))/2]$. Set

$$H = E^\times P_{\ell_r}(r \Leftrightarrow 1) \cdots P_{\ell_2}(1) P_{\ell_1},$$

$$K_i = P_{\ell_r}(r \Leftrightarrow 1) \cdots P_{\ell_{i+1}}(i), \quad 0 \leqslant i \leqslant r \Leftrightarrow 1; \quad K_r = \{1\},$$

$$L_i = P_{\ell_i}(i \Leftrightarrow 1) \cdots P_{\ell_1}, \quad 1 \leqslant i \leqslant r.$$

COROLLARY 3.8. *Let $x \in H^\varphi$. Let $1 \leqslant i \leqslant r$. Then there exist $y \in (E^\times K_i)^\varphi$ and $z \in L_i$ such that $x = yz$.*

Proof. If $i = 1$, apply Lemma 3.6. If $i > 1$, assume that the corollary holds for $1 \leqslant j \leqslant i \Leftrightarrow 1$. Then we can write $x = y'z'$, where

$$y' \in E^\times K_{i-1} = E^\times K_i P_{\ell_i}(i \Leftrightarrow 1), \quad z' \in L_{i-1}, \quad \varphi(y') = y'.$$

The preceding lemma now can be applied to y' to write $y' = yz''$ with $y \in E^\times K_i$ such that $\varphi(y) = y$ and $z'' \in P_{\ell_i}(i \Leftrightarrow 1)$. Since

$$z = z''z' \in P_{\ell_i}(i \Leftrightarrow 1) L_{i-1} = L_i,$$

the corollary follows. □

LEMMA 3.9. *Let $0 \leqslant i \leqslant r$, $j \geqslant 1$, and $\tau \in (H \cap M_i)^\varphi$. Then the map $x \mapsto x\tau\varphi(x)$ from $P_j(i)$ to $(\tau P_j(i))^\varphi$ is onto.*

Proof. Define $\varphi'(X) = \varphi(\tau X \tau^{-1}) = \tau^{-1}\varphi(X)\tau$, for $X \in \mathfrak{gl}_{2n}(F)$. Because $H \cap M_i$ normalizes $\mathcal{B}_j(i)$, it follows from Corollary 3.5(iii) that $\varphi'(\mathcal{B}_j(i)) = \mathcal{B}_j(i)$. Let $g \in P_j(i)^{\varphi'}$; set $X = g \Leftrightarrow 1$. Because $\varphi'(X) = X$, there exists $Y_1 \in \mathcal{B}_j(i)$ such that $Y_1 + \varphi'(Y_1) = X$ (for example, since p is odd, we could take $Y_1 = X/2$). Then

$$X \Leftrightarrow (Y_1 + \varphi'(Y_1) + Y_1\varphi'(Y_1)) = \Leftrightarrow Y_1\varphi'(Y_1)$$

is φ'-invariant and, since $\mathcal{B}_j(i)\mathcal{B}_j(i) \subset \mathcal{B}_{2j}(i)$, lies in $\mathcal{B}_{2j}(i)$.

Suppose that $Y_1, Y_2, \ldots, Y_m \in \mathcal{B}_j(i)$ are such that

$$Y_s \Leftrightarrow Y_{s+1} \in \mathcal{B}_{sj}(i),$$

$$X \Leftrightarrow (Y_s + \varphi'(Y_s) + Y_s\varphi'(Y_s)) \in \mathcal{B}_{(s+1)j}(i) \quad \text{is } \varphi'\text{-invariant}.$$

Choose $W_{m+1} \in \mathcal{B}_{(m+1)j}(i)$ such that

$$W_{m+1} + \varphi'(W_{m+1}) = X \Leftrightarrow (Y_m + \varphi'(Y_m) + Y_m\varphi'(Y_m)).$$

Set $Y_{m+1} = Y_m + W_{m+1}$. Then

$$\begin{aligned} X \Leftrightarrow & (Y_{m+1} + \varphi'(Y_{m+1}) + Y_{m+1}\varphi'(Y_{m+1})) \\ &= Y_m\varphi'(W_{m+1}) + W_{m+1}\varphi'(Y_m) + W_{m+1}\varphi'(W_{m+1}) \end{aligned}$$

is φ'-invariant and belongs to $\mathcal{B}_{(m+1)j}(i)\mathcal{B}_j(i) \subset \mathcal{B}_{(m+2)j}(i)$. The $\mathcal{B}_m(i)$'s form a neighbourhood base of zero in $\mathrm{End}_{E_i}(E)$ and therefore the Y_m's converge to an element Y such that $Y + \varphi'(Y) + Y\varphi'(Y) = X$. Note that $Y \in \mathcal{B}_j(i)$ since $Y_m \in \mathcal{B}_j(i)$ for all $m \geqslant 1$. Thus $y = 1 + Y$ satisfies $y\varphi'(y) = g$, and we have shown that the map $y \mapsto y\varphi'(y)$ from $P_j(i) \to P_j(i)^{\varphi'}$ is onto.

Let $x_1 \in (\tau P_j(i))^{\varphi}$. Then $\varphi(\tau^{-1}x_1) = \tau(\tau^{-1}x_1)\tau^{-1}$. That is, $\tau^{-1}x_1 \in P_j(i)^{\varphi'}$. By the above, there exists $x_2 \in P_j(i)$ such that $\tau^{-1}x_1 = x_2\varphi'(x_2) = x_2\tau^{-1}\varphi(x_2)\tau$. Set $x = \tau x_2 \tau^{-1}$; then $\varphi(\tau) = \tau$ implies that $x_1 = x\tau\varphi(x)$ and $x \in P_j(i)$. $\square$

4. Shahidi's Reducibility Criterion

In this section we set up a type of integral involving matrix coefficients of a supercuspidal representation and state results of Shahidi relating these integrals to the reducibility of induced representations.

Let $G = \mathrm{GL}_{2n}(F)$. For $f \in C_c^\infty(G)$, we set

$$\mathcal{I}_w(f) = \int_{G/\mathrm{Sp}_{2n}(F)} f(gw^{-1}\,{}^t g w)\,\mathrm{d}\dot{g} = \int_{G/\mathrm{Sp}_{2n}(F)} f(g\varphi(g))\,\mathrm{d}\dot{g},$$

where w is the non-singular skew-symmetric matrix defined at the beginning of Section 3.

The quotient $G/\mathrm{Sp}_{2n}(F)$ can be identified with the set of non-singular skew-symmetric matrices in G. Here, the identification is given by $\dot{g} \mapsto gw^{-1}\,{}^t g$, where

$$\mathrm{Sp}_{2n}(F) = \{g \mid gw^{-1}\,{}^t g = w^{-1}\} = \{g \mid {}^t g w g = w\}.$$

The Haar measure on G induces an invariant measure on the set of non-singular skew-symmetric matrices; it is the canonical additive measure on the coordinates above the diagonal divided by $|\det x|^{n-1/2}$. This measure is invariant under $x \mapsto hx{}^t h$, for any $h \in G$. If $x = gw^{-1}\,{}^t g = g\varphi(g)w^{-1}$, then $hx\,{}^t h = h(g\varphi(g))\varphi(h)w^{-1}$. So replacing $g\varphi(g)$ by $h(g\varphi(g))\varphi(h)$ in the integral has no effect.

Let π be an irreducible supercuspidal representation of G whose central character ω has trivial square; let f_π be a matrix coefficient of π. A function $f \in C_c^\infty(G)$ is said to *represent* f_π if

(i) $f_\pi(x) = \int_Z f(zx)\omega^{-1}(z)\,\mathrm{d}z, \quad x \in G,$

(ii) The function $\widetilde{f}$ defined by

$$\widetilde{f}(x) = \int_Z f(z^2x)\,\mathrm{d}z, \quad x \in G,$$

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viewed as a function on G/Z^2, appears in a subspace of $C_c^\infty(G/Z^2)$ which is equivalent to π (viewed as a representation of G/Z^2). Here Z is the centre of G.

Let G' be one of the split groups $\mathrm{SO}_{4n}(F)$, $\mathrm{SO}_{4n+1}(F)$, or $\mathrm{Sp}_{4n}(F)$. Given a maximal parabolic subgroup $P_{\max}$ of G' having Levi component isomorphic to G, extend π trivially across the unipotent radical to obtain a representation $\pi \otimes 1$ of $P_{\max}$ and set $\mathrm{I}(\pi) = \mathrm{Ind}_{P_{\max}}^{G'}(\pi \otimes 1)$.

Define a skew-symmetric matrix in G by

$$w_0 = \begin{pmatrix} & & & & & 1 \\ & & & & -1 & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot & & & & \\ \cdot & & & & & \end{pmatrix}.$$

For $f \in C_c^\infty(G)$, set $\mathcal{I}_{w_0}(f) = \int_{G/\mathrm{Sp}_{2n}(F)} f(gw_0^{-1}\,{}^t g w_0)\,\mathrm{d}\dot{g}$. Shahidi has proved the following result.

THEOREM 4.1 ([Sh], Theorem 5.3). *Let $G' = \mathrm{SO}_{4n}(F)$ or $\mathrm{Sp}_{4n}(F)$. Let π be an irreducible unitary self-contragredient supercuspidal representation of G. Then $\mathrm{I}(\pi)$ is irreducible if and only if there exists a function $f \in C_c^\infty(G)$ representing a matrix coefficient of π such that $\mathcal{I}_{w_0}(f) \neq 0$. Moreover, $\mathcal{I}_{w_0}(f) \neq 0$ implies that $\omega \,|\, F^\times \equiv 1$.*

REMARK. Because $w = xw_0\,{}^t x$ for some $x \in G$, it follows that $\mathcal{I}_{w_0}(f) = \mathcal{I}_w(f')$ where $f'(g) = f({}^t x g\,{}^t x^{-1})$, $g \in G$. Clearly f represents a matrix coefficient of π if and only if f' does, so $\mathcal{I}_{w_0}$ can be replaced by $\mathcal{I}_w$ in the above theorem.

The reducibility criterion for $\mathrm{SO}_{4n+1}(F)$ is dual to that for $\mathrm{SO}_{4n}(F)$ and $\mathrm{Sp}_{4n}(F)$.

LEMMA 4.2 ([Sh], Theorem 1.2). *If π is as in Theorem 4.1, then $\mathrm{I}(\pi)$ is reducible for $G' = \mathrm{SO}_{4n+1}(F)$ if and only if $\mathrm{I}(\pi)$ is irreducible for $G' = \mathrm{SO}_{4n}(F)$ (or $\mathrm{Sp}_{4n}(F)$).*

As in Sections 2 and 3, let E be a tamely ramified degree $2n$ extension of F, and take θ to be a unitary character of $E^\times$ which is admissible over F and satisfies $\theta^{-1} = \theta \circ \sigma$ for some involution $\sigma \in \mathrm{Aut}(E/F)$. Note that $\theta^2 \,|\, F^\times \equiv 1$. If π is the irreducible supercuspidal representation of G associated to θ, then $\pi = \mathrm{Ind}_H^G \kappa$ for some irreducible representation κ of the open compact-mod-centre subgroup H defined in Section 3.

We consider the finite sum of matrix coefficients of π defined as follows, where χ_κ is the character of κ

$$f_\pi(x) = \begin{cases} \chi_\kappa(x), & \text{if } x \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Writing $\mathrm{e} = \mathrm{e}(E/F)$, we let

$$S = \bigcup_{i=0}^{2e-1} \varpi_E^i (H \cap P).$$

Then we let

$$\mathcal{F}(x) = \begin{cases} f_\pi(x), & \text{for } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a nonzero constant c such that $c\mathcal{F}$ represents f_π; so $\mathcal{I}_w(\mathcal{F}) \neq 0$ if and only if $\mathcal{I}_w(c\mathcal{F}) \neq 0$.

Then we consider $\mathcal{I}_w(\mathcal{F}) = \int_{G/\mathrm{Sp}_{2n}(F)} \mathcal{F}(g\varphi(g))\, \mathrm{d}\dot{g}$. In particular, we will show that, under certain conditions on θ, $\mathcal{I}_w(\mathcal{F})$ is nonzero. As w is fixed we drop the subscript and put $\mathcal{I}(\mathcal{F}) = \mathcal{I}_w(\mathcal{F})$.

Note that $x\varphi(x)$ is φ-invariant, so the integral involves values of χ_κ at φ-invariant points. In later sections, we will study properties of χ_κ on points in H^φ.

5. Preliminary Results

Let the subgroups H, K_i, L_i, etc. be defined as in Section 3. Recall ([H2]) that $\pi = \mathrm{Ind}_H^G \kappa$ where κ is an irreducible representation of H. The representation κ is a tensor product $\kappa = \kappa_1 \otimes \cdots \otimes \kappa_r$, where κ_i is defined using the character θ_i. The representation κ_i is first defined on $E^\times K_{i-1}$ and then extended across L_{i-1} by $\psi(\mathrm{tr}(c_i(\cdot \Leftrightarrow 1)))$ to get a representation on all of $H = E^\times K_{i-1} L_{i-1}$.

If $f_E(\theta_r) = 1$, then κ_r is defined in terms of a certain cuspidal representation of $P(r \Leftrightarrow 1)/P_1(r \Leftrightarrow 1)$ parametrized by θ_r. This case will be discussed in Sections 9 and 10.

We remind the reader that $m_i = [(f_E(\theta_i \circ N_{E/E_i}) + 1)/2]$ and $\ell_i = [(f_E(\theta_i \circ N_{E/E_i}))/2]$, $1 \leqslant i \leqslant r$. Let $\det_i$ be the determinant on $M_i = \mathfrak{gl}_{[E:E_i]}(E_i)$. If $i \leqslant r \Leftrightarrow 1$, or if $i = r$ and $f_E(\theta_r) > 1$, define a character ω_i of $E^\times K_i P_{m_i}(i \Leftrightarrow 1) L_{i-1} \subset H$ by

$$\omega_i \mid E^\times K_i = \theta_i \circ \det_i, \quad \text{and}$$

$$\omega_i \mid P_{m_i}(i \Leftrightarrow 1) L_{i-1} = \psi(\mathrm{tr}(c_i(\cdot \Leftrightarrow 1))).$$

The condition $2m_i \geqslant f_E(\theta_i \circ N_{E/E_i})$ guarantees that the two definitions coincide on the intersection $E^\times K_i \cap P_{m_i}(i \Leftrightarrow 1)L_{i-1}$ ([H2]).

The conductoral exponent $f_E(\theta_i \circ N_{E/E_i})$ is even if and only if $m_i = \ell_i$. In this case, $E^\times K_i P_{m_i}(i \Leftrightarrow 1) = E^\times K_i$ and $\kappa_i = \omega_i$. In particular, $\dim \kappa_i = 1$. Otherwise, $m_i = \ell_i + 1$ and a Heisenberg construction is used to define κ_i on $E^\times K_i$ (see Section 6) and $\dim \kappa_i > 1$.

LEMMA 5.1. *Suppose that E is ramified over L. Then $f_E(\theta_r) > 1$.*

Proof. Suppose that $f_E(\theta_r) = 1$. Then $\theta_r|(1 + \mathfrak{p}_E) \equiv 1$. Because E is ramified over L, $\mathcal{O}_E^\times \subset \mathcal{O}_L^\times(1 + \mathfrak{p}_E)$. Thus $\theta_r(\mathcal{O}_E^\times) \subset \theta_r(\mathcal{O}_L^\times)$.

Because $\theta_r^\sigma = \theta_r^{-1}$, $\theta_r \mid L^\times$ has trivial square. Therefore $\theta_r(\mathcal{O}_E^\times) \subset \{\pm 1\}$ and θ_r is not generic over E_{r-1}. This contradiction finishes the proof. □

Next we establish some notation that will help us work with ω_i. Recall the function $\nu(\cdot)$ defined in Section 3: $\nu(x) = j$, where $x \in \mathcal{B}_j \setminus \mathcal{B}_{j+1}$. If E is unramified over L and $x \in H^\varphi$, let

$$\mu(x) = \begin{cases} 1, & \text{if } \ \nu(x) \ \text{ is even,} \\ \varpi_L, & \text{otherwise.} \end{cases}$$

Note that if E is unramified over L and $x \in yP_0$, then $\nu(x) = \nu(y)$, so $\mu(x) = \mu(y)$.

If E is ramified over L, fix a root of unity $\xi \in L$ that is not in $N_{E/L}(E)$. If $x \in H^\varphi$, then, by the above lemma and Corollary 3.8, with $i = r$, $x \in L^\times P_1$. Let

$$\mu(x) = \begin{cases} 1, & \text{if } \ x \in N_{E/L}(E^\times)P_1, \\ \xi, & \text{otherwise.} \end{cases}$$

LEMMA 5.2. *Suppose that $x \in E^\times K_i P_{m_i}(i \Leftrightarrow 1)L_{i-1}$ and $\varphi(x) = x$. If $f_E(\theta_r) = 1$, make the additional assumption that $x \in E^\times P_1$.*

Then $\omega_i(x) = \theta_i \circ N_{E/E_i}(\mu(x))$.

REMARK. In the case $f_E(\theta_r) = 1$, the above result may not hold for certain points in $E^\times P$ (see Lemma 10.2).

Proof. A minor variant of Corollary 3.8 shows that it is possible to write $x = yz$, with $y \in E^\times K_i$ and $z \in P_{m_i}(i \Leftrightarrow 1)L_{i-1}$, and such that $\varphi(y) = y$. Since x is also φ-fixed, a simple calculation shows that $\varphi(z) = yzy^{-1}$.

Since $\varphi(c_i) = \sigma(c_i) = \Leftrightarrow c_i$, we find that

$$\varphi(c_i(z \Leftrightarrow 1)) = \varphi(z \Leftrightarrow 1)\varphi(c_i) = (yzy^{-1} \Leftrightarrow 1)(\Leftrightarrow c_i),$$

so because $y \in M_i$ commutes with c_i, $\mathrm{tr}(c_i(z \Leftrightarrow 1)) = \mathrm{tr}\,\varphi(c_i(z \Leftrightarrow 1)) = \Leftrightarrow \mathrm{tr}(c_i(z \Leftrightarrow 1))$, and $\mathrm{tr}(c_i(z \Leftrightarrow 1)) = 0$. This shows that $\omega_i(x) = \theta_i(\det_i(y))$.

Since $y \in E^\times K_i$ and $\varphi(y) = y$, we can assume that $y \in L^\times K_i$. Write $y = tv$, with $t \in L^\times$ and $v \in K_i \subset P_1(i)$. Since $\varphi(tv) = tv$, we find that

$\varphi(v) = tvt^{-1}$ and $\varphi(\det_i(v)) = \det_i(\varphi(v)) = \det_i(v)$. So $\det_i(v) \in L$. But since $v \in P_1(i)$, we have $\det_i(v) \in 1 + \mathfrak{p}_{E_i \cap L}$. Since $1 + \mathfrak{p}_{E_i \cap L} = N_{E_i/(E_i \cap L)}(1 + \mathfrak{p}_{E_i})$, we find that $\theta_i|_{1+\mathfrak{p}_{E_i \cap L}} = \theta_i \circ N_{E_i/(E_i \cap L)}|_{1+\mathfrak{p}_{E_i}} \equiv 1$. So $\theta_i(\det_i(v)) = 1$ and $\omega_i(x) = \theta_i(\det_i(y)) = \theta_i(\det_i(tv)) = \theta_i(\det_i(t))$. This reduces us to considering $x \in L^{\times}$.

Whether or not E is ramified over L, $x \in \mu(x) N_{E/L}(E^{\times})$. The result follows from the observation that $\theta_i \circ N_{E/E_i}$ is trivial on $N_{E/L}(E^{\times})$ by Lemma 2.5(ii). □

PROPOSITION 5.3. *If* $\dim \kappa = 1$ *and if* $\theta \mid L^{\times} \equiv 1$, *then* $\mathcal{I}(\mathcal{F}) > 0$.

Proof. Since $\dim \kappa = 1$, then for each i, $\dim \kappa_i = 1$ and $m_i = \ell_i$. So $E^{\times} K_i P_{m_i}(i \Leftrightarrow 1) = E^{\times} K_{i-1}$ and $\kappa_i = \omega_i$.

Now, with the function $\mathcal{F}$ defined as in Section 4

$$\mathcal{I}(\mathcal{F}) = \int_{G/\mathrm{Sp}_{2n}(F)} \mathcal{F}(gw^{-1}\,{}^t g w)\,\mathrm{d}\dot{g} = \int_{G/\mathrm{Sp}_{2n}(F)} \mathcal{F}(g\varphi(g))\,\mathrm{d}\dot{g}.$$

The support of $\mathcal{F}$ is $S = \bigcup_{i=0}^{2e-1} \varpi_E^i (H \cap P) \subset H$. Since $\varphi(g\varphi(g)) = g\varphi(g)$, the above lemma applies with $x = g\varphi(g)$ and we find that whenever $g\varphi(g) \in S$,

$$\begin{aligned} \mathcal{F}(g\varphi(g)) &= f_{\pi}(g\varphi(g)) = \chi_{\kappa}(g\varphi(g)) = \prod_{i=1}^{r} \kappa_i(g\varphi(g)) \\ &= \prod_{i=1}^{r} \theta_i \circ N_{E/E_i}(\mu(g\varphi(g))) = \theta(\mu(g\varphi(g))). \end{aligned}$$

Since $\mu(g\varphi(g)) \in L^{\times}$, we see that $\theta(\mu(g\varphi(g))) = 1$ and the integrand is positive. Now for large j, the φ-fixed elements of P_j have positive measure, so the integral is positive. □

The following results allow us to identify certain cases where $\dim \kappa = 1$.

LEMMA 5.4. *Suppose that* $F \subset N_1 \subset N_2 \subset E$, $\sigma(N_h) = N_h$, $h = 1, 2$, *but* $\sigma|N_h \not\equiv 1$. *Assume that* N_2 *is ramified over* $N_2 \cap L$. *Then* N_1 *is ramified over* $N_1 \cap L$ *and* $\mathrm{e}(N_2/N_1)$ *is odd.*

Proof. Suppose N_1 is unramified over $N_1 \cap L$. By the uniqueness of unramified extensions, $f((N_2 \cap L)/(N_1 \cap L))$ is odd. But $f(N_2/(N_1 \cap L)) = 2f(N_2/N_1)$ and $f(N_2/(N_1 \cap L)) = f((N_2 \cap L)/(N_1 \cap L))$, since N_2 is ramified over $N_2 \cap L$. Therefore, N_1 must be ramified over $N_1 \cap L$.

Now suppose $\mathrm{e}(N_2/N_1)$ is even. Let M be the maximal unramified extension of $N_1 \cap L$ contained in $N_2 \cap L$. Then $\mathrm{e}(N_2 \cap L/M) = \mathrm{e}(N_2 \cap L/N_1 \cap L) = \mathrm{e}(N_2/N_1)$, so $\mathrm{e}(N_2 \cap L/M)$ is even.

We can write $N_1 = (N_1 \cap L)(\sqrt{\varpi_{N_1 \cap L}})$ for some prime element $\varpi_{N_1 \cap L}$ in $N_1 \cap L$. Since $\mathrm{e}(M/N_1 \cap L) = 1$, we can assume that $\varpi_M = \varpi_{N_1 \cap L}$.

Since $\mathrm{e}(L/M)$ is even and L is totally ramified over M, there must exist a quadratic ramified extension N of M contained in $N_1 \cap L$. Since $\sqrt{\varpi_{N_1 \cap L}} = \sqrt{\varpi_M} \notin L$, we must have $N = M(\sqrt{\varepsilon_M}\sqrt{\varpi_M})$ for some non-square $\varepsilon_M \in \mathcal{O}_M^\times$.

But since $\sqrt{\varpi_M}$ and $\sqrt{\varepsilon_M}\sqrt{\varpi_M} \in N_2$, we find that $\sqrt{\varepsilon_M} \in N_2$. Since $\sqrt{\varepsilon_M} \notin M$, $f(E/M)$ must be even. In particular $f(N_2/M) > 1$.

But by the definition of M, N_2 is ramified over M, contradicting $f(N_2/M) > 1$. Therefore $\mathrm{e}(N_2/N_1)$ cannot be even. □

LEMMA 5.5. *Suppose E/L is ramified. Then* $\dim \kappa = 1$.

Proof. If $\dim \kappa_i = 1$ for each i, then $\dim \kappa = 1$. But $\dim \kappa_i = 1$ is equivalent to saying that there is no Heisenberg construction for κ_i since $f_E(\theta_r) > 1$, by Lemma 5.1.

Lemma 5.4 shows that E_i is ramified over $E_i \cap L$, for $1 \leqslant i \leqslant r$, and that $\mathrm{e}(E/E_1)$ is odd.

Since $\mathrm{e}(E_i/E_i \cap L) = 2$ and $\sigma(c_i) = \Leftrightarrow c_i$, we see that c_i must generate E_i over $E_i \cap L$. This means $c_i \in \mathfrak{p}_{E_i}^t \setminus \mathfrak{p}_{E_i}^{t+1}$ for some t, which must be odd. But $c_i \in \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+1} \setminus \mathfrak{p}_{E_i}^{-f_{E_i}(\theta_i)+2}$. Therefore $f_{E_i}(\theta_i) \Leftrightarrow 1$ is odd.

Combining these facts, we see that

$$f_E(\theta_i \circ N_{E/E_i}) \Leftrightarrow 1 = \mathrm{e}(E/E_i)(f_{E_i}(\theta_i) \Leftrightarrow 1) \text{ is odd.}$$

Thus $f_E(\theta_i \circ N_{E/E_i})$ is even and a Heisenberg construction is not necessary for κ_i. □

COROLLARY 5.6. *If a Heisenberg construction is required for one of the κ_i's, then E is unramified over L.*

For future reference, we include the following result.

LEMMA 5.7. *If* $f_E(\theta_r) = 1$ *and* $\mathrm{e}(E_{r-1}/(E_{r-1} \cap L)) = 2$, *then* $\dim \kappa_i = 1$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$.

Proof. Let $1 \leqslant i \leqslant r \Leftrightarrow 1$. By Lemma 5.4, $\mathrm{e}(E_i/(E_i \cap L)) = 2$ and $\mathrm{e}(E_{r-1}/E_i)$ is odd. Because $f_E(\theta_r) = 1$, E is unramified over E_{r-1}. Therefore $\mathrm{e}(E/E_i) = \mathrm{e}(E_{r-1}/E_i)$ is odd. As shown above, $\mathrm{e}(E_i/(E_i \cap L)) = 2$ implies that $f_{E_i}(\theta_i) \Leftrightarrow 1$ is odd. Thus $f_E(\theta_i \circ N_{E/E_i}) \Leftrightarrow 1$ is odd. □

6. The Heisenberg construction: part one

Fix i such that $1 \leqslant i \leqslant r$. Suppose that $f_E(\theta_i \circ N_{E/E_i})$ is odd and greater than one. Then a Heisenberg construction is required for the representation κ_i. In this section and the next, we compute the sign of the character value of κ_i at certain φ-invariant elements in the inducing subgroup H. By Lemma 5.1, we must assume that E is unramified over L. In [Mo2], Moy assumes that p does not divide $2n$. The results from [Mo2] which we use still hold under our assumptions; that is, when p is odd and does not divide the ramification degree $\mathrm{e} = \mathrm{e}(E/F)$.

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Let the subgroups H, K_i, L_i, etc., be as defined in Section 3. As we are assuming that $f_E(\theta_i \circ N_{E/E_i})$ is odd,

$$m_i = \ell_i + 1 = \frac{(f_E(\theta_i \circ N_{E/E_i}) + 1)}{2}.$$

Set $H_i = F^\times(1+\mathfrak{p}_E)(K_i P_{\ell_i}(i \Leftrightarrow 1) \cap P_1)$. Then

$$H_i = \begin{cases} F^\times(1+\mathfrak{p}_E)K_i P_{\ell_i}(i \Leftrightarrow 1), & \text{if } f_E(\theta_r) > 1, \\ F^\times P_1(r \Leftrightarrow 1)P_{\ell_{r-1}}(r \Leftrightarrow 2)\dots P_{\ell_i}(i \Leftrightarrow 1), & \text{if } f_E(\theta_r) = 1. \end{cases}$$

Set $H_i' = F^\times(1+\mathfrak{p}_E)(K_i P_{m_i}(i \Leftrightarrow 1) \cap P_1)$. Let ω_i be the character of $E^\times K_i P_{m_i}(i \Leftrightarrow 1)L_{i-1}$ defined in Section 5. As ω_i does not extend to a character of H, a Heisenberg construction is used to produce an irreducible representation κ_i of H. The technical difficulties occur in defining κ_i on $E^\times K_{i-1} = E^\times K_i P_{\ell_i}(i \Leftrightarrow 1)$. After that, if $i \geqslant 2$, κ_i is extended by $\psi(\mathrm{tr}(c_i(\cdot \Leftrightarrow 1)))$ on L_{i-1} to produce a representation of H. Let x be a φ-invariant element of H. By Corollary 3.8, we may write $x = yz$ with $y \in E^\times K_{i-1}$ such that $\varphi(y) = y$ and $z \in L_{i-1}$. Arguing as in the proof of Lemma 5.2, we see that $\varphi(z) = yzy^{-1}$ implies $\mathrm{tr}(c_i(z \Leftrightarrow 1)) = 0$. Therefore, denoting the character of κ_i by χ_i, it follows that $\chi_i(x) = \chi_i(y)$. That is, it suffices to compute χ_i on φ-invariant elements in $E^\times K_{i-1}$. In this section, we deal with the φ-invariant elements in $E^\times H_i$. We remark that if $f_E(\theta_r) > 1$, then $E^\times H_i = E^\times K_{i-1}$. For the purposes of this paper, we do not require values of χ_i when $f_E(\theta_r) = 1$, but, as the proofs do not differ (for points in $E^\times H_i$), in this section and the next we do not place a restriction on $f_E(\theta_r)$.

We now discuss the construction of κ_i (see Sections 3.5–6 of [Mo2] for more details). Both H_i and H_i' are normal subgroups of $E^\times K_{i-1}$. The quotient H_i/H_i' can be made into a symplectic vector space over $\mathbb{F}_p$ by defining

$$\langle x', y' \rangle = \omega_i(x^{-1}y^{-1}xy), \quad x', y' \in H_i/H_i',$$

where x and y are representatives for the cosets x' and y', respectively. The conjugation action of $E^\times H_i$ preserves the symplectic form $\langle \cdot, \cdot \rangle$. This is used to translate to the setting of [H1]. The induced representation $\mathrm{Ind}_{H_i'}^{H_i}\omega_i$ is a multiple of a single irreducible representation κ_i' ([H1]). As indicated in [H1], the oscillator (Weil) representation singles out a unique extension of κ_i' to $E^\times H_i$ parametrized by the character ω_i on $E^\times H_i'$. In particular, the extension κ_i has the property that if $x \in E^\times H_i'$, then $\chi_i(x)$ is equal to $\pm\omega_i(x)$ times the square root of the order of the subspace of H_i/H_i' fixed by x (Proposition 2 of [H1]). In addition, χ_i vanishes on all elements of $E^\times H_i$ whose conjugacy class does not intersect $E^\times H_i'$.

In the process of calculating ε-factors, Moy computes certain of the values of χ_i. In the simplest case, when $i = r = 1$ and n is prime (see Section 3.5 of [Mo2]), Moy shows that all but one of the extensions of κ_i' to $E^\times H_i$ have the same multiplicity in the induced representation $\mathrm{Ind}_{E^\times H_i'}^{E^\times H_i}\omega_i$, and he computes the character of the exceptional component. From the properties of its character (see remarks above), it follow that this component is actually κ_i.

Moy handles the general case as follows (see [Mo2] (3.6.30), (3.6.31)). The vector space $V_i = H_i/H_i'$ decomposes into a direct sum of subspaces $V_i(N)$, where N runs over all subfields of E/E_{i-1} which do not contain E_i

$$V_i = \oplus_N V_i(N).$$

Let $H_i(N)$ be the inverse image of $V(N)$ in H_i. For each N, Moy constructs a representation κ_i^N of $E^\times H_i(N)$. The character χ_i^N of κ_i^N is computed via the same type of argument as in the above mentioned case, and χ_i^H satisfies [Mo2] (3.6.51)

$$\chi_i^N(x) = \begin{cases} 0 & \text{if } \ x \text{ is not conjugate to an element of } \ E^\times H_i', \\ q_{E_{i-1}}^{D(N)}\omega_i(x), & \text{if } \ x \in N^\times H_i', \\ \mathrm{sgn}(N)\omega_i(x), & \text{if } \ x \in E^\times H_i' \Leftrightarrow N^\times H_i'. \end{cases}$$

Here $q_{E_{i-1}}$ is the cardinality of the residue class field of E_{i-1}, $D(N)$ is a positive integer, and $\mathrm{sgn}(N) = \pm 1$. The representation $\kappa_i \mid E^\times H_i$ is a central tensor product of the κ_i^N's as N runs through those intermediate fields containing E_{i-1} but not containing E_i [Mo2] (3.6.31).

LEMMA 6.1 ([Mo2]). *Let $x \in E^\times H_i$. Then*

$$\chi_i(x) \ = \ q_{E_{i-1}}^{\Sigma_{\{N \mid x \in N^\times H_i'\}} D(N)} \left(\prod_{\{N \mid x \notin N^\times H_i'\}} \mathrm{sgn}(N) \right) \omega_i(x),$$

$$\textit{if } \ x \in E^\times H_i'$$

and $\chi_i(x) = 0$ if x is not conjugate to an element of $E^\times H_i'$.

Let $x \in (E^\times H_i)^\varphi$. If $x \in E^\times H_i'$ then by Lemma 5.2, $\omega_i(x) = \theta_i(N_{E/E_i}(\mu(x)))$. If $x \notin E^\times H_i'$ but $y^{-1}xy \in E^\times H_i'$ for some $y \in E^\times H_i$, then $\chi_i(x)$ is a multiple of $\omega_i(y^{-1}xy)$. The element $y^{-1}xy$ is not necessarily φ-invariant. Our goal is to show that $\chi_i(x)$ is real valued and to determine its sign. To do this, we must evaluate $\omega_i(y^{-1}xy)$ and determine the sign $\prod_{\{N \mid y^{-1}xy \notin N^\times H_i'\}} \mathrm{sgn}(N)$.

The next part of this section is devoted to computing $\omega_i(y^{-1}xy)$. Recall that $M_i = \mathfrak{gl}_{[E:E_i]}(E_i)$, $0 \leqslant i \leqslant r$. Let tr_i and $\det_i$ denote the trace and determinant on M_i. For $1 \leqslant i \leqslant r$, set

$$M_i^\perp = \{X \in M_{i-1} \,|\, \mathrm{tr}_{i-1}(XY) = 0 \ \ \forall Y \in M_i\}.$$

LEMMA 6.2. *Let $1 \leqslant i \leqslant r$. Then $\varphi(M_i^\perp) = M_i^\perp$.*

Proof. Note that it follows Lemma 3.4(ii) that $\mathrm{tr}_{i-1}(\varphi(Y)) = \sigma(\mathrm{tr}_{i-1}(Y))$ for $Y \in M_{i-1}$.

The equality

$$\mathrm{tr}_{i-1}(\varphi(X)Y) = \sigma(\mathrm{tr}_{i-1}(\varphi(Y)X))$$

and the fact that $\varphi(M_i) = M_i$ (Corollary 3.5(ii)) yield the desired result. □

COROLLARY 6.3. *Let* $1 \leqslant i \leqslant r$. *Then* $\varphi(\mathcal{B}_j(i \Leftrightarrow 1) \cap M_i^{\perp}) = \mathcal{B}_j(i \Leftrightarrow 1) \cap M_i^{\perp}$.

LEMMA 6.4. *Let* $1 \leqslant i \leqslant r$. *Assume that* $m_i = \ell_i + 1$. *That is, a Heisenberg construction is needed for* κ_i. *Let* $x \in E^{\times}K_iP_{\ell_i}(i \Leftrightarrow 1)$ *be such that* $\varphi(x) = x$ *and* $y^{-1}xy \in E^{\times}K_iP_{m_i}(i \Leftrightarrow 1)$ *for some* $y \in E^{\times}K_iP_{\ell_i}(i \Leftrightarrow 1)$. *Then* $\omega_i(y^{-1}xy) = \theta_i(N_{E/E_i}(\mu(x)))$.

Proof. Write $y^{-1}xy = u(1+W)$, $u \in E^{\times}K_i$, $W \in \mathcal{B}_{m_i}(i \Leftrightarrow 1)$. Because ([H2])

$$\mathcal{B}_j(i \Leftrightarrow 1) = (M_i \cap \mathcal{B}_j(i \Leftrightarrow 1)) \oplus (M_i^{\perp} \cap \mathcal{B}_j(i \Leftrightarrow 1)), \quad j \in \mathbb{Z},$$

we may write $W = W_1 + W_2$, where $W_1 \in M_i \cap \mathcal{B}_{m_i}(i \Leftrightarrow 1)$, $W_2 \in M_i^{\perp} \cap \mathcal{B}_{m_i}(i \Leftrightarrow 1)$. As $M_i^{\perp}$ is invariant under multiplication by elements of M_i, and $(1+W_1)^{-1} \in P_{m_i}(i)$, it follows that $(1+W_1)^{-1}W_2 \in M_i^{\perp} \cap \mathcal{B}_{m_i}(i \Leftrightarrow 1)$. After replacing u by $u(1+W_1)$ and $1+W$ by $1+(1+W_1)^{-1}W_2$, we assume without loss of generality that $W \in M_i^{\perp}$. Thus $\omega_i(y^{-1}xy) = \theta_i \circ \mathrm{det}_i(u)$.

Next we observe that $\omega_i(y^{-1}xy) = \omega_i(\varphi(y^{-1}xy))$ and use this to show that $\theta_i(\mathrm{det}_i(u)) = \pm 1$. As $E^{\times}K_iP_{m_i}\ (i \Leftrightarrow 1)$ is a φ-invariant set, we have

$$\varphi(y^{-1}xy) = \varphi(y)x\varphi(y)^{-1} \in E^{\times}K_iP_{m_i}(i \Leftrightarrow 1).$$

The character ω_i is constant on the set of conjugates of x which lie in $E^{\times}K_iP_{m_i}(i \Leftrightarrow 1)$. Therefore $\omega_i(y^{-1}xy) = \omega_i(\varphi(y)x\varphi(u)^{-1})$.

Observe that

$$\varphi(y^{-1}xy) = \varphi(u(1+W)) = \varphi(u)(1+\varphi(u)^{-1}\varphi(W)\varphi(u)).$$

From $\varphi(u) \in E^{\times}K_i \subset M_i, W \in \mathcal{B}_{m_i}(i \Leftrightarrow 1) \cap M_i^{\perp}$ and Corollary 6.3, it follows that $\varphi(u)^{-1}\varphi(W)\varphi(u) \in \mathcal{B}_{m_i}(i \Leftrightarrow 1) \cap M_i^{\perp}$. Thus, using Lemma 3.4 and the properties of θ_i (Lemma 2.5), we get

$$\omega_i(\varphi(y^{-1}xy)) = \theta_i \circ \mathrm{det}_i(\varphi(u)) = \theta_i \circ \sigma(\mathrm{det}_i(u)) = \theta_i(\mathrm{det}_i(u))^{-1}.$$

Equality of ω_i at $y^{-1}xy$ and $\varphi(y)x\varphi(y)^{-1}$ yields $\theta_i(\mathrm{det}_i(u)) = \theta_i(\mathrm{det}_i(u))^{-1}$.

We want to show that $\theta_i(\mathrm{det}_i(u))$ must equal $\theta_i(N_{E/E_i}(\mu(x)))$. Using Lemma 3.6, write $x = vz$, $v \in E^{\times}K_i$ such that $\varphi(v) = v$, and $z \in P_{\ell_i}(i \Leftrightarrow 1)$. Now $y^{-1}xy = (y^{-1}vy)(y^{-1}zy) \in v'P_{\ell_i}(i \Leftrightarrow 1)$ for some conjugate v' of v in

$E^\times K_i$. Observe that $\varphi(v) = v$ implies that $\det_i(v') = \det_i(v) \in E_i \cap L$. We have $u(1+W) \in v' P_{\ell_i}(i \Leftrightarrow 1)$. Hence $u \in v' P_{\ell_i}(i)$. Therefore

$$\det_i(u) \in \det_i(v) \det_i(P_{\ell_i}(i)) \subset \det_i(v)(1+\mathfrak{p}_{E_i}).$$

From $\det_i(v) \in E_i \cap L$ and the fact that the square of $\theta_i \mid (E_i \cap L)^\times$ is trivial, it follows that $\theta_i(\det_i(v)) = \pm 1$. We have shown above that $\theta_i(\det_i(u)) = \pm 1$. Thus

$$\theta_i(\det_i(u)) \in \theta_i(N_{E/E_i}(\mu(v)))(\theta_i(1+\mathfrak{p}_{E_i}) \cap \{\pm 1\}).$$

As θ_i is trivial on $1+\mathfrak{p}_{E_i}^{f_{E_i}(\theta_i)}$ and $(1+\mathfrak{p}_{E_i})/(1+\mathfrak{p}_{E_i}^{f_{E_i}(\theta_i)})$ is a p-group, oddness of p does not allow θ_i to take the value $\Leftrightarrow 1$ on $1+\mathfrak{p}_{E_i}$. Therefore $\theta_i(\det_i(u)) = \theta_i(N_{E/E_i}(\mu(v)))$. Observe that $\mu(x) = \mu(v)$. Thus $\omega_i(y^{-1}xy) = \theta_i(N_{E/E_i}(\mu(x)))$. $\square$

Set

$$\mathcal{S}_i = \{N \mid E_{i-1} \subset N, E_i \not\subset N\}.$$

Suppose that $x \in E^\times H_i \Leftrightarrow E^\times H_i'$ is such that $y^{-1}xy \in E^\times H_i'$ for some $y \in E^\times H_i$. We want to compute the quantity

$$\prod_{\{N \in \mathcal{S}_i \mid y^{-1}xy \notin N^\times H_i'\}} \operatorname{sgn}(N).$$

Suppose that $F \subset N \subset E$. Let ζ_N denote the set of roots of unity in N of order prime to p. We assume that a uniformizer $\varpi_N \in N$ is chosen so that $\varpi_N^{\mathrm{e}(N/F)} \in \varpi\zeta_F$, where ϖ is a uniformizer in F. Let C_N be the subgroup of $N^\times$ generated by ϖ_N and ζ_N.

LEMMA 6.5. *Let x and y be as above. Assume that $\varphi(x) = x$. Then there exists a unique $c_L(x) \in C_L$ such that $x \in c_L(x)(H_i \cap P_1)$. Furthermore, given any subfield N of E containing F,*

$$y^{-1}xy \in N^\times H_i' \iff c_L(x) \in N^\times.$$

Proof. By Lemma 3.6, there exists $u \in L^\times$ such that $x \in u(H_i \cap P_1)$. By [H2], p. 438, there exists a unique $c_L(x) \in C_L$, the 'standard representative' of x, such that $u \in c_L(x)(1+\mathfrak{p}_L)$. Since $1+\mathfrak{p}_L \subset H_i \cap P_1$, we have

$$x \in c_L(x)(1+\mathfrak{p}_L)(H_i \cap P_1) = c_L(x)(H_i \cap P_1).$$

Set $z = c_L(x)^{-1}x$. Let N be an intermediate extension. Then

$$y^{-1}xy = c_L(x)(c_L(x)^{-1}y^{-1}xy) = c_L(x)(c_L(x)^{-1}y^{-1}c_L(x)y)(y^{-1}zy)$$

and $c_L(x)^{-1}y^{-1}c_L(x)y$, $y^{-1}zy \in H_i \cap P_1$. Together with $y^{-1}xy \in E^\times H_i'$, this implies that $c_L(x)^{-1}(y^{-1}xy) \in H_i' \cap P_1$. Therefore

$$y^{-1}xy \in N^\times H_i' \Longleftrightarrow c_L(x) \in N^\times(H_i' \cap P_1) \cap E^\times = N^\times(1+\mathfrak{p}_E).$$

The standard representative of an element v of $N^\times(1+\mathfrak{p}_E)$ in $E^\times$ is just the standard representative in N of any $v' \in N^\times$ such that $v \in v'(1+\mathfrak{p}_E)$. By uniqueness of standard representative, it follows that $c_L(x) \in N^\times(1+\mathfrak{p}_E)$ if and only if $c_L(x) \in C_N \subset N^\times$. □

By the above lemma, we must determine

$$\mathrm{sgn}(\alpha) \stackrel{\text{def}}{=\!=} \prod_{\{N \in \mathcal{S}_i \mid \alpha \notin N^\times\}} \mathrm{sgn}(N), \quad \alpha \in C_L.$$

This will be done in the next section.

7. The Heisenberg construction: part two – computing signs

Let the notation be as in the previous section. We continue to assume that $m_i = \ell_i + 1$, and $f_E(\theta_r) > 1$ if $i = r$. In this section we compute sgn(α) for $\alpha \in C_L$. In Proposition 7.12, we give a formula for the character χ_i on φ-fixed elements in $E^\times H_i$.

We begin with a brief summary of definitions and results from [Mo2] which will be used later. We remark that results in [Mo2] are stated for the case $i = 1$, that is, $E_{i-1} = F$. To apply them, we must replace F by E_{i-1}. Recall that

$$\begin{aligned} V_i = H_i/H_i' &\simeq P_{\ell_i}(i \Leftrightarrow 1)/P_{\ell_i}(i)P_{\ell_i+1}(i \Leftrightarrow 1) \\ &\simeq \mathcal{B}_{\ell_i}(i \Leftrightarrow 1)/(\mathcal{B}_{\ell_i}(i) + \mathcal{B}_{\ell_i+1}(i \Leftrightarrow 1)). \end{aligned}$$

Given a subfield N of E/E_{i-1}, let $R(N)$ be the residue class field of N, let $\mathcal{B}_j^N$ be the set of matrices in $\mathcal{B}_j$ which commute with N. Note that $E_{i-1} \subset N$ implies that $\mathcal{B}_j^N \subset \mathcal{B}_j(i \Leftrightarrow 1)$. Set

$$\Omega_i(N) = (\mathcal{B}_{\ell_i}^N + \mathcal{B}_{\ell_i+1}(i \Leftrightarrow 1))/\mathcal{B}_{\ell_i+1}(i \Leftrightarrow 1) \simeq \mathcal{B}_{\ell_i}^N/\mathcal{B}_{\ell_i+1}^N.$$

The set $\Omega_i(N)$ is an $R(N)$-vector space and a $U_i = E^\times/E_{i-1}^\times(1+\mathfrak{p}_E)$-module. For future reference, we note that

$$\dim_{R(N)} \Omega_i(N) = \mathrm{e}(E/N)f(E/N)^2 = f(E/N)[E:N]. \tag{7.1}$$

The set $V_i(N)$ is defined to be the U_i-complement in $\Omega_i(N)$ of the $R(N)U_i$-module

$$\sum_{\{M \mid N \subset M \subset E\}} \Omega_i(M).$$

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Define $A(N) = \dim_{R(N)}(V_i(N))/2$. If $N \subset \mathcal{S}_i$, then by [Mo2] (3.6.45), $V_i(N)$ can be identified with a subspace of V_i. The following result [Mo2] (3.6.43) is useful for computing dimensions

$$\Omega_i(N) = V_i(N) \bigoplus \bigoplus_{\{M \mid N \subset M \subset E, N \neq M\}} V_i(M). \tag{7.2}$$

LEMMA 7.3 ([Mo2] Proposition 3.6.55, 3.6.60). *Let $N \in \mathcal{S}_i$.*

(i) *If $f(E/N) > 2$, then* $\mathrm{sgn}(N) = 1$.
(ii) *If $f(E/N) = 2$, then* $\mathrm{sgn}(N) = (-1)^{A(N)}$.
(iii) *If $f(E/N) = 1$ and $f(E/E_{i-1})$ is even, then* $\mathrm{sgn}(N) = 1$.
(iv) *If $f(E/N) = 1$ and $f(E/E_{i-1})$ is odd, then*
  (a) *If $[E:N]$ is divisible by two distinct odd primes, or by 4 and an odd prime, then* $\mathrm{sgn}(N) = 1$.
  (b) *If $[E:N] = \ell^r$ or $2\ell^r$ for some odd prime ℓ, then* $\mathrm{sgn}(N)$ *is the Legendre symbol* $\left(\frac{q_{E_{i-1}}}{\ell}\right)$.
  (c) *If $[E:N] = 2^m$, then $m \geqslant 2$ and*

$$\mathrm{sgn}(N) = \begin{cases} 1 & \textit{if } m > 2, \\ 1 & \textit{if } m = 2 \textit{ and } q_{E_{i-1}} \equiv 1 \bmod 4, \\ -1 & \textit{if } m = 2 \textit{ and } q_{E_{i-1}} \equiv -1 \bmod 4. \end{cases}$$

REMARK. In general, $\mathrm{sgn}(N)$ depends on i. Therefore, so does $\mathrm{sgn}(\alpha)$, $\alpha \in C_L$.

Recall that E must be unramified over L by Lemma 5.1. Let σ be the nontrivial element of $\mathrm{Gal}(E/L)$. The notation L_{un} will be used to denote the unramified extension of F of degree $f(E/F)/2$. Choose $\varepsilon \in \zeta_{L_{un}}$ such that ε is not a square in L_{un}. Then $E = L(\sqrt{\varepsilon})$. Let ϖ_L be a uniformizer in L.

LEMMA 7.4. *Let $N \in \mathcal{S}_i$ be such that $f(E/N) = 1$. Then* $\mathrm{sgn}(N) = 1$.

Proof. If $i = 1$, then $f(E/E_{i-1}) = f(E/F)$ is even, so by Lemma 7.3(iii), $\mathrm{sgn}(N) = 1$. Similarly if $i > 1$ and $f(E/E_{i-1})$ is even.

Assume that $i > 1$ and $f(E/E_{i-1})$ is odd. Then $f(E/(E_{i-1} \cap L)) = 2\,f(L/(E_{i-1} \cap L)) = f(E/E_{i-1})f(E_{i-1}/(E_{i-1} \cap L))$ implies that E_{i-1} is unramified over $E_{i-1} \cap L$. Thus $q_{E_{i-1}} = (q_{E_{i-1} \cap L})^2$ and $q_{E_{i-1}} \equiv 1 \bmod 4$. Apply Lemma 7.3(iv) to complete the proof. □

LEMMA 7.5. *Assume that* $\mathrm{e}(E/F)$ *is even and $F \subset N \subset E$. Let $L' = L_{un}(\varpi_L\sqrt{\varepsilon})$.*

(i) *If $[E:N] = f(E/N) = 2$ and $\sigma(N) = N$, then $N \in \{L, L'\}$.*
(ii) *If $\sigma(N) = N$ and $N \not\subset L$, then $N \subset L'$ if and only if* $\mathrm{e}(E/N)$ *is odd and N is ramified over $N \cap L$.*

Proof. Let N be as in (i). If $\sigma \mid N \equiv 1$ then, since $L = E^\sigma$, $N \subset L$. Because $[E:L] = [E:N] = 2$, we must have $N = L$. Suppose that $\sigma \mid N \not\equiv 1$. Then

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$N^\sigma = N \cap L$ and $[N\colon N \cap L] = 2$. Because $f(E/N) = 2$, we have $L_{un} \subset N$. But $L_{un} \subset L$. Thus $L_{un} \subset N \cap L$. From $f(E/F) = 2f(N/(N \cap L))f(L/F)$, it follows that N is a ramified quadratic extension of $N \cap L$.

Note that $L_{un}(\varpi_L^2)$ is a totally ramified extension of L_{un} of degree $\mathrm{e}(E/F)/2 = \mathrm{e}(L/L_{un})/2$. Since $(\varpi_L\sqrt{\varepsilon})^2 = \varpi_L^2\varepsilon$ is a uniformizer in $L_{un}(\varpi_L^2)$, $L' = L_{un}(\varpi_L\sqrt{\varepsilon})$ is a ramified quadratic extension of $L_{un}(\varpi_L^2)$ which is not contained in L and is fixed by σ. Note that $[E : L'] = f(E/L') = 2$.

Because $N \cap L$ is a totally ramified extension of L_{un} of degree $\mathrm{e}(L/L_{un})/2$, we must have $N \cap L = L_{un}(\varpi_L^2)$. As there are only two ramified quadratic extensions of $L_{un}(\varpi_L^2)$, that is, L and L', the condition $\sigma \mid N \not\equiv 1$ forces $N = L'$.

(ii) Assume that $N \not\subset L$ but $N \subset L'$. By Lemma 5.4, since L' is ramified over $L' \cap N$, N is ramified over $N \cap L$ and $\mathrm{e}(L'/N) = \mathrm{e}(E/N)$ is odd.

Now assume that N is ramified over $N \cap L$ and $\mathrm{e}(E/N)$ is odd. Observe that $f(E/N) = f(E/(N \cap L)) = 2f(L/(N \cap L))$ guarantees that $f(E/N)$ is even. Let N' be an unramified extension of N of degree $f(E/N)/2$. From $\sigma(N) = N$ and uniqueness of unramified extensions, it follows that $\sigma(N') = N'$. Note that $\mathrm{e}(E/N) = \mathrm{e}(E/N')$. As a consequence of $f(E/N') = 2$, we have $L_{un} \subset N' \cap L$ and therefore $f(E/(N \cap L')) = 2 = f(E/N')$. Thus N' is a ramified quadratic extension of $N' \cap L$. Because $N \subset N'$ and N' satisfies the hypotheses, there is no loss of generality in replacing N by N'. Therefore we may assume that $f(E/N) = 2$, so $L_{un} \subset N \cap L$. Note that $\mathrm{e}(L/(N \cap L)) = 2\,\mathrm{e}(E/N)$. Set $m = \mathrm{e}(E/N)$. Both $L_{un}(\varpi_L^{2m})$ and $N \cap L$ are totally ramified extensions of L_{un} of degree $\mathrm{e}(L/L_{un})/(2m)$ contained in L. Therefore $N \cap L = L_{un}(\varpi_L^{2m})$. The field $L_{un}(\varpi_L^m) = (N \cap L)(\varpi_L^m)$ is a ramified quadratic extension of $N \cap L$ contained in L. The other ramified quadratic extension of $N \cap L$ is $L_{un}(\varpi_L^m\sqrt{\varepsilon}) = (N \cap L)(\varpi_L^m\sqrt{\varepsilon})$. Thus $N = L_{un}(\varpi_L^m\sqrt{\varepsilon})$. As $m = \mathrm{e}(E/N)$ is odd, we have $(\varpi_L\sqrt{\varepsilon})^m = (\varpi_L^m\sqrt{\varepsilon})\varepsilon^{(m-1)/2}$, which implies that $\varpi_L^m\sqrt{\varepsilon} \in L_{un}(\varpi_L\sqrt{\varepsilon}) = L'$. $\square$

PROPOSITION 7.6. *Assume that* $N \in \mathcal{S}_i$ *and* $f(E/N) = 2$.

(i) *If* $[E : N] = 2$, *then* $\mathrm{sgn}(N) = \Leftrightarrow 1$.
(ii) *If* $[E : N] > 2$ *and* $\sigma(N) = N$, *then* $\mathrm{sgn}(N) = 1$.

Proof. By definition, $\Omega_i(E) = V_i(E) \simeq \mathfrak{p}_{E_i}^{\ell_i}/\mathfrak{p}_{E_i}^{\ell_i+1}$. Since $f(E/N) = 2$, we have $\dim_{R(N)}(V_i(E)) = 2$.

Assume that $[E : N] = 2$. By (7.1) and (7.2),

$$\dim_{R(N)}(V_i(N)) = \dim_{R(N)}(\Omega_i(N)) \Leftrightarrow \dim_{R(N)}(V_i(E)) = 4 \Leftrightarrow 2 = 2.$$

Thus $A(N) = 1$ and therefore by Lemma 7.3(ii), $\mathrm{sgn}(N) = \Leftrightarrow 1$.

Assume that N is as in (ii). By Lemma 7.3(ii), we must show that $\dim_{R(N)}(V_i(N)) \equiv 0 \bmod 4$. We will prove a slightly more general result

$$E_{i-1} \subset N \subset E,\ \ [E : N] > 2,\ \ f(E/N) = 2,\ \ \sigma(N) = N$$
$$\Longrightarrow \dim_{R(N)}(V_i(N)) \equiv 0 \bmod 4. \tag{7.7}$$

Even though N may not belong to $\mathcal{S}_i$ (N might contain E_i), $V_i(N)$ is still defined because $E_{i-1} \subset N$. By (7.1), $\dim_{R(N)}(\Omega_i(N)) \equiv 0 \bmod 4$. Therefore, by (7.2), it suffices to show that

$$\sum_{\{M \mid N \subset M \subset E,\, N \neq M\}} \dim_{R(N)}(V_i(M)) \equiv 0 \bmod 4.$$

Suppose that $E_{i-1} \subset M \subset E$ and $f_E(E/M) = 1$. By Lemma 3.6.58 of [Mo2],

$$\dim_{R(M)}(V_i(M)) = \phi(\mathrm{e}(E/M)),$$

where ϕ denotes the Euler ϕ-function. If in addition, $M \supset N$, then $f(E/N) = 2$ implies $[R(M) : R(N)] = 2$, so

$$\dim_{R(N)}(V_i(M)) = 2\,\phi(\mathrm{e}(E/M)).$$

If $\mathrm{e}(E/E_{i-1})$ is even, let N_0 denote the unique extension of E_{i-1} in E such that $f(E/N_0) = 1$ and $\mathrm{e}(E/N_0) = 2$. In this case, $N \subset N_0$ is equivalent to $\mathrm{e}(E/N)$ being even. By the above remarks

$$M \supset N,\;\; f(E/M) = 1,\;\; \dim_{R(N)}(V_i(M)) \equiv 2 \bmod 4$$
$$\implies M \in \begin{cases} \{E, N_0\}, & \text{if } \mathrm{e}(E/N) \text{ is even,} \\ \{E\}, & \text{if } \mathrm{e}(E/N) \text{ is odd.} \end{cases}$$

If $M \supset N$ and $\sigma(M) \neq M$, then $\sigma(M) \supset \sigma(N) = N$. It is not difficult to see that $\dim_{R(M)}(V_i(M)) = \dim_{R(\sigma(M))}(V_i(\sigma(M)))$. If in addition, $f(E/M) = 2$, then $\dim_{R(M)}(V_i(M))$ is even ([Mo2]). Also $R(M) \simeq R(\sigma(M)) = R(N)$. Thus

$$\dim_{R(N)}(V_i(M) \oplus V_i(\sigma(M))) \equiv 0 \bmod 4,$$
$$\sigma(M) \neq M,\;\; f(E/M) = 2,\;\; M \supset N.$$

We may now conclude that what we need to show is

$$\sum_{\{M \mid N \subset M \subset E,\, N \neq M, f(E/M)=2, \sigma(M)=M\}} \dim_{R(N)}(V_i(M))$$
$$\equiv 2\,\mathrm{e}(E/N) \bmod 4. \tag{7.8}$$

Suppose that $\mathrm{e}(E/N) = \ell$ is prime. Let M be as in (7.8). If such an M exists, then $[E{:}\, M] = f(E/M) = 2$, and as we saw in the proof of (i), $\dim_{R(N)}(V_i(M)) = \dim_{R(M)}(V_i(M)) = 2$.

Suppose $\ell = 2$. Then we may apply Lemma 7.5(i) to conclude that $M \in \{L, L'\}$. If $N \not\subset L$, then $M \neq L$. However, since $\mathrm{e}(E/N) = 2$, Lemma 7.5(ii) implies $N \not\subset L'$. Therefore there are no M as in (7.8) when $\mathrm{e}(E/N) = 2$ and

$N \not\subset L$. Hence (7.8) must hold. If $\mathrm{e}(E/N) = 2$ and $N \subset L$, then it is easy to see that $N = L_{un}(\varpi_L^2) \subset L'$. Hence the left side of (7.8) equals

$$\dim_{R(N)}(V_i(L)) + \dim_{R(N)}(V_i(L')) = 4 = 2\,\mathrm{e}(E/N).$$

Assume that ℓ is odd. If $N \not\subset L$, then $f(E/N) = 2$ implies that N is ramified over $L \cap N$. Thus $\mathrm{e}(E/F)$ is even and Lemma 7.5 applies. By Lemma 7.5, $N \subset L'$ and hence $M = L'$. The left side of (7.8) equals 2, and $2\ell = 2\,\mathrm{e}(E/N) \equiv 2 \bmod 4$, so (7.8) (hence (7.7)) holds. If $N \subset L$, and $\mathrm{e}(E/F)$ is even, then Lemma 7.5(i) applies, and $M \in \{L, L'\}$. However $\mathrm{e}(E/N)$ odd and $N \subset L$ imply that $N \not\subset L'$. Thus $M = L$, and (7.8) holds. Finally, if $\mathrm{e}(E/F)$ is odd, then L is the only σ-stable subfield of E of which E is a quadratic unramified extension. Thus $M = L$, and again (7.8) holds.

We have shown that (7.7) holds for all $N \supset E_{i-1}$ such that $\mathrm{e}(E/N)$ is prime, $f(E/N) = 2$, and $\sigma(N) = N$.

Now by induction, we assume that (7.7) holds for all M as in (7.8) such that $1 < \mathrm{e}(E/M) < \mathrm{e}(E/N)$. Then the left side of (7.8) is congruent modulo 4 to twice the quantity

$$\#\{M \mid N \subset M \subset E,\ \ f(E/M) = [E:M] = 2,\ \ \sigma(M) = M\},$$

where # denotes cardinality. To complete the proof, it suffices to show that this cardinality has the same parity as $\mathrm{e}(E/N)$.

If $\mathrm{e}(E/F)$ is odd, then, as we saw in the case $\mathrm{e}(E/N)$ prime, $N \subset L$ and the only M as above is L.

If $\mathrm{e}(E/F)$ is even and $\mathrm{e}(E/N)$ is odd, then it is easy to check that N belongs to precisely one of L and L'. Similarly, if $\mathrm{e}(E/F)$ is even and $\mathrm{e}(E/N)$ is even, then by Lemma 7.5(ii), N belongs to L if and only if N belongs to L'. □

LEMMA 7.9.

(i) *If E_1 is unramified over $E_1 \cap L$, then $\mathrm{e}(E_1/F)$ must be odd.*

(ii) *If E_j is unramified over $E_j \cap L$ for some $j \leqslant r \Leftrightarrow 1$, then $f(E/E_j)$ is odd and E_h is unramified over $E_h \cap L$ for $j \leqslant h \leqslant r$.*

Proof. (i) Let $c_1' = c_{E_1}(c_1) \in C_{E_1}$ be the standard representative of c_1. Choose $\varepsilon \in \zeta_{E_1 \cap L}$ which is not a square in $E_1 \cap L$. Then $E_1 = (E_1 \cap L)(\sqrt{\varepsilon})$ and $\sigma \mid E_1 \not\equiv 1$ imply $\sigma(\sqrt{\varepsilon}) = \Leftrightarrow\sqrt{\varepsilon}$. Because $\sigma(c_1) = \Leftrightarrow c_1$, and standard representatives are unique, it follows that $\sigma(c_1') = \Leftrightarrow c_1'$. Choose a uniformizer ϖ_{E_1} in E_1 which is also a uniformizer in $E_1 \cap L$. Then we have

$$c_1' = \varpi_{E_1}^m \eta \sqrt{\varepsilon},$$

for some $\eta \in \zeta_{E_1 \cap L}$ and some integer m.

Since $c_1' \in c_1(1 + \mathfrak{p}_{E_1})$, it follows that c_1' represents θ_1 on $1 + \mathfrak{p}_{E_1}^{f_{E_1}(\theta_1)-1}$ and hence genericity of θ_1 implies that c_1' generates E_1 over F.

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Suppose that $\mathrm{e}(E_1/F)$ is even. Then $f_{E_1/F}(\theta_1) \Leftrightarrow 1 = \Leftrightarrow m$ must be prime to $\mathrm{e}(E_1/F)$ (otherwise c_1' wouldn't generate E_1 over F), so m must be odd. Because E_1 is unramified over $E_1 \cap L$, we can apply Lemma 7.5 with E, L and ϖ_L replaced by E_1, $E_1 \cap L$, and ϖ_{E_1}, respectively. Let L_1 be the unramified extension of F of degree $f(E_1/F)/2$. Then as we saw in the proof of Lemma 7.5(ii), $L_1(\varpi_{E_1}^m\sqrt{\varepsilon})$ is a subfield of the proper subfield $L_1(\varpi_{E_1}\sqrt{\varepsilon})$ of E_1. But this is impossible because $E_1 = F(c_1') = F(\varpi_{E_1}^m\sqrt{\varepsilon}\eta)$ and $\eta \in \zeta_{E_1\cap L} = \zeta_{L_1}$ implies that $E_1 = L_1(\varpi_{E_1}^m\sqrt{\varepsilon})$.

(ii) From $f(E/E_j) = f(E/(E_j \cap L))/2 = f(L/(E_j \cap L))$, it follows that if $f(E/E_j)$ were even, then there would be a quadratic unramified extension of $E_j \cap L$ contained in L. By uniqueness of unramified extensions, this is impossible as E_j is a quadratic unramified extension of $E_j \cap L$ which is not contained in L. Thus $f(E/E_j)$ is odd. Suppose that $j < h < r$. Then $f(E/(E_h \cap L)) = 2f(L/(E_h \cap L))$ and $f(E/E_h)$ odd forces $f(E_h/(E_h \cap L)) = 2$. □

We are now ready to compute $\mathrm{sgn}(\alpha)$ for $\alpha \in C_L$. If $\sigma(N) \neq N$, then from $\sigma(\alpha) = \alpha$, it follows that $\alpha \notin N$ if and only if $\alpha \notin \sigma(N)$. As $\mathrm{sgn}(N) = \mathrm{sgn}(\sigma(N))$,

$$\mathrm{sgn}(\alpha) = \prod_{\{N \in \mathcal{S}_i \,|\, \alpha \notin N,\, \sigma(N) = N\}} \mathrm{sgn}(N).$$

PROPOSITION 7.10. *Let* $\alpha \in C_L$. *If* $\mathrm{e}(E/F)$ *is even, define* L' *as in Lemma* 7.5.

(i) *If* $\mathrm{e}(E/F)$ *is odd, then* $\mathrm{sgn}(\alpha) = 1$.
(ii) *If* $\mathrm{e}(E/F)$ *is even and* $L' \notin \mathcal{S}_i$, *then* $\mathrm{sgn}(\alpha) = 1$.
(iii) *If* $\mathrm{e}(E/F)$ *is even and* $L' \in \mathcal{S}_i$, *then* $\mathrm{sgn}(\alpha) = (\Leftrightarrow 1)^{\nu(\alpha)}$.

Proof. By Lemma 7.4 and Proposition 7.6,

$$\mathrm{sgn}(\alpha) = (\Leftrightarrow 1)^{\#\{N \in \mathcal{S}_i \,|\, f(E/N) = [E:N] = 2,\, \sigma(N) = N,\, \alpha \notin N\}}.$$

If $\mathrm{e}(E/F)$ is odd, then the only field such that $f(E/N) = 2 = [E:N]$ and $\sigma(N) = N$ is $N = L$. Since $\alpha \in N$, $\mathrm{sgn}(\alpha) = 1$.

If $\mathrm{e}(E/F)$ is even, then by Lemma 7.5(i), there are two possibilities for N, namely L and L'. Note that $\alpha \in L'$ if and only if $\nu(\alpha)$ is even. Since $\alpha \in L$, (ii) and (iii) now follow. □

COROLLARY 7.11.

(i) *Suppose that one of the following holds*
  (a) $\mathrm{e}(E/F)$ *is even,* $i = 1$, *and* E_1 *is unramified over* $E_1 \cap L$,
  (b) $i = 1$, E_1 *is ramified over* $E_1 \cap L$, *and* $\mathrm{e}(E/E_1)$ *is even,*
  (c) $i > 1$, $\mathrm{e}(E/E_{i-1})$ *is odd,* E_{i-1} *is ramified over* $E_{i-1} \cap L$, *and* E_i *is unramified over* $E_i \cap L$.

  Then $m_i = \ell_i + 1$ *and* $\mathrm{sgn}(\alpha) = (\Leftrightarrow 1)^{\nu(\alpha)}$, $\alpha \in C_L$.
(ii) *If none of the three conditions* (a)–(c) *holds, but* $m_i = \ell_i + 1$, *then* $\mathrm{sgn}(\alpha) = 1$ $\forall \alpha \in C_L$.

Proof. If (a) holds, then by Lemma 7.9(i), $\mathrm{e}(E_1/F)$ must be odd. Hence $\mathrm{e}(E/E_1)$ is even. If (b) holds, then $\mathrm{e}(E/E_1)$ is even by assumption. Thus if (a) or (b) holds, we have

$$f_E(\theta_1 \circ N_{E/E_1}) = \mathrm{e}(E/E_1)(f_{E_1}(\theta_1) - 1) + 1$$

is odd. That is, $m_1 = \ell_1 + 1$.

If (c) holds, then it suffices to show that $f_{E_i}(\theta_i)$ is odd, because that implies

$$f_E(\theta_i \circ N_{E/E_i}) = \mathrm{e}(E/E_i)(f_{E_i}(\theta_i) - 1) + 1$$

is odd.

The argument is similar to that in the proof of Lemma 7.9(i). Let $L_{i,un}$ be the maximal unramified extension of $E_{i-1} \cap L$ contained in $E_i \cap L$. Choose $\varepsilon \in \zeta_{L_{i,un}}$ such that ε is not a square in $E_i \cap L$. Then $E_i = (E_i \cap L)(\sqrt{\varepsilon})$ and $\sigma(\sqrt{\varepsilon}) = -\sqrt{\varepsilon}$. Let ϖ_i be a prime element in $E_i \cap L$. Let $c_i' = c_{E_i}(c_i) \in C_{E_i}$ be the standard representative of c_i. Then $\sigma(c_i') = -c_i'$ and $E_{i-1}(c_i') = E_i$. Write $c_i' = \varpi_i^h \sqrt{\varepsilon}\eta$, where $h = -f_{E_i}(\theta_i) + 1$ and $\eta \in \zeta_{E_i \cap L} = \zeta_{L_{i,un}}$. Let $L_i' = L_{i,un}(\varpi_i \sqrt{\varepsilon})$.

Assume that h is odd. Then $(\varpi_i \sqrt{\varepsilon})^h = (\varpi_i^h \sqrt{\varepsilon})\varepsilon^{(h-1)/2}$ and $\eta \in L_{i,un}$ imply that $\varpi_i^h \sqrt{\varepsilon} \in L_i'$. Also, $\eta \in L_{i,un} \subset L_i'$. Therefore $c_i' \in L_i'$. We can apply Lemma 7.5 with E, L, F, L_{un} and L' replaced by E_i, $E_i \cap L$, $E_{i-1} \cap L$, $L_{i,un}$ and L_i', respectively. By Lemma 7.5(i), since $\mathrm{e}(E_i/(E_{i-1} \cap L)) = 2\,\mathrm{e}(E/E_i)$, it follows that $f(E_i/L_i') = [E_i : L_i'] = 2$. Also, since E_{i-1} is ramified over $E_{i-1} \cap L$ and $\mathrm{e}(E_i/E_{i-1})$ is odd, $E_{i-1} \subset L'$. Thus $E_i = E_{i-1}(c_i') \subset L_i'$. Contradiction. Therefore $h = -f_{E_i}(\theta_i) + 1$ must be even if (c) holds.

For the remainder of the proof, we may suppose that $m_i = \ell_i + 1$. As we already know that $\mathrm{sgn}(\alpha)$ equals 1 if $\mathrm{e}(E/F)$ is odd (Proposition 7.10(i)), we assume that $\mathrm{e}(E/F)$ is even.

If $i = 1$, then $E_{i-1} = F \subset L'$. Therefore $L' \in \mathcal{S}_1'$ if and only if $E_1 \not\subset L'$. By Lemma 7.5(ii), $E_1 \not\subset L'$ if and only if E_1 is unramified over $E_1 \cap L$, or $\mathrm{e}(E/E_1)$ is even and E_1 is ramified over $E_1 \cap L$. Thus $L' \in \mathcal{S}_1'$ if and only if one of (a) and (b) holds.

Suppose that $i > 1$. By Lemma 7.5(ii), $L' \in \mathcal{S}_i'$ if and only if E_{i-1} is ramified over $E_{i-1} \cap L$, $\mathrm{e}(E/E_{i-1})$ is odd, and E_i is unramified over $E_i \cap L$. By Proposition 7.10(ii) and (iii), $\mathrm{sgn}(\alpha) = (-1)^{\nu(\alpha)}$ if and only if (c) holds. □

PROPOSITION 7.12. *Assume that $m_i = \ell_i + 1$, and $f_E(\theta_r) > 1$ if $i = r$. Suppose that $x \in (E^\times H_i)^\varphi$. There exists a positive integer d_x such that*

(i) *If $i = 1$ and $\mathrm{e}(E/E_1)$ is even, or if $i > 1$, $\mathrm{e}(E/E_{i-1})$ is odd, and $f(E_i/(E_i \cap L)) = \mathrm{e}(E_{i-1}/(E_{i-1} \cap L)) = 2$, then*

$$\chi_i(x) = \begin{cases} q_{E_{i-1}}^{d_x} (-1)^{\nu(x)}\, \theta_i(N_{E/E_i}(\mu(x))), \\ \qquad \text{if } x \text{ is conjugate to an element of } E^\times H_i', \\ 0, \quad \text{otherwise.} \end{cases}$$

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(ii) *In all other cases,*

$$\chi_i(x) = \begin{cases} q_{E_{i-1}}^{d_x}\,\theta_i(N_{E/E_i}(\mu(x))), \\ \quad \text{if } x \text{ is conjugate to an element of } E^\times H_i', \\ 0, \quad \text{otherwise}. \end{cases}$$

Proof. By Lemma 6.1, if x is not conjugate to an element of $E^\times H_i'$, then $\chi_i(x) = 0$.

Otherwise, choose $y \in E^\times H_i$ such that $y^{-1}xy \in E^\times H_i'$. Next, set $d_x = \sum_{\{N \mid y^{-1}xy \in N^\times H_i'\}} D(N)$. By Lemmas 6.1, 6.4 and 6.5 and the definition of $\mathrm{sgn}(c_L(x))$, we have

$$\chi_i(x) = q_{E_{i-1}}^{d_x}\,\mathrm{sgn}(c_L(x))\,\theta_i(N_{E/E_i}(\mu(x))).$$

Suppose that $i = 1$. By Lemma 7.9(i), if E_1 is unramified over $E_1 \cap L$, then $\mathrm{e}(E_1/F)$ is even. Thus in this case, $\mathrm{e}(E/F)$ is even if and only if $\mathrm{e}(E/E_1)$ is even. From this it follows that, if $i = 1$, then one of (a) and (b) of Corollary 7.11(i) holds if and only if $\mathrm{e}(E/E_1)$ is even.

Therefore the conditions of (i) are precisely the conditions (a)–(c) of Corollary 7.11(i), and the proposition is a consequence of Corollary 7.11 and $\nu(x) = \nu(c_L(x))$. □

There is a simple way to determine exactly when the type of behaviour in Proposition 7.12(i) can occur for some κ_i.

LEMMA 7.13. *Suppose that E is unramified over L. If E_{r-1} is ramified over $E_{r-1} \cap L$, assume that $f_E(\theta_r) > 1$.*

(i) *If $[E\colon E_1]$ is even, then there exists exactly one i, $1 \leqslant i \leqslant r$, such that one of the conditions of Proposition* 7.12(i) *holds. Furthermore, for this i, m_i must equal $\ell_i + 1$.*

(ii) *If $[E\colon E_1]$ is odd, then neither of the conditions of Proposition* 7.12(i) *hold for any i, $1 \leqslant i \leqslant r$.*

Proof. Suppose that $\mathrm{e}(E/E_1)$ is even. By Lemma 7.9(ii), if E_1 is unramified over $E_1 \cap L$, then E_h is unramified over $E_h \cap L$ for $2 \leqslant h \leqslant r$. By Corollary 7.11(i), $m_1 = \ell_1 + 1$. Then (i) follows by Proposition 7.12(i).

Assume that $\mathrm{e}(E/E_1)$ is even and E_1 is ramified over $E_1 \cap L$. By Corollary 7.11(i), $m_1 = \ell_1 + 1$. By Lemmas 5.4 and 7.9(ii), there exists a unique j, $2 \leqslant j \leqslant r$, such that E_j is unramified over $E_j \cap L$ and E_{j-1} is ramified over $E_{j-1} \cap L$. Furthermore, by Lemma 5.4, $\mathrm{e}(E_{j-1}/E_1)$ is odd. This forces $\mathrm{e}(E/E_{j-1})$ to be even. Thus the conditions of Proposition 7.12(i) apply only for $i = 1$.

Assume that $\mathrm{e}(E/E_1)$ is odd. Then the conditions of Proposition 7.12(i) do not apply for $i = 1$. If $f(E/E_1)$ is even, then by Lemma 7.9(ii), E_1 cannot be unramified over $E_1 \cap L$. Thus there exists a unique j, $2 \leqslant j \leqslant r$, as above. By assumption, $\mathrm{e}(E/E_{j-1})$ is odd and therefore the conditions of Proposition 7.12(i) apply for $i = j$. By Corollary 7.11(i), $m_j = \ell_j + 1$.

Assume that $[E : E_1]$ is odd. Then the conditions of Proposition 7.12(i) cannot apply for $i = 1$. From $f(E/E_1)f(E_1/(E_1 \cap L)) = 2\, f(L/(E_1 \cap L))$ and $f(E/E_1)$ odd, it follows that E_1 must be unramified over $E_1 \cap L$. By Lemma 7.9(ii), E_i is unramified over $E_i \cap L$ for $1 \leqslant i \leqslant r$. Thus the conditions of Proposition 7.12(i) cannot apply for $i > 1$. $\square$

8. Deligne–Lusztig characters

Digne and Michel ([DM]) have developed a Deligne–Lusztig theory for complex characters of non-connected reductive groups over finite fields. Below we state a particular case of a character formula of theirs which will be applied in the case $f_E(\theta_r) = 1$ in Sections 9 and 10.

Fix an integer $d \geqslant 2$. For q a positive integral power of the odd prime p, let $\mathcal{G}^\circ = \mathrm{GL}_d(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field of order q. Suppose η is an automorphism of $\mathcal{G}^\circ$ of order two. Set $\mathcal{G} = \mathcal{G}^\circ \rtimes \langle\eta\rangle$. Given an η-stable maximal torus $\mathcal{T}^\circ$ in $\mathcal{G}^\circ, \mathcal{T} = \mathcal{T}^\circ \rtimes \langle\eta\rangle$ is a maximal torus in $\mathcal{G}$ (cf. Definition 1.2, [DM]). Fix an η-stable character $\bar\theta$ of $\mathcal{T}^\circ$ and an extension $\bar\theta$ to $\mathcal{T}$ (note: $\bar\theta(\eta) = \pm 1$). Let $R^{\mathcal{G}}_{\mathcal{T}}(\bar\theta)$ denote the corresponding Deligne–Lusztig (virtual) character of $\mathcal{G}$ defined by Digne and Michel (Definition 2.2, [DM]).

The following notation is needed for the character formula for $R^{\mathcal{G}}_{\mathcal{T}}(\bar\theta)$. If $\mathcal{T}'$ is the group of $\mathbb{F}_q$-rational points of a maximal torus of a connected reductive group over $\mathbb{F}_q$ with $\mathbb{F}_q$-rational points $\mathcal{G}'$, let $\mathcal{T}'^\circ$ and $\mathcal{G}'^\circ$ be the $\mathbb{F}_q$-rational points of their identity components. Denote the Green function attached to $\mathcal{T}'^\circ$ and $\mathcal{G}'^\circ$ by $Q^{\mathcal{G}'^\circ}_{\mathcal{T}'^\circ}\colon \mathcal{U}_{\mathcal{G}'^\circ} \to \mathbb{C}$, where $\mathcal{U}_{\mathcal{G}'^\circ}$ is the unipotent subset of $\mathcal{G}'^\circ$. Given a semisimple element $s \in \mathcal{G}'$, let $\mathcal{G}'^s$ be the centralizer of s in $\mathcal{G}'$. For $x \in \mathcal{G}'$, ${}^x\mathcal{T}'$ denotes $x\mathcal{T}'x^{-1}$. If χ is a character of $\mathcal{T}'$, let ${}^x\chi$ be the character of ${}^x\mathcal{T}'$ defined by ${}^x\chi(s) = \chi(x^{-1}sx)$, for $s \in {}^x\mathcal{T}'$.

PROPOSITION 8.1 ([DM], Proposition 2.6(i)). *Let $g \in \mathcal{G}$ have Jordan decomposition $g = su$. Then*

$$R^{\mathcal{G}}_{\mathcal{T}}(\bar\theta)(g) = |\mathcal{T}|^{-1}|(\mathcal{G}^s)^\circ|^{-1} \sum_{\{x \in \mathcal{G} \mid s \in {}^x\mathcal{T}\}} Q^{(\mathcal{G}^s)^\circ}_{(({}^x\mathcal{T})^s)^\circ}(u)\, {}^x\bar\theta(s).$$

REMARK. It follows immediately from a comparison of the Deligne–Lusztig character formula for connected groups ([DL], Theorem 4.2) and the restriction of the above formula to $\mathcal{G}^\circ$ that

$$R^{\mathcal{G}}_{\mathcal{T}}(\bar\theta)\Big|_{\mathcal{G}^\circ} = R^{\mathcal{G}^\circ}_{\mathcal{T}^\circ}(\bar\theta),$$

where $R^{\mathcal{G}^\circ}_{\mathcal{T}^\circ}(\bar\theta)$ is the Deligne–Lusztig (virtual) character of $\mathcal{G}^\circ$ corresponding to the restriction of $\bar\theta$ to $\mathcal{T}^\circ$.

9. The case $f_E(\theta_r) = 1$: part one

In this section and the next, we consider the case $f_E(\theta_r) = 1$. After stating the definition of κ_r, we discuss properties of the map induced by φ on $P(r \Leftrightarrow 1)/P_1(r \Leftrightarrow 1)$. Then we prove some results concerning certain types of elements in general linear groups over finite fields. At the end of the section, these results are applied to compute the signs of certain sums of values of the character χ_r of κ_r.

Let $\overline{E}$ and $\overline{E}_{r-1}$ denote the residue class fields of E and E_{r-1}. By the definition of H and P

$$(H \cap P)/(H \cap P_1) \cong P(r \Leftrightarrow 1)/P_1(r \Leftrightarrow 1) \cong \mathrm{GL}_{[E:E_{r-1}]}(\overline{E}_{r-1}).$$

Let $\overline{H} = \mathrm{GL}_{[E:E_{r-1}]}(\overline{E}_{r-1})$. Since $f_E(\theta_r) = 1$, the character θ_r determines a character $\bar{\theta}_r$ of the elliptic maximal torus $\overline{E}^{\times}$. The character $\bar{\theta}_r$ corresponds (via Deligne–Lusztig induction) to an irreducible cuspidal representation $\overline{\kappa}_r$ of $\overline{H}$. The restriction of κ_r to $H \cap P$ is the unique representation of $H \cap P$ which is trivial on $H \cap P_1$ and induces $\overline{\kappa}_r$ on $\overline{H}$. To define κ_r on all of H, set $\kappa_r(\varpi_{r-1}) = \theta_r(\varpi_{r-1})\kappa_r(1)$, where ϖ_{r-1} is a prime element in E_{r-1}.

Our definition of the matrix s (see Section 3) depended on a choice of basis of L over F. When $f_E(\theta_r) = 1$, it is convenient to choose a basis that makes it easy to determine the map induced by φ on $P(r \Leftrightarrow 1)/P_1(r \Leftrightarrow 1)$.

Suppose $F_1 \subset F_2 \subset F_3$ is a tower of fields, and let $\alpha = \{a_i\}$ be a basis of F_2 over F_1 and $\beta = \{b_j\}$ a basis of F_3 over F_2. Write $\alpha^* = \{a_i^*\}$, $\beta^* = \{b_i^*\}$ for the corresponding dual bases. Then $\beta\alpha = \{b_1a_1, b_1a_2, \ldots; b_2a_1, b_2a_2, \ldots\}$ is a basis for F_3 over F_1, and the corresponding dual basis is easily seen to be $(\beta\alpha)^* = \{b_1^*a_1^*, b_1^*a_2^*, \ldots; b_2^*a_1^*, b_2^*a_2^*, \ldots\}$. Let s_α be the transition matrix from the basis α to the dual basis α^*, and similarly for s_β and $s_{\beta\alpha}$.

LEMMA 9.1. (i) *The entries of s_α are given by $(s_\alpha)_{ij} = \mathrm{tr}_{F_2/F_1}(a_ia_j)$. In particular, s_α is a symmetric matrix.*

(ii) *The transition matrices defined above are related as follows*

$$s_{\beta\alpha} = \begin{pmatrix} s_\alpha & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & s_\alpha \end{pmatrix} s_\beta,$$

where there are $[F_3 : F_2]$ diagonal blocks in the matrix on the left and s_β is interpreted as a matrix over F using the basis α.

Proof. (i) $(s_\alpha)_{ij} = \langle s_\alpha(a_i), (a_j^*)^*\rangle = \langle s_\alpha(a_i), a_j\rangle = \mathrm{tr}_{F_2/F_1}(a_ia_j)$.

(ii) Using (i), we see that the $(ij),(k\ell)$-entry of the transition matrix $s_{\beta\alpha}$ is given by

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$$\begin{aligned}
&\mathrm{tr}_{F_3/F_1}(b_i a_j b_k a_\ell)\\
&\quad = \mathrm{tr}_{F_2/F_1}(a_j a_\ell \mathrm{tr}_{F_3/F_2}(b_i b_k)) = \mathrm{tr}_{F_2/F_1}(a_j a_\ell (s_\beta)_{ik})\\
&\quad = \mathrm{tr}_{F_2/F_1}\left(a_j \sum_r (s_\alpha)_{r\ell} a_r^* (s_\beta)_{ik}\right) = \sum_r (s_\alpha)_{\ell r} [(s_\beta)_{ik}]^\alpha_{rj}.
\end{aligned}$$

(Here we wrote $[(s_\beta)_{ik}]^\alpha_{rj}$ for the rjth entry of the matrix with respect to the basis α of the element $(s_\beta)_{ik} \in F_2$ and also used the symmetry of s_α).

This last formula is precisely the required entry in the matrix product. $\square$

Recall that $f_E(\theta_r) = 1$ implies that E is unramified over L (Lemma 5.1) and over E_{r-1}. Fix $\varepsilon \in \zeta_L$ such that ε is not a square in L. As before, σ denotes the non-trivial element of $\mathrm{Gal}(E/L)$. Let ϖ_0 be a prime element in $E_{r-1} \cap L$. Let $f_0 = f\big(E_{r-1}/(E_{r-1} \cap L)\big)$, $e_0 = \mathrm{e}(E_{r-1}/(E_{r-1} \cap L))$.

If $e_0 = 1$, set $\varpi_E = \varpi_L = \varpi_0$. If $e_0 = 2$, then $\varpi_E = \sqrt{\varpi_0}$ is a prime element in E which generates E_{r-1} over $E_{r-1} \cap L$ and such that $\sigma(\varpi_E) = \Leftrightarrow\varpi_E$. The element $\varpi_L = \sqrt{\varepsilon\varpi_0} = \sqrt{\varepsilon}\varpi_E$ is a prime element in L. Note that in the above definition of κ_r we can take $\varpi_{r-1} = \varpi_E$.

Let $d_0 = f\big(L/(E_{r-1} \cap L)\big)$. Let $M \subset L$ be the unramified extension of $E_{r-1} \cap L$ of degree d_0. We will use bars to denote residue class fields. Choose a basis $\xi = \{\xi_1, \dots, \xi_{d_0}\}$ of M over $E_{r-1} \cap L$ such that $\xi_j \in \mathcal{O}_M^\times$ and the images of $\xi_1, \dots \xi_{d_0}$ in $\overline{M}$ form a basis of $\overline{M}$ over $\overline{E_{r-1} \cap L}$. If $L = M$, set $\beta = \xi$. Otherwise, $e_0 = \mathrm{e}(E_{r-1}/(E_{r-1} \cap L)) = \mathrm{e}(L/(E_{r-1} \cap L)) = [L\colon M] = 2$, and $\beta \overset{\mathrm{def}}{=\!=} \{\xi_1, \dots \xi_{d_0}, \varpi_L\xi_1, \dots \varpi_L\xi_{d_0}\}$ is a basis of L over $E_{r-1} \cap L$. If $r > 1$, let $\alpha = \{a_1, \dots, a_k\}$ be a basis of $E_{r-1} \cap L$ over F.

Applying Lemma 9.1 in the case where $r > 1$, with $F_3 = L$, $F_2 = E_{r-1} \cap L$, $F_1 = F$, and bases as defined above, we find that the corresponding transition matrices are related as follows

$$s = s_{\beta\alpha} = \begin{pmatrix} s_\alpha & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & s_\alpha \end{pmatrix} s_\beta,$$

where there are $d = [E : E_{r-1}]$ diagonal blocks in the matrix on the left and s_β is interpreted as a matrix over F using the basis α. If $r = 1$, then $E_{r-1} \cap L = F$ and we let $s = s_\beta$.

Because s_α has been chosen so that $s_\alpha^{-1}\,{}^t x s_\alpha = x$, where x is an element of $E_{r-1} \cap L$ viewed as a matrix over F via the basis α, it follows that if $X \in \mathfrak{gl}_d(E_{r-1} \cap L)$, then

$$s^{-1}\,{}^t X s = s_\beta^{-1}\,{}^T X s_\beta,$$

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where ${}^{T}X$ refers to the transpose over $E_{r-1} \cap L$. When $r = 1$, the two transposes ${}^{t}X$ and ${}^{T}X$ are the same, and $s = s_\beta$.

When $r > 1$, we compare this situation with that of Lemma 3.4(ii), with N' in that lemma replaced by E_{r-1} and N_0' replaced by $E_{r-1} \cap L$. The above expression says that the matrix $\mathcal{S}$ given in Lemma 3.4 can be taken to be s_β.

Continuing the comparison, we see that τ in Lemma 3.4 corresponds to $\sqrt{\varepsilon}$ while τ_N corresponds to $\sqrt{\varpi_0}$ if $e_0 = 2$ and to $\sqrt{\varepsilon}$ otherwise. Since $\varpi_L = \sqrt{\varpi_0 \varepsilon}$, we see that ω in the proof of Lemma 3.4(ii) corresponds to $\varpi_L \varepsilon^{-1}$ if $e_0 = 2$ and to 1 otherwise. Recall that $f_0 = f(E_{r-1}/(E_{r-1} \cap L))$.

LEMMA 9.2. (i) *If* $f_0 = 1$, *then the map induced by* φ *on* $\mathrm{GL}_d(\overline{E}_{r-1})$ *is* $\varphi(x) = w^{-1}\,{}^{t}x w$, *with* w *a skew-symmetric matrix.*

(ii) *If* $f_0 = 2$, *then the map induced by* φ *on* $\mathrm{GL}_d(\overline{E}_{r-1})$ *is* $\varphi(x) = h^{-1}\,{}^{t}\sigma(x)h$, *with* h *a matrix that is hermitian relative to* $\overline{E}_{r-1}/(\overline{E_{r-1} \cap L})$.

Proof. (i) The case where $r = 1$ is immediate from the original definition of φ. Now suppose $r > 1$ and $e_0 = 2$. If we write $[\varepsilon]$ for the matrix of ε with respect to the basis ξ, then the matrix of ε with respect to the given basis of L over $E_{r-1} \cap L$ is $\begin{pmatrix} [\varepsilon] & 0 \\ 0 & [\varepsilon] \end{pmatrix}$, while the matrix of ϖ_L is $\begin{pmatrix} 0 & [\varepsilon]\varpi_0 \\ \mathrm{I} & 0 \end{pmatrix}$, (here I means the $d_0 \times d_0$ identity matrix).

Accordingly, the matrix of $\varepsilon\varpi_L^{-1}$ is $\begin{pmatrix} 0 & [\varepsilon] \\ \varpi_0^{-1}\mathrm{I} & 0 \end{pmatrix}$. Using the lemma above and also the analogous result obtained by applying Lemma 9.1 with $E_{r-1} \cap L \subset M \subset L$ as $F_1 \subset F_2 \subset F_3$, we find that, in the notation of Lemma 3.4

$$\begin{aligned} \mathcal{S}_0 \;&=\; \mathcal{S}\omega^{-1} = s_\beta \varepsilon \varpi_L^{-1} = \begin{pmatrix} s_\xi & 0 \\ 0 & s_\xi \end{pmatrix} s_{\{1,\varpi_L\}} \varepsilon \varpi_L^{-1} \\ &= \begin{pmatrix} s_\xi & 0 \\ 0 & s_\xi \end{pmatrix} \begin{pmatrix} 2\mathrm{I} & 0 \\ 0 & 2[\varpi_L^2] \end{pmatrix} \begin{pmatrix} 0 & [\varepsilon] \\ \varpi_0^{-1}\mathrm{I} & 0 \end{pmatrix} \\ &= \begin{pmatrix} s_\xi & 0 \\ 0 & s_\xi \end{pmatrix} \begin{pmatrix} 0 & 2[\varepsilon] \\ 2[\varepsilon] & 0 \end{pmatrix}. \end{aligned}$$

Without loss of generality, we can remove the constant 2 and let

$$\mathcal{S}_0 = \begin{pmatrix} 0 & s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix}.$$

So for $X \in \mathfrak{gl}_d(E_{r-1})$, $\varphi(X) = \mathcal{S}_0^{-1}\,{}^{T}\sigma(X)\mathcal{S}_0$, where ${}^{T}(\cdot)$ means the transpose over E_{r-1} and $\sigma = \sigma_{r-1}$ is the non-trivial conjugation of E_{r-1} over $E_{r-1} \cap L$, applied to the entries of a matrix. Note that $\mathcal{S}_0$ is symmetric.

To realize M_{r-1} in a form in which it will be easy to reduce modulo the prime ideal $\mathfrak{p}_{E_{r-1}}$, we consider conjugating elements of M_{r-1} by the diagonal matrix

$$D = \begin{pmatrix} \varpi_E & 0 \\ 0 & \mathrm{I} \end{pmatrix} \in \mathfrak{gl}_{2d}(E_{r-1});$$

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here I is the $d_0 \times d_0$ identity matrix. Note that

$$\begin{aligned}\sigma(D^{-1})\mathcal{S}_0 &= \begin{pmatrix} -\varpi_E^{-1} & 0 \\ 0 & \mathrm{I} \end{pmatrix} \begin{pmatrix} 0 & s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\varpi_E^{-1} s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} & 0 \\ 0 & \varpi_E^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix} D \begin{pmatrix} \varpi_E^{-1} & 0 \\ 0 & \varpi_E^{-1} \end{pmatrix}.\end{aligned}$$

If $X \in \mathfrak{gl}_d(E_{r-1})$, then

$$\begin{aligned}\varphi(D^{-1}XD) &= \mathcal{S}_0^{-1}\,{}^T\sigma(D^{-1}XD)\mathcal{S}_0 = \mathcal{S}_0^{-1}\sigma(D)\,{}^T\sigma(X)\sigma(D^{-1})\mathcal{S}_0 \\ &= D^{-1}\left[\begin{pmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix}^{-1} {}^T\sigma(X) \begin{pmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{pmatrix}\right] D.\end{aligned}$$

This shows that conjugation by D takes φ into the map given by composing the transpose over E_{r-1}, the automorphism $\sigma = \sigma_{r-1}$, and conjugation by the skew-symmetric matrix $\left(\begin{smallmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{smallmatrix}\right)$. Because of the way we chose the basis ξ, we find that not only does s_ξ have integer entries, but it is an element of $\mathrm{GL}_{d_0}(\mathcal{O}_{E_{r-1}\cap L}) \subset \mathrm{GL}_{d_0}(\mathcal{O}_{E_{r-1}})$. So $\left(\begin{smallmatrix} 0 & -s_\xi[\varepsilon] \\ s_\xi[\varepsilon] & 0 \end{smallmatrix}\right)$ is an element of $\mathrm{GL}_d(\mathcal{O}_{E_{r-1}})$ (since $d = 2d_0$). We can also think of its reduction modulo $1 + \mathfrak{gl}_d(\mathfrak{p}_{E_{r-1}})$ as a skew-symmetric element of the finite group $\mathrm{GL}_d(\overline{E}_{r-1})$. Since σ is trivial on $\mathcal{O}_{E_{r-1}}/\mathfrak{p}_{E_{r-1}}$, this finishes the proof of (i).

(ii) Suppose $f_0 = 2$; then $e_0 = 1$, $M = L$ and $\beta = \xi$. As remarked above, $\mathcal{S} = s_\beta$, $\omega = 1$, so

$$\mathcal{S}_0 = \mathcal{S}\omega^{-1} = s_\beta = s_\xi.$$

Note that $\mathcal{S}_0 \in \mathfrak{gl}_{d_0}(E_{r-1} \cap L)$ is symmetric, and as above, $s_\xi \in \mathrm{GL}_{d_0}(\mathcal{O}_{E_{r-1}\cap L}) \subset \mathrm{GL}_{d_0}(\mathcal{O}_{E_{r-1}})$. In particular, $\mathcal{S}_0$ is fixed by σ_{r-1}. So its reduction modulo $\mathfrak{p}_{E_{r-1}}$ is a symmetric matrix that is fixed by σ_{r-1}, and in particular it is hermitian. $\square$

Set $\mathcal{G}^\circ = \mathrm{GL}_d(\mathbb{F}_q)$, where $q = q_{E_{r-1}}$ is the cardinality of $\overline{E}_{r-1}$. Then $\mathcal{T}^\circ = \overline{E}^\times$ is a maximal torus in $\mathcal{G}^\circ$. Recall that $\overline{\kappa}_r$ is the irreducible cuspidal representation of $\mathcal{G}^\circ \cong \overline{H}$ corresponding via Deligne–Lusztig induction to $\bar{\theta}_r$. The notation χ_r, which has already been used for the character of κ_r, will also be used to denote the character of $\overline{\kappa}_r$. Then in the notation of Section 8, using Theorems 4.2 and 7.1 of [DL] to obtain the sign

$$\chi_r(x) = (-1)^{d-1} R_{\mathcal{T}^\circ}^{\mathcal{G}^\circ}(\bar{\theta}_r)(x), \quad x \in \mathcal{G}^\circ. \tag{9.3}$$

Let η be the automorphism of $\mathcal{G}^\circ = \overline{H}$ given by $\eta(x) = \varphi(x^{-1})$. Comparing character formulas ([DL]) results in

$$R^{\mathcal{G}^\circ}_{\mathcal{T}^\circ}(\bar{\theta}_r)(\eta(x)) = R^{\mathcal{G}^\circ}_{\mathcal{T}^\circ}(\bar{\theta}_r \circ \eta)(x), \quad x \in \mathcal{G}^\circ.$$

Also, by properties of θ_r (Lemma 2.5(ii)), $\bar{\theta}_r$ is fixed by η. Thus $\chi_r = \chi_r \circ \eta$, that is, $\overline{\kappa}$ is equivalent to $\overline{\kappa}_r \circ \eta$. Choosing an operator A_η which intertwines $\overline{\kappa}_r$ with $\overline{\kappa}_r \circ \eta$ and whose square is the identity, we extend $\overline{\kappa}_r$ to a representation of $\mathcal{G} = \mathcal{G}^\circ \rtimes \langle \eta \rangle$ by setting $\overline{\kappa}_r(\eta) = A_\eta$.

Since the maximal torus $\mathcal{T}^\circ$ is η-stable and $\bar{\theta}_r$ is fixex by η, $\bar{\theta}_r$ extends (in two ways) to a character (also denoted $\bar{\theta}_r$) of the (non-abelian) maximal torus $\mathcal{T} = \mathcal{T}^\circ \rtimes \langle \eta \rangle$ of $\mathcal{G}$. Let $R^{\mathcal{G}}_{\mathcal{T}}(\bar{\theta}_r)$ be the Deligne–Lusztig virtual character of $\mathcal{G}$ defined by Digne and Michel ([DM], Definition 2.2). From the character formula of Digne and Michel (Proposition 8.1), remarks following, and (9.3), $(\Leftrightarrow 1)^{d-1} R^{\mathcal{G}}_{\mathcal{T}}(\bar{\theta}_r)$ is a virtual character of $\mathcal{G}$ which coincides with χ_r on $\mathcal{G}^\circ$. Thus, replacing $\bar{\theta}_r(\eta)$ by $\Leftrightarrow \bar{\theta}_r(\eta)$ if necessary, we may assume that $(\Leftrightarrow 1)^{d-1} R^{\mathcal{G}}_{\mathcal{T}}(\bar{\theta}_r)$ is the character of the extension of $\overline{\kappa}_r$ to $\mathcal{G}$ given by $\overline{\kappa}_r(\eta) = A_\eta$.

Let $\mathcal{C}_{\overline{E}}$, resp. $\mathcal{C}_{\overline{L}}$, be the set of elements in $\overline{H}$ whose semisimple part is conjugate to an element of $\overline{E}^\times$, resp. $\overline{L}^\times$. The images of ε and $\sqrt{\varepsilon}$ in $\overline{L}$ and $\overline{E}$ will also be denoted by ε and $\sqrt{\varepsilon}$. Similarly, σ will be used to denote the non-trivial element of $\mathrm{Gal}(\overline{E}/\overline{L})$.

LEMMA 9.4. *Let $g \in \overline{H}^\varphi$. Then the semisimple part γ of g belongs to $\overline{H}^\varphi$. If $\gamma \in \mathcal{C}_{\overline{E}}$, then $\{\, x^{-1}\gamma x : x \in \overline{H}\} \cap \overline{E}^\times \subset \overline{L}^\times$. In particular, $g \in \mathcal{C}_{\overline{L}}$.*

Proof. If $g = \gamma u$ is the multiplicative Jordan decomposition of g, then $g = \gamma + \gamma(u \Leftrightarrow 1)$ is the additive Jordan decomposition, i.e., $\gamma(u \Leftrightarrow 1)$ is nilpotent, because γ and u commute. The additive Jordan decomposition of $\varphi(g)$ is $\varphi(\gamma + \gamma(u \Leftrightarrow 1)) = \gamma + \gamma(u \Leftrightarrow 1)$, so equating semisimple parts, $\gamma \in \overline{H}^\varphi$.

First we assume that $f_0 = 1$. By Lemma 9.2(i), there exists a skew-symmetric $w \in \overline{H}$ such that $\varphi(g) = w^{-1t}gw, g \in \overline{H}$. Choose $x \in \overline{H}$ such that $\gamma_1 = x^{-1}\gamma x \in \overline{E}^\times$. Let $y = \varphi(x)x$. If $\delta \in \overline{E}_{r-1}(\gamma_1)$, then $z\delta x^{-1} \in \overline{E}_{r-1}(\gamma)$ implies that $\varphi(x\delta x^{-1}) = x\delta x^{-1}$. That is, $\varphi(\delta) = y\delta y^{-1}$. The action of φ on $\overline{E}^\times$ is given by σ, so

$$y\delta y^{-1} = \sigma(\delta), \quad \delta \in \overline{E}_{r-1}(\gamma_1). \tag{9.5}$$

Assume that $\gamma_1 \notin \overline{L}$. Then $\gamma_1 \Leftrightarrow \sigma(\gamma_1) = a\sqrt{\varepsilon} \in \overline{E}_{r-1}(\gamma_1)$ for some $a \in \overline{L}^\times$. Note that $\varphi(y) = \varphi(\varphi(x)) = y$. Therefore yw^{-1} is skew-symmetric. By (9.5) with $\delta = a\sqrt{\varepsilon}$, $ya\sqrt{\varepsilon} \in \overline{H}^\varphi$, and hence $ya\sqrt{\varepsilon}w^{-1}$ is also skew-symmetric. As the determinant of a skew-symmetric matrix in $\overline{H}$ is a square in $\overline{E}^\times_{r-1}$, it follows that $\det(a\sqrt{\varepsilon}) \in (\overline{E}^\times_{r-1})^2$. Observe that $\det(a\sqrt{\varepsilon}) = N_{\overline{L}/\overline{E}_{r-1}}(\Leftrightarrow a^2\varepsilon) = (\Leftrightarrow 1)^{d/2} N_{\overline{L}/\overline{E}_{r-1}}(a)^2 N_{\overline{L}/\overline{e}_{r-1}}(\varepsilon)$. As $\varepsilon \notin (\overline{L}^\times)^2$, we have $N_{\overline{L}/\overline{E}_{r-1}}(\varepsilon) \notin (\overline{E}^\times_{r-1})^2$. Therefore $(\Leftrightarrow 1)^{d/2} \notin (\overline{E}^\times_{r-1})^2$. In particular, $d/2$ is odd.

As $d/2$ is odd, there exists $b \in \overline{E}_{r-1}$ such that $b\sqrt{\varepsilon}$ generates a quadratic extension of $\overline{E}_{r-1}$ (which is not contained in $\overline{L}$). It follows from (9.5) that

$$yw^{-1t}(b\sqrt{\varepsilon})wy^{-1} = b\sqrt{\varepsilon}. \tag{9.6}$$

There exists a matrix $z \in \mathrm{GL}_d(\overline{E}_{r-1}(b\sqrt{\varepsilon}))$ such that

$$zb\sqrt{\varepsilon}z^{-1} = \begin{pmatrix} b\sqrt{\varepsilon}I)_{d/2} & 0 \\ 0 & \Leftrightarrow b\sqrt{\varepsilon}I_{d/2} \end{pmatrix},$$

where $I_{d/2}$ denotes the $(d/2) \times (d/2)$ identity matrix. Since wy^{-1} is skew-symmetric, the matrix $\mathcal{A} = zwy^{-1t}z \in \mathrm{GL}_d(\overline{E}_{r-1}(b\sqrt{\varepsilon}))$ is skew-symmetric. However, it is a consequence of (9.6), and the definitions of z and $\mathcal{A}$ that $\mathcal{A}$ commutes with the (symmetric) matrix $zb\sqrt{\varepsilon}z^{-1}$. That is $\mathcal{A} \in \mathrm{GL}_{d/2}(\overline{E}_{r-1}(b\sqrt{\varepsilon})) \times \mathrm{GL}_{d/2}(\overline{E}_{r-1}(b\sqrt{\varepsilon}))$. This is a contradiction, as $d/2$ odd implies such matrices cannot be skew-symmetric. Thus $\gamma_1 \in \overline{L}$.

Now assume that $f_0 = 2$. In this case, $\overline{E}_{r-1} = (\overline{E}_{r-1} \cap \overline{L})(\sqrt{\varepsilon})$, and $\sigma \mid \overline{E}_{r-1}$ generates the corresponding Galois group. Suppose that $x \in \overline{H}$ and $\gamma_1 = x^{-1}\gamma x \in \overline{E}$. By Lemma 3.4(ii)

$$\det(\gamma_1) = \det(\gamma) = \det(\varphi(\gamma)) = \det(\sigma(\gamma)) = \sigma(\det(\gamma)),$$

which implies that $\det(\gamma_1) = N_{\overline{E}/\overline{E}_{r-1}}(\gamma_1) \in \overline{E}_{r-1} \cap \overline{L}$. If $\gamma_1 \notin \overline{L}$, then $\overline{E} = \overline{L}(\gamma_1)$ implies that $N_{\overline{E}/\overline{E}_{r-1}}(\overline{E}^{\times}) \subset \overline{E}_{r-1} \cap \overline{L}$, which is impossible. Thus $\gamma_1 \in \overline{L}$. □

LEMMA 9.7. *Suppose* $e_0 = 2$. *Recall that* $q = |\overline{E}_{r-1}|$. *Then*

(i) $(\mathcal{G}^{\eta})^{\circ}$ *is the* $d \times d$ *symplectic group* $\mathrm{Sp}_d(\mathbb{F}_q)$.
(ii) $(\mathcal{G}^{\sqrt{\varepsilon}\eta})^{\circ}$ *is the* $d \times d$ *special orthogonal group of* $\mathbb{F}_q$*-rank equal to* $(d/2) \Leftrightarrow 1$.
(iii) *If* $g \in \overline{H}^{-\varphi} \cap (\mathcal{C}_{\overline{E}} \backslash \mathcal{C}_{\overline{L}})$, *then* $g = x\sqrt{\varepsilon}\varphi(x)$, *for some* $x \in \overline{H}$.
(iv) *If* $g \in \overline{H}^{-\varphi}$ *is not of the form* $g = x\sqrt{\varepsilon}\varphi(x)$, *for some* $x \in \overline{H}$, *then the* $\mathcal{G}$*-conjugacy class of* $g\eta$ *does not intersect* $\mathcal{T}$.

Proof. Since $f_0 = 1$, part (i) follows from Lemma 9.2(i).

For (ii), note that $g \in (\mathcal{G}^{\sqrt{\varepsilon}\eta})^{\circ}$ if and only if $g(\sqrt{\varepsilon}w^{-1})^t g = \sqrt{\varepsilon}w^{-1}$, where w is the skew-symmetric matrix given by Lemma 9.2(i). Observe that $\sqrt{\varepsilon} \in \overline{H}^{-\varphi}$ implies that $\sqrt{\varepsilon}w^{-1}$ is symmetric. Fix a non-square $\nu \in \mathbb{F}_q^{\times}$. By [C], a symmetric matrix determines the special orthogonal group of $\mathbb{F}_q$-rank $(d/2) \Leftrightarrow 1$ if and only if its determinant belongs to $(\Leftrightarrow 1)^{d/2}\nu(\mathbb{F}_q^{\times})^2$. It is simple matter to check that the fact that ε is a non-square in $\overline{L}^{\times}$ implies that $\det(\sqrt{\varepsilon}) \in (\Leftrightarrow 1)^{d/2}\nu(\mathbb{F}_q^{\times})^2$. Since w is skew-symmetric, $\det(w) \in (\mathbb{F}_q^{\times})^2$.

By Lemma 9.4, $g^2 \in \mathcal{C}_{\overline{L}}$. Therefore, as $g \in \mathcal{C}_{\overline{E}} \backslash \mathcal{C}_{\overline{L}}$, the semisimple part of g is conjugate to an element of $\sqrt{\varepsilon}(\overline{L}^{\times})^2$. This implies that $\det(gw^{-1}) \in$

$\det(\sqrt{\varepsilon}w^{-1})(\mathbb{F}_q^\times)^2$. Note that by Lemma 9.2(i), $g \in \overline{H}^{-\varphi}$ implies that gw^{-1} is symmetric. Therefore, gw^{-1} and $\sqrt{\varepsilon}w^{-1}$ belong to the same equivalence class of symmetric matrices. Part (iii) now follows from Lemma 9.2(i).

Since $\mathcal{T}$ is η-stable, the $\mathcal{G}$-conjugacy class of $g\eta$ intersects $\mathcal{T}$ if and only if the $\overline{H}$-conjugacy class of $g\eta$ intersects $\mathcal{T}$. Suppose that $xg\eta x^{-1} \in \mathcal{T}$, for some $x \in \overline{H}$. Then $xg\eta x^{-1}\eta^{-1} = xg\varphi(x) \in \mathcal{T}^\circ = \overline{E}^\times$. But $xg\varphi(x) \in \overline{H}^{-\varphi}$, so $xg\varphi(x) \in \sqrt{\varepsilon}\overline{L}^\times$. Since $\det(x\varphi(x)) = \det(x)^2 \in (\mathbb{F}_q^\times)^2$, it follows that $\det(g) \in \det(\sqrt{\varepsilon})(\mathbb{F}_q^\times)^2$. Arguing as above, this implies that $g = x\sqrt{\varepsilon}\varphi(x)$, for some $x \in \overline{H}$, a contradiction. $\square$

LEMMA 9.8. *Let $g \in \overline{H}$. Then*

$$\sum_{x\in\overline{H}} \chi_r(xg\varphi(x)) = |\overline{H}|\chi_r(g\eta)\chi_r(\eta)\chi_r(1)^{-1}.$$

Proof. Note that for $x, g \in \overline{H}, xg\varphi(x) = xg\eta x^{-1}\eta^{-1}$. The operator $\sum_{x\in\overline{H}} \overline{\kappa}_r(xg\eta x^{-1})$ commutes with $\overline{\kappa}_r$, so by Schur's Lemma is a scalar multiple λ of the identity operator. Evaluating the trace gives $\lambda = |\overline{H}|\chi_r(g\eta)\chi_r(1)^{-1}$. Thus

$$\begin{aligned}\sum_{x\in\overline{H}} \chi_r(xg\varphi(x)) &= \operatorname{trace}\left(\sum_{x\in\overline{H}} \overline{\kappa}_r(xg\eta x^{-1}\eta^{-1})\right)\\ &= \operatorname{trace}\left(\sum_{x\in\overline{H}} \overline{\kappa}_r(xg\eta x^{-1})\overline{\kappa}_r(\eta^{-1})\right)\\ &= \operatorname{trace}(|\overline{H}|\chi_r(g\eta)\chi_r(1)^{-1}\overline{\kappa}_r(\eta^{-1}))\\ &= |\overline{H}|\chi_r(g\eta)\chi_r(1)^{-1}\chi_r(\eta^{-1}).\end{aligned}$$ $\square$

PROPOSITION 9.9.

(i) $\sum_{g\in\overline{H}^\varphi} \chi_r(g) > 0$.

(ii) *Suppose that $e_0 = 2$. Then*

$$\theta_r(\sqrt{\varepsilon}) \sum_{g\in\overline{H}^{-\varphi}\cap(\mathcal{C}_{\overline{E}}\setminus\mathcal{C}_{\overline{L}})} \chi_r(g) < 0 \quad \textit{and} \quad \sum_{g\in\overline{H}^{-\varphi}\cap\mathcal{C}_{\overline{L}}} \chi_r(g) = 0.$$

Proof. Take $g \in \overline{H}^\varphi$. If $f_0 = 1$, by Lemma 9.2(i) there exists a skew-symmetric $w \in \overline{H}$ such that $g = \varphi(g) = w^{-1}\,{}^tgw$. Thus gw^{-1} is skew-symmetric. It follows that there exists $x \in \overline{H}$ such that $gw^{-1} = xw^{-1}\,{}^tx$. That is $g = x\varphi(x)$. If $f_0 = 2$, a similar argument shows that $g = x\varphi(x)$ for some $x \in \overline{H}$, using hermitian matrices rather than skew-symmetric matrices (see Lemma 9.2(ii)), and the fact that there is one equivalence class of hermitian matrices in $\overline{H}$ ([C]). The sum in (i) is equal to

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$|\overline{H}^\eta| \sum_{x\in\overline{H}} \chi_r(x\varphi(x))$, which by Lemma 9.8 is a positive multiple of $\chi_r(\eta)^2$. By the remarks preceding Lemma 9.4, and Proposition 8.1

$$\begin{aligned}
\chi_r(\eta) &= (\Leftrightarrow 1)^{d-1} R^{\mathcal{G}}_{\mathcal{T}}(\bar{\theta}_r)(\eta)\\
&= |\mathcal{T}|^{-1}|\overline{H}^\eta|^{-1} 2 \sum_{\{x\in\overline{H} \mid x\varphi(x)\in\mathcal{T}^\circ\}} Q^{(\mathcal{G}^\eta)^\circ}_{(({}^x\mathcal{T})^\eta)^\circ}(1)\bar{\theta}_r(x^{-1}\varphi(x^{-1})\eta)\\
&= 2|\mathcal{T}|^{-1}|\overline{H}^\eta|^{-1}\left(\sum_{\{x\in\overline{H} \mid x\varphi(x)\in\mathcal{T}^\circ\}} Q^{(\mathcal{G}^\eta)^\circ}_{(({}^x\mathcal{T})^\eta)^\circ}(1)\right)\bar{\theta}_r(\eta). \qquad (9.10)
\end{aligned}$$

Here we have used the fact that for $x \in \overline{H}, \eta \in {}^x\mathcal{T}$ if and only if $\eta \in {}^{x\eta}\mathcal{T}$ if and only if $x\varphi(x) \in \mathcal{T}^\circ$. For such x, ${}^x\bar{\theta}_r(\eta) = \bar{\theta}_r(x^{-1}\varphi(x^{-1})\eta)$, which equals $\bar{\theta}_r(\eta)$ since $(\mathcal{T}^\circ)^\varphi = \overline{L}^\times$ and $\bar{\theta}_r \mid \overline{L}^\times \equiv 1$. The tori appearing in (9.10) are elliptic in the semisimple group $(\mathcal{G}^\eta)^\circ$, hence have $\mathbb{F}_q$-rank zero. Therefore all of the above Green functions take real values at the identity with sign determined by the $\mathbb{F}_q$-rank of $(\mathcal{G}^\eta)^\circ$ (Theorem 7.1 of [DL]). This, together with $\bar{\theta}_r(\eta) = \pm 1$, implies that $\chi_r(\eta)^2 > 0$. Hence (i).

Suppose $e_0 = 2$ and $g \in \overline{H}^{-\varphi}$. Arguing as in the proof of Lemma 9.7(iii), gw^{-1} is symmetric, where w is the skew-symmetric matrix given in Lemma 9.2(i). By Lemma 9.7(iii), if $g \in \mathcal{C}_{\overline{E}}\backslash\mathcal{C}_{\overline{L}}$, then $g = x\sqrt{\varepsilon}\varphi(x)$, for some $x \in \overline{H}$. So, using Lemma 9.8

$$\begin{aligned}
&\bar{\theta}_r(\sqrt{\varepsilon}) \sum_{g\in\overline{H}^{-\varphi}\cap(\mathcal{C}_{\overline{E}}\backslash\mathcal{C}_{\overline{L}})} \chi_r(g)\\
&= \bar{\theta}_r(\sqrt{\varepsilon}) \mid (\mathcal{G}^{\sqrt{\varepsilon}\eta})^\circ|^{-1} \sum_{x\in\overline{H}} \chi_r(x\sqrt{\varepsilon}\varphi(x))\\
&= \bar{\theta}_r(\sqrt{\varepsilon}) \mid (\mathcal{G}^{\sqrt{\varepsilon}\eta})^\circ|^{-1}|\overline{H}|\chi_r(\sqrt{\varepsilon}\eta)\chi_r(\eta)\chi_r(1)^{-1}. \qquad (9.11)
\end{aligned}$$

By Lemma 9.7(i), the $\mathbb{F}_q$-rank of $(\mathcal{G}^\eta)^\circ$ is $d/2$. The tori $(({}^x\mathcal{T})^\eta)^\circ$ are elliptic and have $\mathbb{F}_q$-rank zero. Therefore, by Theorem 7.1 of [DL], the sign of the value at the identity of each Green function occurring in (9.10) is $(\Leftrightarrow 1)^{d/2}$. So $(\Leftrightarrow 1)^{d/2}\bar{\theta}_r(\eta)\chi_r(\eta) > 0$. Applying Proposition 8.1 to express $\chi_r(\sqrt{\varepsilon}\eta)$ in terms of $\bar{\theta}_r(\sqrt{\varepsilon}\eta)$ and Green functions for elliptic tori in $(\mathcal{G}^{\sqrt{\varepsilon}\eta})^\circ$, and then using Lemma 9.7(ii) to determine the signs of the Green functions, we obtain $(\Leftrightarrow 1)^{(d/2)-1}\bar{\theta}_r(\sqrt{\varepsilon}\eta)\chi_r(\sqrt{\varepsilon}\eta) > 0$. We can now conclude that $\bar{\theta}_r(\sqrt{\varepsilon})\chi_r(\sqrt{\varepsilon}\eta)\chi_r(\eta)$ is a positive multiple of

$$\bar{\theta}_r(\sqrt{\varepsilon})(\Leftrightarrow 1)^{(d/2)-1}\bar{\theta}_r(\sqrt{\varepsilon}\eta)(\Leftrightarrow 1)^{d/2}\bar{\theta}_r(\eta) = \Leftrightarrow 1.$$

The first part of (ii) now follows from (9.11).

It remains to deal with the case $\overline{H}^{-\varphi} \cap \mathcal{C}_{\overline{L}} \neq \emptyset$. Fix a symmetric matrix y which is not in the same equivalence class as $\sqrt{\varepsilon}w^{-1}$. By Lemma 9.7(iii), if $g \in \overline{H}^{-\varphi} \cap \mathcal{C}_{\overline{L}}$, then $g = xy\varphi(x)$, for some $x \in \overline{H}$. Applying Lemma 9.8

$$\sum_{g \in \overline{H}^{-\varphi} \cap \mathcal{C}_{\overline{L}}} \chi_r(g) = |(\mathcal{G}^{y\eta})^\circ|^{-1} |\overline{H}| \chi_r(y\eta)\chi_r(\eta)\chi_r(1)^{-1}.$$

By Lemma 9.7(iv), the $\mathcal{G}$-conjugacy class of $y\eta$ does not intersect $\mathcal{T}$. Since $y \in \overline{H}^{-\varphi}, (y\eta)^2 = y\varphi(y^{-1}) = \Leftrightarrow 1$, which implies that $y\eta$ is semisimple. By Proposition 8.1, $\chi_r(y\eta) = 0$. □

10. The case $f_E(\theta_r) = 1$: part two

In this section, we consider the case $f_E(\theta_r) = 1$ and $\dim \kappa_i = 1$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$. We show that certain conditions on θ imply that $\mathcal{I}(\mathcal{F}) > 0$ (Theorem 10.7). In order to determine $\mathcal{I}(\mathcal{F})$ in this case, it suffices to consider the values of χ_κ on $(\varpi_E^j(H \cap P))^\varphi$, $j = 0, 1$ (Lemma 10.1). Because $\dim \kappa_i = 1$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$, χ_i is easily expressed in terms of θ_i (Lemma 10.2). Lemma 10.3 and Proposition 10.5 combine results from Section 9 describing the map on $\overline{H}$ induced by φ with Lemma 10.2 to obtain relations between $\mathcal{I}(\mathcal{F})$ and sums of χ_r over certain subsets of $\overline{H}$. The signs of these sums were determined in Proposition 9.9, as a consequence of the expression of χ_r in terms of Green functions of finite reductive groups, and of the results of Section 8.

If $C \subset G$, let $\mathbf{1}_C$ be the characteristic function of C. Set $\mathcal{F}_0 = \chi_\kappa \mathbf{1}_{H \cap P}$ and $\mathcal{F}_1 = \chi_\kappa \mathbf{1}_{\varpi_E(H \cap P)}$. Recall that $e_0 = \mathrm{e}(E_{r-1}/(E_{r-1} \cap L))$ and $f_0 = f(E_{r-1}/(E_{r-1} \cap L))$.

LEMMA 10.1. *Suppose that* $f_E(\theta_r) = 1$ *and* $\dim \kappa_i = 1$ *for* $1 \leqslant i \leqslant r \Leftrightarrow 1$. *Then* $\mathcal{I}(\mathcal{F}) = \mathrm{e}(\mathcal{I}(\mathcal{F}_0) + \mathcal{I}(\mathcal{F}_1))$.

Proof. Let $x \in H \cap P$. Recall that $\varphi(\varpi_E) = \sigma(\varpi_E) = (\Leftrightarrow 1)^{f_0(e_0-1)}\varpi_E$. It follows that

$$\varpi_E^j x \varphi(\varpi_E^j) = \varpi_E^j \sigma(\varpi_E^j)(\varpi_E^{-j} x \varpi_E^j) \in \varpi_E^{2j}(H \cap P).$$

Thus the map $x \mapsto \varpi_E^j x \varphi(\varpi_E^j)$ is a measure-preserving bijection from $(H \cap P)^\varphi$ to $(\varpi_E^{2j}(H \cap P))^\varphi$. Also

$$\chi_r(\varpi_E^j x \varphi(\varpi_E^j)) = \theta_r(\varpi_E \sigma(\varpi_E))^j \, \chi_r(\varpi_E^{-j} x \varpi_E^j).$$

Since $\varpi_E \in E_{r-1}$, conjugation by ϖ_E has no effect on $\mathrm{GL}_{[E:E_{r-1}]}(E_{r-1})$, so the image of $\varpi_E^{-j} x \varpi_E^j$ in $\overline{H}$ is the same as the image of x in $\overline{H}$. By Lemma 2.5(ii), $\theta_r(\sigma(\varpi_E)) = \theta_r(\varpi_E)^{-1}$. We conclude that

$$\chi_r(\varpi_E^j x \varphi(\varpi_E^j)) = \chi_r(x), \quad x \in H \cap P.$$

For $1 \leqslant i \leqslant r \Leftrightarrow 1$, since κ_i is one dimensional,

$$\begin{aligned}\chi_i(\varpi_E^j x \varphi(\varpi_E^j)) &= \kappa_i(\varpi_E^j \sigma(\varpi_E)^j x)\\ &= \theta_i(N_{E/E_i}(\varpi_E \sigma(\varpi_E)^j))\kappa_i(x) = \chi_i(x),\end{aligned}$$

the last equality following from $\theta_i \circ N_{E/E_i} \circ \sigma = (\theta_i \circ N_{E/E_i})^{-1}$.

Consequently, $\mathcal{F}(\varpi_E^j x \varphi(\varpi_E)^j) = \mathcal{F}(x)$ for $x \in H \cap P$ and, given the above remarks regarding measures,

$$\mathcal{I}(\chi_\kappa \mathbf{1}_{\varpi_E^{2j}(H\cap P)}) = \mathcal{I}(\mathcal{F}_0), \quad 0 \leqslant j \leqslant e \Leftrightarrow 1.$$

By a similar argument

$$\mathcal{I}(\chi_\kappa \mathbf{1}_{\varpi_E^{2j+1}(H\cap P)}) = \mathcal{I}(\mathcal{F}_1), \quad 0 \leqslant j \leqslant e \Leftrightarrow 1.$$

The lemma now follows from $\mathcal{F} = \chi_\kappa \mathbf{1}_S$ and $\mathbf{1}_S = \sum_{j=0}^{2e-1} \mathbf{1}_{\varpi_E^j(H\cap P)}$, with S as in Section 4. □

Let $\mathcal{S}_L$, resp. $\mathcal{S}_{E-L}$, be the set of $x \in H \cap P$ whose image in $(H \cap P)/(H \cap P_1)$ belongs to $\mathcal{C}_{\overline{L}}$, resp. $\mathcal{C}_{\overline{E}} \backslash \mathcal{C}_{\overline{L}}$. Recall that, if $\dim \kappa_i = 1$, $1 \leqslant i \leqslant r \Leftrightarrow 1$, then $m_i = \ell_i$ and $\kappa_i = \omega_i$, where ω_i is the character of H defined in Section 5.

LEMMA 10.2. *Suppose that* $f_E(\theta_r) = 1$. *Fix* $1 \leqslant i \leqslant r \Leftrightarrow 1$ *and assume that* $\dim \kappa_i = 1$.

(i) *If* $x \in (H \cap P)^\varphi \cap \mathcal{S}_L$, *then* $\chi_i(x) = 1$.
(ii) *If* $x \in (\varpi_E(H \cap P))^\varphi$ *is such that* $\varpi_E^{-1} x \in \mathcal{S}_L$, *then* $\chi_i(x) = \theta_i(N_{E/E_i}(\varpi_E))$.
(iii) *If* $x \in (\varpi_E(H \cap P))^\varphi$ *is such that* $\varpi_E^{-1} x \in \mathcal{S}_{E-L}$, *then* $\chi_i(x) = \theta_i(N_{E/E_i}(\sqrt{\varepsilon}\varpi_E))$.

Proof. As remarked above, $\dim \kappa_i = 1$ implies that $\chi_i(x) = \omega_i(x)$ for $x \in H$. Let $x \in (H \cap P)^\varphi \cup (\varpi_E(H \cap P))^\varphi$. As shown in the proof of Lemma 5.2, there exists $y \in (\varpi_E^{\nu(x)} K_i)^\varphi$ such that $x \in y L_i$ and $\omega_i(x) = \omega_i(y)$. Therefore we may assume that $x \in (\varpi_E^{\nu(x)} K_i)^\varphi$.

Suppose that $\varpi_E^{-\nu(x)} x \in \mathcal{S}_L$. Let $\lambda \in \mathcal{O}_E^\times$ be such that the semisimple part of the image of $\varpi_E^{-\nu(x)} x$ in $\overline{H}$ is conjugate to the image of $N_{E/L}(\lambda)$ in $\overline{L}^\times$. Then $\det_i(\varpi_E^{-\nu(x)} x) \in \det_i(N_{E/L}(\lambda)) \det_i(P_1(i))$. Set $g = \varpi_E^{-\nu(x)} N_{E/L}(\lambda^{-1}) x$. Recall that $\varphi(\varpi_E) = (\Leftrightarrow 1)^{f_0(e_0-1)} \varpi_E$. Thus, since $\varphi(x) = x$,

$$\varphi(g) = (\Leftrightarrow 1)^{f_0(e_0-1)} \varpi_E^{\nu(x)} N_{E/L}(\lambda^{-1})\, g\, N_{E/L}(\lambda) \varpi_E^{-\nu(x)},$$

which implies that

$$\sigma(\det_i(g)) = \det_i \varphi(g) = (\Leftrightarrow 1)^{f_0(e_0-1)[E:E_{r-1}]} \det_i(g) = \det_i(g).$$

That is, $\det_i(g) \in E_i \cap L$. Together with $\det_i(g) \in \det_i(P_1(i)) \in 1 + \mathfrak{p}_{E_i}$ and $\theta_i \,|\, (1 + \mathfrak{p}_{E_i \cap L}) \equiv 1$, this implies that $\theta_i(\det_i(g)) = 1$. Therefore, using $\theta_i \circ N_{E/(E_i \cap L)} \equiv 1$ (Lemma 2.5(ii)),

$$\begin{aligned}\theta_i(\det_i(x)) &= \theta_i(\det_i(\varpi_E^{\nu(x)} N_{E/L}(\lambda)))\\ &= \theta_i(N_{E/E_i}(\varpi_E^{\nu(x)}) N_{E/(E_i\cap L)}(\lambda)) = \theta_i(N_{E/E_i}(\varpi_E^{\nu(x)})).\end{aligned}$$

By definition of ω_i, since $x \in E^\times K_i$, $\omega_i(x) = \theta_i(\det_i(x))$. Parts (i) and (ii) now follow.

Let x be as in (iii). Then $\overline{\varpi_E^{-1}x}$, the image of $\varpi_E^{-1}x$ in $\overline{H}$, is in $\overline{H}^{c\varphi}$, where $c = (\Leftrightarrow 1)^{f_0(e_0-1)}$. If $c = 1$, then Lemma 9.4 implies that $\overline{\varpi_E^{-1}x} \in \mathcal{C}_{\overline{L}}$. This forces $\varpi_E^{-1}x \in \mathcal{S}_L$, which is a contradiction. Therefore $c = \Leftrightarrow 1$ and $\overline{\varpi_E^{-1}x} \in \mathcal{C}_{\overline{E}} \backslash \mathcal{C}_{\overline{L}}$. By Lemma 9.4, $(\overline{\varpi_E^{-1}x})^2 \in \overline{H}^{\varphi}$ implies that $(\overline{\varpi_E^{-1}x})^2 \in \mathcal{C}_{\overline{L}}$. From this it follows that the semisimple part of $\overline{\varpi_E^{-1}x}$ is conjugate to some $\gamma_1 \in \overline{E}^\times$ such that $\sigma(\gamma_1) = \Leftrightarrow \gamma_1$. Thus there exists $\lambda \in \mathcal{O}_E^\times$ such that the image of $\sqrt{\varepsilon} N_{E/L}(\lambda)$ in $\overline{E}^\times$ is equal to γ_1. Now set $g = \sqrt{\varepsilon}^{-1} \varpi_E^{-1} N_{E/L}(\lambda)^{-1} x$ and argue as for parts (i) and (ii). □

LEMMA 10.3. *Assume that* $f_E(\theta_r) = 1$.

(i) *Suppose that* $x \in P(r \Leftrightarrow 1)^\varphi$. *Then there exists* $g \in P(r \Leftrightarrow 1)$ *such that* $x = g\varphi(g)$.

(ii) *Suppose that* $x \in (\varpi_E P(r \Leftrightarrow 1))^\varphi$. *If* $e_0 = 2$ *and* $\varpi_E^{-1}x \in \mathcal{S}_{E-L}$, *or if* $e_0 = 1$, *then there exists* $g \in P(r \Leftrightarrow 1)$ *such that* $x = g\varpi_L\varphi(g)$.

(iii) *Suppose that* $e_0 = 2$ *and* $f(L/(E_{r-1} \cap L))$ *is even. Fix* $\delta \in P(r \Leftrightarrow 1) \cap \mathcal{S}_L$ *such that* $\varphi(\varpi_E\delta) = \varpi_E\delta$. *If* $x \in (\varpi_E P(r \Leftrightarrow 1))^\varphi$ *and* $\varpi_E^{-1}x \in \mathcal{S}_L$, *then there exists* $g \in P(r \Leftrightarrow 1)$ *such that* $x = g\varpi_E\delta\varphi(g)$.

REMARK. In (iii), the assumption $f(L/(E_{r-1} \cap L))$ even is necessary for $(\varpi_E P(r \Leftrightarrow 1))^\varphi$ to intersect $\varpi_E \mathcal{S}_L$ nontrivially.

Proof. Let $\tau = 1$, ϖ_L, and $\varpi_E\delta$ in cases (i), (ii), and (iii), respectively. Suppose that

$$\text{There exists } g_1 \in P(r \Leftrightarrow 1) \text{ such that } g_1^{-1} x \varphi(g_1^{-1}) \in (\tau P(r \Leftrightarrow 1))^\varphi. \qquad (10.4)$$

Since $\tau \in (E^\times P(r \Leftrightarrow 1))^\varphi$, by Lemma 3.9 applied with $j = 1$ and $i = r \Leftrightarrow 1$, there exists $g_2 \in P_1(r \Leftrightarrow 1)$ such that $g_1^{-1} x \varphi(g_1^{-1}) = g_2 \tau \varphi(g_2)$. It follows that $x = g\tau\varphi(g)$, for $g = g_1 g_2$. Thus it suffices to prove (10.4).

Given $y \in P(r \Leftrightarrow 1)$, let $\overline{y}$ denote the image of y in $\overline{H} \simeq P(r \Leftrightarrow 1)/P_1(r \Leftrightarrow 1)$. As in previous sections, the map on $\overline{H}$ induced by φ will also be denoted by φ.

Let $x \in P(r \Leftrightarrow 1)^\varphi$. If $f_0 = 1$, by Lemma 9.2(i), there exists a skew-symmetric $\mathcal{W} \in \overline{H}$ such that $\overline{x} = \varphi(\overline{x}) = \mathcal{W}^{-1}\,{}^t\overline{x}\mathcal{W}$. Thus $\overline{x}\mathcal{W}^{-1}$ is skew-symmetric. It follows that there exists $z \in \overline{H}$ such that $\overline{x}\mathcal{W}^{-1} = z\mathcal{W}^{-1}\,{}^t z$. That is, $\overline{x} = z\varphi(z)$. Choosing $g_1 \in P(r \Leftrightarrow 1)$ such that $\overline{g}_1 = z$, we obtain (10.4). If $f_0 = 2$, the argument is similar, except that it involves hermitian matrices rather than skew-symmetric matrices (see Lemma 9.2(ii)), and the fact that there is one equivalence class of hermitian matrices in $\overline{H}$ ([C]).

Next, let $x \in (\varpi_E P(r \Leftrightarrow 1))^{\varphi}$. As ϖ_E was chosen to be in E_{r-1}, and $\varphi(\varpi_E) = c\,\varpi_E$, $c = (\Leftrightarrow 1)^{f_0(e_0-1)}$, it follows that $\varpi_E^{-1}x \in P(r \Leftrightarrow 1)^{c\varphi}$.

If $e_0 = 1$, then $\varpi_E = \varpi_L$ and $\varpi_L^{-1}x \in P(r \Leftrightarrow 1)^{\varphi}$, so it follows from (i) that $\varpi_L^{-1}x = g\varphi(g)$ for some $g \in P(r \Leftrightarrow 1)$. Thus $x = g\varpi_L\varphi(g)$.

Let $e_0 = 2$. Then, setting $y = \varpi_E^{-1}x$, we have $\overline{y} \in \overline{H}^{-\varphi}$. By Lemma 9.2(i), there exists a skew-symmetric matrix $\mathcal{W} \in \overline{H}$ such that $\overline{y}\mathcal{W}^{-1}$ is symmetric. As $[E : E_{r-1}]$ is even, there are two equivalence classes of symmetric matrices in $\overline{H}$, and they can be distinguished by the coset of $(\overline{E}_{r-1}^{\times})^2$ in $\overline{E}_{r-1}^{\times}$ in which their determinants lie ([C]). Note that $\sqrt{\overline{\varepsilon}}\mathcal{W}^{-1}$ is symmetric.

If $y \in \mathcal{S}_{E-L}$, then as remarked in the proof of Lemma 10.2, the semisimple part of $\overline{y}$ is conjugate to an element in $\sqrt{\overline{\varepsilon}}\,\overline{L}^{\times}$. This implies that $\det(y) \in \det(\sqrt{\overline{\varepsilon}})(\overline{E}_{r-1}^{\times})^2$, from which it follows that $y = z\sqrt{\overline{\varepsilon}}\mathcal{W}^{-1\,t}z$ for some $z \in \overline{H}$. Choosing $g_1 \in P(r \Leftrightarrow 1)$ such that $\overline{g}_1 = z$, we obtain $g_1^{-1}y\varphi(g_1^{-1}) \in \sqrt{\varepsilon}P_1(r \Leftrightarrow 1)$. That is, $g_1^{-1}x\varphi(g_1^{-1}) \in \varpi_E\sqrt{\varepsilon}P_1(r \Leftrightarrow 1) = \varpi_L P_1(r \Leftrightarrow 1)$ and (10.4) holds in case (ii).

If $y \in \mathcal{S}_L$, then by definition of δ, there exists $z \in \overline{H}$ such that $\overline{y}\mathcal{W}^{-1} = z\overline{\delta}^t z$. The remainder of the argument is as for case (ii), with ϖ_L replaced by $\varpi_E\delta$. $\square$

As in Section 9, the notation χ_r is used for the character of κ_r and also for the character of $\overline{\kappa}_r$.

PROPOSITION 10.5. *Suppose that* $f_E(\theta_r) = 1$. *If* $f_0 = 2$, *assume that* $\dim \kappa_i = 1$ *for* $1 \leqslant i \leqslant r \Leftrightarrow 1$.

(i) $\mathcal{I}(\mathcal{F}_0) = \mathcal{I}(\mathbf{1}_{H\cap P_1})\left(\sum_{x\in\overline{H}^{\varphi}} \chi_r(x)\right)$.

(ii) *If* $e_0 = 1$, *then* $\mathcal{I}(\mathcal{F}_1) = \theta(\varpi_L)\,\mathcal{I}(\mathcal{F}_0)$.

(iii) *If* $e_0 = 2$ *and* $f(L/(E_{r-1}\cap L))$ *is odd, then*

$$\mathcal{I}(\mathcal{F}_1) = \mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)})\,\theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1}\left(\sum_{x\in\overline{H}^{-\varphi}} \chi_r(x)\right).$$

(iv) *If* $e_0 = 2$ *and* $f(L/(E_{r-1}\cap L))$ *is even, let* δ *be as in Lemma* 10.3(iii). *Then*

$$\mathcal{I}(\mathcal{F}_1) = \mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)})\theta(\varpi_L)\theta_r(\sqrt{\varepsilon})^{-1}\left(\sum_{x\in(\mathcal{C}_{\overline{E}}\setminus\mathcal{C}_{\overline{L}})\cap\overline{H}^{-\varphi}} \chi_r(x)\right)$$

$$+\mathcal{I}(\mathbf{1}_{\varpi_E\delta(H\cap P_1)})\,\theta(\varpi_E)\left(\sum_{x\in\mathcal{C}_{\overline{L}}\cap\overline{H}^{-\varphi}}\chi_r(x)\right).$$

Proof. Note that if $e_0 = 2$, then by Lemma 5.7, $\dim\kappa_i = 1$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$. Thus Lemma 10.2 applies in every case.

Let $x \in (H\cap P)^{\varphi}\cup(\varpi_E(H\cap P))^{\varphi}$. Then

$$\nu(x) = 1 \Longrightarrow \varphi(\varpi_E^{-\nu(x)}x) = c\varpi_E(\varpi_E^{-1}x)\varpi_E, \quad c = (\Leftrightarrow 1)^{f_0(e_0-1)}$$

and, since conjugation by $\varpi_E \in E_{r-1}$ has no effect on $\mathrm{GL}_{[E:E_{r-1}]}(E_{r-1})$, the image of $\varpi_E^{-1}x$ in $\overline{H}$ belongs to $\overline{H}^{c\varphi}$ if $\nu(x) = 1$.

By (9.3) and Proposition 8.1, χ_r vanishes at points in $H\cap P$ which do not lie in $\mathcal{S}_L\cup\mathcal{S}_{E-L}$. Putting this together with $\chi_r(x) = \theta_r(\varpi_E^{\nu(x)})\,\chi_r(\varpi_E^{-\nu(x)}x)$ and Lemma 10.2, we obtain

$$\chi_\kappa(x) = \begin{cases} \theta(\varpi_E^{\nu(x)})\,\chi_r(\varpi_E^{-\nu(x)}x), & \text{if } \varpi_E^{-\nu(x)}x\in\mathcal{S}_L,\\ \theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1}\,\chi_r(\varpi_E^{-1}x), & \text{if } \nu(x)=1 \text{ and } \varpi_E^{-1}x\in\mathcal{S}_{E-L},\\ 0, & \text{if } \varpi_E^{-\nu(x)}x\notin\mathcal{S}_L\cup\mathcal{S}_{E-L}. \end{cases}$$

By Corollary 3.8, there exists $y \in (\varpi_E^{\nu(x)}(P(r\Leftrightarrow 1))^{\varphi}$ and $z\in L_{r-1}$ such that $x = yz$. Note that

$$\varpi_E^{-\nu(x)}x(H\cap P_1) = \varpi_E^{-\nu(x)}y(H\cap P_1), \quad \text{and}$$

$$\varpi_E^{-\nu(x)}y \in \begin{cases} P(r\Leftrightarrow 1)^{\varphi}, & \text{if } \nu(x) = 0,\\ P(r\Leftrightarrow 1)^{c\varphi}, & \text{if } \nu(x) = 1.\end{cases}$$

Thus, given a coset of $H\cap P_1$ in $H\cap P$ which contains elements u such that $\varphi(u) = u$ or $\varphi(u) = c\,\varpi_E u\varpi_E^{-1}$, we can (and do) choose a coset representative in $P(r\Leftrightarrow 1)$ which transforms the same way under φ. Let $\{y_i \mid i\in\mathrm{I}_j\}$, $j = 1,2,3$, resp., be a set of such representatives of those cosets containing elements u of $\mathcal{S}_L$ if $j = 1,2$, resp. of $\mathcal{S}_{E-L}$ if $j = 3$, which satisfy $\varphi(u) = u$ if $j = 1$ and $\varphi(u) = c\,\varpi_E^{-1}u\varpi_E$ if $j = 2,3$. Observe that, by Lemma 9.4, there are no cosets containing φ-invariant elements of $\mathcal{S}_{E-L}$. By definition of I_j, $j = 1,2,3$, and the above formula for $\chi_\kappa(y_i)$, $i\in\mathrm{I}_1$ and $\chi_\kappa(\varpi_E y_i)$, $i\in\mathrm{I}_j$, $j = 2,3$,

$$\mathcal{I}(\mathcal{F}_0) = \sum_{i\in\mathrm{I}_1}\chi_r(y_i)\,\mathcal{I}(\mathbf{1}_{y_i(H\cap P_1)}),$$

$$\mathcal{I}(\mathcal{F}_1) = \theta(\varpi_E)\sum_{i\in\mathrm{I}_2}\chi_r(y_i)\mathcal{I}(\mathbf{1}_{\varpi_E y_i(H\cap P_1)})$$

$$+\theta(\varpi_L)\,\theta_r(\sqrt{\varepsilon})^{-1}\sum_{i\in\mathrm{I}_3}\chi_r(y_i)\mathcal{I}(\mathbf{1}_{\varpi_E y_i(H\cap P_1)}). \tag{10.6}$$

Set $\tau_1 = 1$, $\tau_2 = \varpi_L$, and $\tau_3 = \varpi_E\delta$. By Lemma 10.3, if $i \in \mathrm{I}_j$, since $\varpi_E^{\nu(\tau_j)} y_i \in (\varpi_E^{\nu(\tau_j)} P(r \Leftrightarrow 1))^{\varphi}$, by definition of I_j, there exists $g_i(j) \in P(r \Leftrightarrow 1)$ such that $\varpi_E^{\nu(\tau_j)} y_i = g_i(j)\tau_j\varphi(g_i(j))$. It follows that $v \mapsto g_i(j)v\varphi(g_i(j))$ from $\tau_j(H \cap P_1))^{\varphi}$ to $(\varpi_E^{\nu(\tau_j)} y_i(H \cap P_1))^{\varphi}$ is a measure-preserving bijection. Thus, given $i \in \mathrm{I}_j$,

$$\mathcal{I}(\mathbf{1}_{\varpi_E^{\nu(\tau_j)} y_i(H\cap P_1)}) = \begin{cases} \mathcal{I}(\mathbf{1}_{H\cap P_1}), & \text{if } j = 1, \\ \mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)}), & \\ \quad \text{if } e_0 = 2 \text{ and } j = 3, \text{ or if } e_0 = 1 \text{ and } j = 2, & \\ \mathcal{I}(\mathbf{1}_{\varpi_E\delta(H\cap P_1)}), & \text{if } e_0 = 2 \text{ and } j = 2. \end{cases}$$

Observe that if $e_0 = 1$, then $c = 1$, $\varpi_E = \varpi_L$ and I_2 is nonempty, and I_3 is empty (Lemma 9.4). If $e_0 = 2$ then $c = \Leftrightarrow 1$ and, if $f(L/(E_{r-1} \cap L))$ is odd, I_2 is empty and I_3 nonempty, otherwise both I_2 and I_3 are nonempty.

Now (10.6) can be rewritten as

$$\begin{aligned} \mathcal{I}(\mathcal{F}_0) &= \mathcal{I}(\mathbf{1}_{H\cap P_1}) \left(\sum_{i\in \mathrm{I}_1} \chi_r(y_i) \right), \\ \mathcal{I}(\mathcal{F}_1) &= \theta(\varpi_L)\mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)}) \left(\sum_{i\in \mathrm{I}_2} \chi_r(y_i) \right), \quad \text{if } e_0 = 1, \\ \mathcal{I}(\mathcal{F}_1) &= \theta(\varpi_E)\mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)}) \left(\sum_{i\in \mathrm{I}_3} \chi_r(y_i) \right) \\ &\qquad + \theta(\varpi_L)\theta_r(\sqrt{\varepsilon})^{-1}\mathcal{I}(\mathbf{1}_{\varpi_E\delta(H\cap P_1)}) \sum_{i\in \mathrm{I}_2} \chi_r(y_i), \quad \text{if } e_0 = 2. \end{aligned}$$

Identifying each y_i, $i \in \mathrm{I}_j$, with its image in $\overline{H}^{\varphi}$, resp. $\overline{H}^{c\varphi}$, if $j = 1$, resp. $j = 2$ or 3, results in (i), (iii), (iv) and, if $e_0 = 1$,

$$\begin{aligned} \mathcal{I}(\mathcal{F}_1) &= \theta(\varpi_L)\,\mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)}) \left(\sum_{x\in \overline{H}^{\varphi}} \chi_r(x) \right) \\ &= \mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)})\mathcal{I}(\mathbf{1}_{H\cap P_1})^{-1}\mathcal{I}(\mathcal{F}_0). \end{aligned}$$

To finish the proof of (ii), note that if $r > 1$ then $f_0 = 2$. If E_1 is ramified over $E_1 \cap L$, then $f(E/E_1)$, hence $[E : E_1]$ is even, and by Lemma 7.13(i), $f_0 = 2$ implies that $m_i = \ell_i + 1$ for some $i \leqslant r \Leftrightarrow 1$. As we have assumed that $m_i = \ell_i$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$, it follows that E_1 is unramified over $E_1 \cap L$. Thus by Lemma 7.9(i), $\mathrm{e}(E_1/F)$ is odd. As $m_1 = \ell_1$, $\mathrm{e}(E/E_1)$ is odd. Thus if $r > 1$, $e = \mathrm{e}(E/F)$ is odd. If $r = 1$, then $\mathrm{e}(E/F) = 1$. As $e = \mathrm{e}(L/F)$, there exists $\lambda \in \mathcal{O}_L^{\times}$ such that $\varpi_L^e = \lambda\varpi_F$, where ϖ_F is a prime element in F. Choose $\eta \in \mathcal{O}_E^\times$ such that $\lambda = N_{E/L}(\eta)$. Because $e+1$ is even, the map

$$\begin{aligned}
&(H \cap P_1)^\varphi \to (\varpi_L^{e+1}\lambda^{-1}(H \cap P_1))^\varphi = (\varpi_F\varpi_L(H \cap P_1))^\varphi \\
&v \mapsto \eta^{-1}\varpi_L^{(e+1)/2} v\varphi(\eta^{-1}\varpi_L^{(e+1)/2}) \\
&\qquad = \varpi_F\varpi_L\,(\varphi(\eta\varpi_L^{-(e+1)/2})v\varphi(\eta^{-1}\varpi_L^{(e+1)/2}))
\end{aligned}$$

is a measure-preserving bijection. Furthermore, by definition of the measure, $v \mapsto \varpi_F v$ is measure preserving. Thus

$$\mathcal{I}(\mathbf{1}_{H\cap P_1}) = \mathcal{I}(\mathbf{1}_{\varpi_F\varpi_L(H\cap P_1)}) = \mathcal{I}(\mathbf{1}_{\varpi_L(H\cap P_1)}),$$

completing the proof of (ii). □

THEOREM 10.7. *Suppose that* $f_E(\theta_r) = 1$. *If* $f_0 = 2$, *assume that* $m_i = \ell_i$ *for* $1 \leqslant i \leqslant r \Leftrightarrow 1$.

(i) *If* $e_0 = 1$ *and* $\theta \mid L^\times \equiv 1$, *then* $\mathcal{I}(\mathcal{F}) > 0$.
(ii) *If* $e_0 = 2$ *and* $\theta \mid L^\times \not\equiv 1$, *then* $\mathcal{I}(\mathcal{F}) > 0$.

Proof. Note that since E/L is unramified and $\theta \circ \sigma = \theta^{-1}$, the condition $\theta \mid L^\times \equiv 1$ is equivalent to $\theta(\varpi_L) = 1$. By Lemma 10.1, it suffices to show that $\mathcal{I}(\mathcal{F}_i) > 0$, $i = 1, 2$.

If $e_0 = 1$ and $\theta(\varpi_L) = 1$, (i) is a consequence of Proposition 10.5(i) and (ii), and Proposition 9.9.

If $e_0 = 2$ and $\theta(\varpi_L) = \Leftrightarrow 1$, (ii) is a consequence of Proposition 10.5(i),(iii) and (iv) and Proposition 9.9. □

11. Main results

Recall that E is a tamely ramified degree $2n$ extension of F and θ is a unitary character of $E^\times$, admissible over F, having the property that $\theta \circ \sigma = \theta^{-1}$ for some involution σ in $\mathrm{Aut}(E/F)$. We continue to assume that the residue characteristic p of F is odd. The fixed field of σ is denoted by L and E_1 is a subfield of E appearing in the Howe factorization of θ (see (2.1)). If $r = 1$ (that is, θ is generic over F) then $E_1 = E$. Let $f_E(\theta)$ be the conductoral exponent of θ. If $f_E(\theta) > 1$, let c_1 be as in (2.3). It follows from remarks preceding Lemma 2.5 that c_1 represents θ on $1 + \mathfrak{p}_E^{f_E(\theta)-1}$, that is, $\theta(1+x) = \psi(\mathrm{tr}_{E/F}(c_1 x))$ for $x \in \mathfrak{p}_E^{f_E(\theta)-1}$. If $c \in E$ also represents θ on $1 + \mathfrak{p}_E^{f_E(\theta)-1}$, then $F(c) \supset F(c_1) = E_1$. Thus E_1 is minimal among those subfields of E generated by elements which represent θ on $1 + \mathfrak{p}_E^{f_E(\theta)-1}$. Our main results are stated in terms of the values of θ on $L^\times$ and, if E is unramified over L, the degree $[E : E_1]$.

The function $\mathcal{F} \in C_c^\infty(\mathrm{GL}_{2n}(F))$ defined in Section 4 represents a finite sum of matrix coefficients of the unitary supercuspidal representation π associated

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to θ. Results of Shahidi state (see Section 4) that non-vanishing of the integral $\mathcal{I}(\mathcal{F})$ defined in Section 4 is related to reducibility of the representation of a classical group induced from the extension of π to a maximal parabolic subgroup. In Theorem 11.1, we show that certain conditions on $\theta \,|\, L^\times$ force $\mathcal{I}(\mathcal{F})$ to be positive. In Theorem 11.4 this is translated into statements about reducibility.

THEOREM 11.1. *If θ satisfies one of the following conditions, then $\mathcal{I}(\mathcal{F}) > 0$.*

(i) *E is ramified over L and $\theta \,|\, L^\times \equiv 1$,*

(ii) *E is unramified over L and $\theta \,|\, L^\times = (\Leftrightarrow 1)^{\nu(\cdot)([E:E_1]-1)}$, with the additional assumption that if $r > 1$ and $f_E(\theta_r) = 1$, then $m_i = \ell_i$, $1 \leqslant i \leqslant r \Leftrightarrow 1$.*

Proof. If (i) holds, the result follows from Proposition 5.3 and Lemma 5.5.

Suppose that (ii) holds. Because $f(E/L) = 2$, $\theta \circ \sigma = \theta^{-1}$ implies that $\theta(\tau) = \theta(\varpi_L)^{\nu(\tau)}$, $\tau \in E^\times$, so

$$\theta(\varpi_L) = (\Leftrightarrow 1)^{[E:E_1]-1} \Longleftrightarrow \theta \,|\, L^\times = (\Leftrightarrow 1)^{\nu(\cdot)([E:E_1]-1)}. \tag{11.2}$$

If $f_E(\theta_r) = 1$, then $\mathcal{I}(\mathcal{F}) > 0$ by Theorem 10.7. Thus we may assume that $f_E(\theta_r) > 1$. Let $1 \leqslant i \leqslant r$. As shown in Lemma 5.2,

$$\chi_i(x) = \kappa_i(x) = \theta_i(N_{E/E_i}(\mu(x))), \quad \text{if } \ m_i = \ell_i. \tag{11.3}$$

Suppose that $m_i = \ell_i + 1$. Let H_i and H_i' be as in Section 6. Because $f_E(\theta_r) > 1$, we know that $H_i = F^\times(1 + \mathfrak{p}_E)K_{i-1}$. By Corollary 3.8, there exist $y \in (E^\times K_{i-1})^\varphi = (E^\times H_i)^\varphi$ and $z \in L_{i-1}$ such that $x = yz$. As shown at the beginning of Section 6, $\chi_i(x) = \chi_i(y)$ and by Lemma 6.1, $\chi_i(y) \neq 0$ implies that y is conjugate to an element of $E^\times H_i'$. By definition of the functions ν and μ, $\nu(x) = \nu(y)$ and $\mu(x) = \mu(y)$.

By (11.3), Proposition 7.12, and Lemma 7.13, if $x \in H^\varphi$ is such that $\chi_\kappa(x) \neq 0$, then there exists $a_x > 0$ such that

$$\begin{aligned}
\chi_\kappa(x) \ &= \ a_x\, (\Leftrightarrow 1)^{\nu(x)([E:E_1]-1)} \prod_{i=1}^{r} \theta_i(N_{E/E_i}(\mu(x))) \\
&= \ (\Leftrightarrow 1)^{\nu(x)([E:E_1]-1)}\, \theta(\mu(x)).
\end{aligned}$$

In view of (11.2), it follows from the assumption on θ that if $x \in H^\varphi$ is such that $\chi_\kappa(x) \neq 0$, then $\chi_\kappa(x) > 0$. In particular, $\chi_\kappa(x) > 0$ for $x \in (\varpi_E^j P_{m_r}(r \Leftrightarrow 1) \ldots P_{m_1}(0))^\varphi$, $j \in \mathbb{Z}$. The subset of φ-invariant points in this set has positive measure in $(\varpi_E^j(H \cap P))^\varphi$. As $\mathcal{F} = \chi_\kappa$ on $\bigcup_{j=0}^{2e-1} \varpi_E^j(H \cap P)$, and zero elsewhere, it follows that $\mathcal{I}(\mathcal{F}) > 0$. □

REMARKS. (i) If $\dim \kappa = 1$, or if θ is generic, then $\theta \,|\, L^\times \equiv 1$ implies that $\mathcal{I}(\mathcal{F}) > 0$. If $\dim \kappa = 1$, this is Proposition 5.3, and when θ is generic, it follows from Theorem 11.1 since $E = E_1$ and $r = 1$.

(ii) The case excluded from the theorem, $r > 1$, $f_E(\theta_r) = 1$ and $m_i = \ell_i + 1$ for some $i \leqslant r \Leftrightarrow 1$ can occur only if E_{r-1} is unramified over $E_{r-1} \cap L$ (Lemma 5.7). In this case, the sign of χ_κ on H^φ will be influenced by both the sign of the cuspidal representation κ_r of the finite general linear group $\mathrm{GL}_{[E:E_{r-1}]}(\overline{E}_{r-1})$ and by the signs of the characters of the Heisenberg representations for those $i \leqslant r \Leftrightarrow 1$ such that $m_i = \ell_i + 1$. Therefore, in order to compute the sign of χ_κ, it would be necessary to find a way to combine the techniques in Sections 6–10 in such a way that sums of products $\chi_{\kappa_r}\chi_{\kappa_i}$ over certain sets could be computed.

Let π be the irreducible unitary self-contragredient supercuspidal representation of $G = \mathrm{GL}_{2n}(F)$ associated to θ via Howe's construction ([H2]). Let $\mathrm{I}(\pi)$ be the induced representation of G' defined in Section 4, where $G' = \mathrm{SO}_{4n}(F)$, $\mathrm{SO}_{4n+1}(F)$ or $\mathrm{Sp}_{4n}(F)$. The following theorem is an immediate consequence of Theorem 11.1 and results of Shahidi (see Theorem 4.1 and Lemma 4.2).

THEOREM 11.4. *Suppose that the admissible character θ associated to π satisfies* (i) *or* (ii) *of Theorem* 11.1. *Then the representation $\mathrm{I}(\pi)$ is irreducible if $G' = \mathrm{SO}_{4n}(F)$ or $\mathrm{Sp}_{4n}(F)$ and reducible if $G' = \mathrm{SO}_{4n+1}(F)$.*

A non-unitary irreducible self-contragredient supercuspidal representation of G arising via the construction of Howe is of the form $\pi \otimes |\det(\cdot)|^\alpha$, for some real number α and some π as above. The admissible character corresponding to such a representation is $|N_{E/F}(\cdot)|^\alpha\theta$. Given an admissible character θ' of $E^\times$ such that $\theta' \circ \sigma = \overline{\theta'}$ for some involution σ in $\mathrm{Aut}(E/F)$, there exists a unitary admissible θ having the same property relative to σ and a real number α such that $\theta' = |N_{E/F}(\cdot)|^\alpha\theta$.

COROLLARY 11.5. *Assume that θ satisfies* (i) *or* (ii) *of Theorem* 11.1.

(i) *If $G' = \mathrm{SO}_{4n}(F)$ or $\mathrm{Sp}_{4n}(F)$, then $\mathrm{I}(\pi \otimes |\det(\cdot)|^\alpha)$ is reducible for $\alpha = \pm\frac{1}{2}$ and irreducible for other real values of α.*

(ii) *If $G' = \mathrm{SO}_{4n+1}(F)$, then $\mathrm{I}(\pi \otimes |\det(\cdot)|^\alpha)$ is irreducible for all non-zero real values of α.*

Proof. Both (i) and (ii) follow from Theorem 11.4 and [Sh], Theorem 5.3, which relates reducibility of $\mathrm{I}(\pi)$ to reducibility of $\mathrm{I}(\pi \otimes |\det(\cdot)|^\alpha)$. □

Given an irreducible unitary supercuspidal representation π' of G, let $\rho(\pi')$ denote the conjectural irreducible $2n$-dimensional representation of the Weil group W_F parametrizing π' ([B], [T]). Let π be as above (unitary and self-contragredient). Henceforth, in order to avoid stating cases, we assume that $G' = \mathrm{SO}_{4n}(F)$ or $\mathrm{Sp}_{4n}(F)$. As indicated by Shahidi ([Sh]), as a consequence of properties of L-functions attached to representations of W_F, it is expected that $\mathrm{I}(\pi)$ is irreducible if and only if $\rho(\pi)$ factors through $\mathrm{Sp}_{2n}(\mathbb{C})$. Otherwise, ρ should factor through $\mathrm{SO}_{2n}(\mathbb{C})$ and $\mathrm{I}(\pi)$ should be reducible.

From now on, we assume that p does not divide $2n$. In this case, Moy ([Mo2]) has shown that every irreducible supercuspidal representation π' of G arises via Howe's construction ([H2]) from an admissible character θ' of the multiplicative group of a tamely ramified degree $2n$ extension of F. The map

$$\theta' \mapsto \mathbf{r}(\theta') = \mathrm{Ind}_{W_E}^{W_F} \theta'$$

induces a bijection between (equivalence classes of) admissible quasi-characters θ' as above and (equivalence classes of) irreducible $2n$-dimensional representations of W_F ([Mo2], Theorem 2.2.2). Thus we have a bijection $\pi' \leftrightarrow \mathbf{r}(\theta')$.

Necessary and sufficient conditions for $\mathbf{r}(\theta')$ to be symplectic or orthogonal are known.

LEMMA 11.6 ([Mo1], Theorem 1). *Let K/F be an extension of degree $2n$. Suppose that θ' is a unitary character of $K^\times$, admissible over F and of finite order. Then the representation $\mathbf{r}(\theta')$ is orthogonal, resp. symplectic, if and only if there exists an involution $\tau \in \mathrm{Aut}(K/F)$ such that $\theta' \circ \tau = \theta'^{-1}$ and $\theta'|K^\tau \equiv 1$*, resp. $\theta'|K^\tau \not\equiv 1$.

REMARK. Moy's result is stated for Galois representations. Such representations can be identified with a subset of the representations of the Weil group W_F ([T], (2.2)). A representation of W_F is a Galois representation if and only if it has finite order. Note that the condition $\theta \circ \sigma = \theta^{-1}$ guarantees that θ has finite order.

Assuming that the conjectural representation $\rho(\pi)$ does exist, it cannot be equal to $\mathbf{r}(\theta)$ because $\mathbf{r}(\theta)$ does not satisfy the required functoriality properties; in particular, π and $\mathbf{r}(\theta)$ do not have the same local constants (see [Mo2], [R]). In Section 4 of [Mo2], Moy defines a character Ω of $E^\times$ (depending on θ) such that that π and $\mathbf{r}(\Omega\theta)$ have the same local constants. (There is a misprint in Moy's paper: the ramification degree $\mathrm{e}(E_1/F)$, not $\mathrm{e}(E/E_1)$, should appear in the definition of Ω). We have checked that $\Omega \,|\, L^\times = \mathrm{sgn}_{E/L}^a(\theta \,|\, L^\times)$, where $a = 1$ if $f(E/L) = 1$ and $a = [E : E_1]$ if $f(E/L) = 2$. Here, $\mathrm{sgn}_{E/L}$ denotes the character of $L^\times$ associated by class field theory to the quadratic extension E/L. Therefore if $\rho(\pi)$ were equal to $\mathbf{r}(\theta\Omega)$, by Lemma 11.6 and remarks above, we would have a criterion for reducibility of $\mathrm{I}(\pi)$ in terms of $f(E/L)$, parity of $[E\colon E_1]$, and $\theta \,|\, L^\times$, as follows.

CONJECTURE.

$$\mathrm{I}(\pi) \text{ is irreducible} \Leftrightarrow \theta \,|\, L^\times \begin{cases} \equiv 1, & \text{if } f(E/L) = 1, \\ = (\Leftrightarrow 1)^{\nu(\cdot)([E:E_1]-1)}, & \text{if } f(E/L) = 2. \end{cases}$$

Thus, as a complement to Theorem 11.4, we would like to prove

$$\begin{array}{l} \text{If } \theta \text{ satisfies one of the following conditions,} \\ \text{then } \mathrm{I}(\pi) \text{ is reducible (for } G' = \mathrm{SO}_{4n}(F) \text{ or } \mathrm{Sp}_{4n}(F)) \\ \quad \text{(i)}' \;\; E \text{ is ramified over } L \text{ and } \theta \mid L^\times \not\equiv 1. \\ \quad \text{(ii)}' \;\; E \text{ is unramified over } L \text{ and } \theta \mid L^\times = (\Leftrightarrow 1)^{\nu(\cdot)[E:E_1]}. \end{array} \tag{11.7}$$

In order to to prove (11.7) using Shahidi's theorem (Theorem 4.1), it would be necessary to show that $\mathcal{I}(f) = 0$ for every $f \in C_c^\infty(G)$ representing a matrix coefficient of π, for all choices of a matrix coefficient. We remark that if (i)$'$ is satisfied and $f(E/F) = f(L/F)$ is odd, or if (ii)$'$ is satisfied and $\mathrm{e}(E/F) = \mathrm{e}(L/F)$ is even, then (11.7) holds because in both these cases $\theta \mid F^\times$ is non-trivial, so $\mathrm{I}(\pi)$ must be reducible by Theorem 4.1.

In some cases, if we assume that $\rho(\pi)$ does exist, then using Theorem 11.4 and properties of $\rho(\pi)$, we can see that (11.7) holds. If $f(E/L) = 2$, or if $f(E/L) = 1$ and $(q \Leftrightarrow 1)/\mathrm{gcd}(e, q \Leftrightarrow 1)$ is even, there exists a character χ of $F^\times$ (of finite order) such that $\chi \circ N_{E/F} \mid L^\times = \mathrm{sgn}_{E/L}$. Thus θ satisfies condition (ii)$'$ if and only if $(\Leftrightarrow 1)^{\nu(\cdot)}\theta = (\chi \circ N_{E/F})\theta$ satisfies the first two parts of condition (ii) of Theorem 11.1 (that is, drop the additional assumption on the m_i's). Similarly, if we assume that $(q \Leftrightarrow 1)/\mathrm{gcd}(e, q \Leftrightarrow 1)$ is even, then θ satisfies condition (i)$'$ if and only if $\chi(N_{E/F}(\cdot))\theta$ satisfies (ii) of Theorem 11.1. One of the expected properties of $\pi \leftrightarrow \rho(\pi)$ is $\rho(\pi \otimes \chi \circ \det) = \rho(\pi) \otimes \chi$ ([Mo2]). Also, the supercuspidal representation corresponding to $(\chi \circ N_{E/F})\theta$ is $\pi \otimes \chi \circ \det$. Hence it follows from the definition of χ that $\rho(\pi)$ is orthogonal if and only if $\rho(\pi \otimes \chi \circ \det)$ is symplectic. In view of Theorem 11.4, we get the following result.

COROLLARY 11.8. *Assume that the representation $\rho(\pi)$ exists. Suppose that one of the following holds*

(a) $(q \Leftrightarrow 1)/\mathrm{gcd}(e, q \Leftrightarrow 1)$ *is even and θ satisfies* (i)$'$;
(b) *θ satisfies* (ii)$'$, *together with the additional condition that if $r > 1$ and $f_E(\theta_r) = 1$, then $m_i = \ell_i$ for $1 \leqslant i \leqslant r \Leftrightarrow 1$.*

Then $\mathrm{I}(\pi)$ is reducible and $\mathrm{I}(\pi \otimes |\det(\cdot)|^\alpha)$ is irreducible for every nonzero real number α.

In Section 7 of [Sh], Shahidi interprets the reducibility criterion of Theorem 4.1 in terms of the theory of twisted endoscopy ([KS1], [KS2]). The group SO_{2n+1} is a twisted endoscopic group of GL_{2n} ([Sh] Section 3) and has $\mathrm{Sp}_{2n}(\mathbb{C}) \times W_F$ as its L-group. When $\rho(\pi)$ factors through $\mathrm{Sp}_{2n}(\mathbb{C})$, which should correspond to $\mathrm{I}(\pi)$ being irreducible ([Sb]), then $\rho(\pi)$ should parametrize an L-packet of discrete series representations of $\mathrm{SO}_{2n+1}(F)$. That is, the L-packet $\{\pi\}$ of $\mathrm{GL}_{2n}(F)$ should come via twisted endoscopic transfer from the L-packet of $\mathrm{SO}_{2n+1}(F)$ parametrized by $\rho(\pi)$. Thus if θ is as in Theorem 11.4 and $G' = \mathrm{SO}_{4n}(F)$ or $\mathrm{Sp}_{4n}(F)$, then π should come from an L-packet of discrete series representations of $\mathrm{SO}_{2n+1}(F)$.

Similarly ([Sh], Sections 3 and 7), a quasi-split SO_{2n} is a twisted endoscopic group of GL_{2n}. If $\mathrm{I}(\pi)$ is reducible then, since $\rho(\pi)$ should factor through $\mathrm{SO}_{2n}(\mathbb{C})$, π should come via twisted endoscopic transfer from an L-packet of a quasi-split $\mathrm{SO}_{2n}(F)$.

Therefore if the above conjecture holds, and the theory of twisted endoscopy holds, then we have a criterion, in terms of θ, for determining whether an irreducible unitary self-contragredient representation π comes via twisted endoscopy from an L-packet of $\mathrm{SO}_{2n+1}(F)$ or of a quasi-split $\mathrm{SO}_{2n}(F)$.

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