ABELIAN ERGODIC THEOREMS FOR VECTOR-VALUED FUNCTIONS

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This note contains extensions of the Abelian ergodic theorems in [3] and [6] to functions which take their values in a Banach space. The results are based on an adaptation of Rota's maximal ergodic theorem for Abel limits [8]. Convergence theorems for continuous parameter semigroups are deduced by the approximation technique developed in [3], [6]. A direct application of the resolvent equation also enables us to deduce a convergence theorem for pseudo-resolvents.

Let (Ω, β, μ) denote a σ-finite complete measure space and let X be a Banach space. As in [6], we call an operator T with domain dense in $L_X^1 \equiv L^1(\Omega, \beta, \mu, X)$ a *Dunford-Schwartz* operator if $\|T\|_1 \leqq 1$ and $\|T\|_\infty \leqq 1$. By the Riesz–Thorin convexity theorem [2, Chapter V] we may then extend T to a contraction on each space $L_X^p(1 \leqq p \leqq \infty)$. We shall deal exclusively with Dunford–Schwartz operators. The operator $R_\rho \equiv \sum_{k=0}^{\infty} \rho^k T^k (0 \leqq \rho < 1)$ is again Dunford–Schwartz and the discussion in [3] of measurable representations of the map $f \to R_\rho f (f \in L_X^p)$ extends easily to this situation.

Similarly we may consider a class (E) semigroup (see [5]) $\{T_t : t \geqq 0\}$ of Dunford–Schwartz operators on L_X^p and adapt the discussion in [3] of *admissible measurable representations* of the map $f \to J_\lambda f \equiv \int_0^\infty e^{-\lambda t} T_t f dt$ to the present case. Thus the symbol $(J_\lambda f)(w)$ will denote a well-defined element of X for each $\lambda > 0$ and for almost all $w \in \Omega$.

1. Maximal ergodic theorems. Let T be a Dunford-Schwartz operator on L_X^p. Given $f \in L_X^p$ and $a > 0$, define

$$\Omega_{f,a} = \{w \in \Omega : (1-\rho)\|(R_\rho f)(w)\|_X > a\}.$$

THEOREM 1.

$$\int_{\Omega_{f,a}} (\|f(w)\|_X - a)\mu(dw) \geqq 0.$$

Proof. Let $h_0 \in L^p$ and suppose that e_0 is a strongly measurable function from Ω into the unit ball of X. Define sequences $\{h_n\}$ and $\{e_n\}$ by setting, for $n \geqq 0$,

$$h_{n+1} = \|T(h_n^+ e_n)\|_X - h_n^-$$

$$e_{n+1} = \begin{cases} 0 , & \text{when } T(h_n^+ e_n) = 0 \\ \dfrac{T(h_n^+ e_n)}{\|T(h_n^+ e_n)\|_X}, & \text{otherwise} \end{cases}$$

Neveu [7] proved that

(i) $\{h_n^-\}_n$ decreases in L^p,

(ii) $\{\int_H h_n \, d\mu\}_n$ decreases, where $H = \bigcup_{n \geq 0} \{w : h_n(w) > 0\}$.

Hence we obtain

$$\int_H h_0 \, d\mu \geqq 0. \tag{1}$$

Now for each $m \geqq 0$ consider the identity

$$T(h_m^+ e_m) = h_{m+1}^+ e_{m+1} + (h_m^- - h_{m+1}^-) e_{m+1}, \tag{2}$$

which follows from the definition of h_{m+1}. Fix $\rho \in (0, 1)$ and, for each m, multiply both sides of (2) by ρ^{m+1} and add the resulting equations for $m = 0, 1, 2, \ldots$. We obtain

$$(I - \rho T) \sum_{k=0}^{\infty} \rho^k h_k^+ e_k = h_0^+ e_0 - \sum_{k=1}^{\infty} \rho^k (h_{k-1}^- - h_k^-) e_k. \tag{3}$$

Now apply this equation with $h_0(.) = \|f(.)\|_X - a$ and

$$e_0(.) = \frac{f(.)}{\|f(.)\|_X}$$

to obtain

$$f = (I - \rho T) \sum_{k=0}^{\infty} \rho^k h_k^+ e_k + (a - h_0^-) e_0 + \sum_{k=1}^{\infty} \rho^k (h_{k-1}^- - h_k^-) e_k. \tag{4}$$

By telescoping the sum of the last two terms in (4) has norm (in X) less than or equal to a for almost all $w \in \Omega$. Hence

$$\left\| R_\rho (a - h_0^-) e_0 + \sum_{k=1}^{\infty} \rho^k (h_{k-1}^- - h_k^-) e_k \right\| \leqq \sum_{k=0}^{\infty} \rho^k \|T\|_\infty \cdot a \leqq \frac{a}{1-\rho}.$$

Using this estimate in (4) we find that

$$\Omega_{f,a} \subseteq \bigcup_{0<\rho<1} \left\{ w : \sum_{k=0}^{\infty} \rho^k (h_k^+ e_k)(w) \neq 0 \right\} \subseteq H.$$

As $\Omega_{f,a} \supseteq \{w : \|f(w)\|_X > a\}$ we have proved that

$$\int_{\Omega_{f,a}} (\|f(w)\|_X - a) \mu(dw) > 0.$$

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REMARK. By Theorem 1, the function $f^* = \sup_{0<\rho<1} ((1-\rho)\|R_\rho f\|_X)$ is a.e. (μ) finite for each $f \in L_X^p$. (cf. [2, VIII. 6.5]). Almost everywhere convergence of $(1-\rho)(R_\rho f)(w)$ is easily obtained on the set $L = L_1 \oplus L_2$, where L_1 is the set of fixed points of T and $L_2 = \{g \in L_X^p : g = g_1 - Tg_1 \text{ for } g_1 \in L_X^p \cap L_X^\infty\}$. *If X is reflexive*, The Yosida-Kakutani mean ergodic theorem implies that L is dense in L_X^p (cf. [1]). Hence, by the Banach convergence theorem, Theorem 1 implies the a.e. (μ) convergence of the averages $(1-\rho)(R_\rho f)(w)$ as $\rho \uparrow 1$, for all $f \in L_X^p$, $1 \leqq p < \infty$.

A continuous-parameter version of Theorem 1 can be obtained for class (C_0) semi-groups by the approximation arguments of [3], [6]. We omit the proof.

THEOREM 2. *Let X be a Banach space and let $\{T_t\}_{t \geqq 0}$ be a class (C_0) semigroup of Dunford–Schwartz operators on L_X^1. If $a > 0, f \in L_X^p$, $1 \leqq p \leqq \infty$, and if $\Omega_{f,a}^* = \{w : \sup_{\lambda>0} \|(\lambda J_\lambda f)(w)\|_X > a\}$, then $\int_{\Omega_{f,a}^*} (\|f(w)\|_X - a) d\mu \geqq 0$.*

Finally, Theorem 2 leads to the following convergence theorem by the technique developed in [3]:

THEOREM 3. *If X is reflexive and if $f \in L_X^p$, then the averages $(\lambda J_\lambda f)(w)$ converge in X as $\lambda \downarrow 0$, for almost all $w \in \Omega$.*

2. Convergence of pseudo-resolvents. Let $\{R_\lambda\}_{\lambda>0}$ be a pseudo-resolvent (cf. [5]) on L_X^p. The operators $\{R_\lambda\}$ satisfy the *first resolvent equation*:

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu. \tag{5}$$

It is well-known that the $\{R_\lambda\}$ have a common range and kernel and commute pairwise.

Putting $\mu = 1$ in (5) we obtain immediately

$$R_\lambda = \sum_{k=0}^{\infty} (1-\lambda)^k R_1^{k+1} \quad \text{for} \quad 0 < \lambda < 1. \tag{6}$$

(This was first used in [4].)

In order to discuss the a.e. convergence of $(R_\lambda f)(w)$ for $f \in L_X^p$, we may adapt the usual discussion of measurable representations of the map $\lambda \to R_\lambda f$ and *choose* the representation $H(\lambda, w) = \sum_{k=0}^{\infty} (1-\lambda)^k (R_1^{k+1} f)(w)$. We wish our representation to include the previous definition of $(J_\lambda f)(w)$. The following simple lemma shows that our choice of H is then μ-essentially unique.

LEMMA. *Let $\lambda \to G(\lambda)$ be Bochner-integrable on $(0, 1]$, with values in L_X^p, and let H_1 and H_2 denote two measurable representations of $\lambda \to G(\lambda)$. Suppose that H_1 and H_2 are continuous on $(0, 1]$ for almost all $w \in \Omega$. Then there is a μ-null set N with $H_1(\lambda, w) = H_2(\lambda, w)$ for all $\lambda \in (0, 1]$ and $w \in \Omega \backslash N$.*

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THEOREM 4. *Suppose that X is reflexive. If $\{R_\lambda\}_{0<\lambda\leq 1}$ is a pseudo-resolvent on L^1_X such that R_1 is a Dunford–Schwartz operator, then for all $f \in L^p_X$, the family $(\lambda R_\lambda f)(w)$ converges to an element of X for almost all $w \in \Omega$, as $\lambda \downarrow 0$.*

Proof. Let $\lambda = 1-\rho$, $R_1 f = g$; then $(\lambda R_\lambda f)(w) = (1-\rho)\sum_{k=0}^{\infty} \rho^k (R_1^k g)(w)$, and the convergence assertion is a consequence of the remark following Theorem 1.

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