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Some Results on Surfaces of General Type

Dedicated to M. Pavaman Murthy

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Abstract. In this article we prove some new results on projective normality, normal presentation and higher syzygies for surfaces of general type, not necessarily smooth, embedded by adjoint linear series. Some of the corollaries of more general results include: results on property N_p associated to $K_S \otimes B^{\otimes n}$ where B is base-point free and ample divisor with $B \otimes K^*$ *nef*, results for pluricanonical linear systems and results giving effective bounds for adjoint linear series associated to ample bundles. Examples in the last section show that the results are optimal.

Introduction

In this article we prove new results on higher syzygies associated to adjunction bundles for a surface of general type. To motivate the results, we need to introduce some definitions, notations and concepts.

Let L be a very ample line bundle on a variety X and let

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R \longrightarrow 0$$

be the minimal graded free resolution of the coordinate ring R of the image of X by the embedding induced by L. Let I_X be the ideal defining X under the embedding given by L. The property N_p is defined as follows

– L satisfies the property N_0 (or embeds X as a projectively normal variety) if R is normal.

– L satisfies the property N_1 (or is normally generated) if in addition I_X is generated by quadrics, that is, if the entries of the matrix of φ_1 have degree 2.

– L satisfies the property N_p if in addition to satisfying property N_1, the resolution is linear from the second step until the pth step, *i.e.*, if the matrices of $\varphi_2, \dots, \varphi_p$ have linear entries.

Several precise results on projective normality, normal presentation and higher syzygies have been proved for the case of an algebraic curve. For algebraic surfaces and higher dimensional varieties, the terrain of higher syzygies and its connections to geometry are not well charted.

We will now mention some basic questions in this area. Reider [R] showed that for an algebraic surface S, $K_S \otimes A^{\otimes n}$ is very ample for all $n \geq 4$ if A is ample. This motivated the following conjecture of Mukai, which is a two dimensional analogue of Green's result [G2] for curves: Let S be an algebraic surface and A an ample line

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bundle on S, then $K_S \otimes A^{\otimes n}$ satisfies property N_p for all $n \geq p+4$. Not even the first case is settled in all its generality: $p = 0$. A closely related question is the following (Q1): What happens if you replace an ample bundle in Mukai's conjecture with an ample and base-point free bundle? This question is still open for a surface of positive Kodaira dimension. An interesting weaker question would be to ask for an effective bound towards Mukai's conjecture analogous to effective Matsusaka's results on freeness and very ampleness. More explicitly, it would be nice to have an answer to the following (Q2): Given an ample line bundle A on an algebraic surface S, can one prove $K_S \otimes A^{\otimes n}$ satisfies N_p with n depending for instance on the Hilbert function of A? These questions are addressed in various generality in this article for a surface of general type.

We will start with some of the known results that are in the spirit of this paper. For an algebraic variety of arbitrary dimension, in [EL] the authors prove a beautiful general result on adjoint linear series associated to a *very ample* line bundle. The article [Bu] deals with higher syzygies for ruled varieties over a curve obtaining a uniform bound in the line of Mukai's conjecture. In [Hb] the author has proved results on projective normality for rational surfaces. For Abelian varieties, results on syzygies related to multiples of ample bundles can be found in [Pa] (see also [K].) For projective spaces, higher syzygy results have been proved in [OP]. In [GP2], the authors obtain uniform bounds towards Mukai's conjecture and also answer (Q1) for a surface of Kodaira dimension zero and prove results on syzygies of pluricanonical embeddings of surfaces of general type. In [GLM], the authors study questions on projective normality of Enriques surfaces. In [GP1] it is shown that a strong conjecture, which recovers Mukai's conjecture as a special case, is shown to be true for rational surfaces. Also in [GP1], the authors show the connections between algebra of free resolutions and the geometry for a rational surface (see also [V] for results connecting the geometry and algebra.) In [GP4, GP5], the authors study N_p property of arbitrary bundles (not necessarily adjoint linear systems) over an elliptic ruled surface (see also [Ho1, Ho2].)

In this article we prove new results on projective normality, normal presentation and higher syzygies for a surface of general type, not necessarily smooth, embedded by adjoint linear series associated to an ample and base-point free line bundle. In Section 3, we prove some precise results on projective normality and normal presentation for adjoint linear systems. From more general results namely, Theorem 3.3, Theorem 3.4, Theorem 3.9 and Theorem 3.11, we obtain in particular an answer to a stronger version of (Q1) for ample and base-point free line bundles B with $B \otimes K_S^*$ *nef*. Other corollaries of the above mentioned results include new cases, missing in [GP2], on projective normality and normal presentation of pluricanonical linear systems. Besides proving new cases, these results also improve the bounds on the results in [GP2] and unify results of other authors including [Bo, Ci, G1].

In Section 4, we prove results on higher syzygies associated to adjoint bundles; we answer (Q1) whenever $B \otimes K_S^*$ is *nef*. This result generalizes to higher syzygies the results in Section 3 and results in [GP2]. Further applications of these theorems include higher syzygy results for pluricanonical embeddings of surfaces of general type, which recover results in [GP2] and answer to (Q2) giving effective bounds for adjoint linear systems satisfying property N_p associated to ample line bundles.

In Section 5, we construct some examples to show that the results in the previous sections are optimal in several ways.

For proving results on higher syzygies for surfaces and higher dimensional varieties, some methods are available (see [EL, Pa, OP, Bu]). The methods of this article are different from those used in the above works and have some common features with [GP2]. The techniques for proving results on projective normality and normal presentation in Section 3 differ from those in [GP2]. But the methods to prove results on higher syzygies build upon the methods in [GP2], where results that are similar in spirit to the present article are proved for Kodaira dimension zero surfaces. The situation for surfaces of general type, needless to say, is more involved due to the "big and nef-ness" of K_S. The proofs involve vanishing of a Koszul cohomology group. Standard methods using Castelnuovo–Mumford regularity do not work. To study curves lying on the surface is important for us in the context at hand. Going to curves on a surface to understand the geometry of the surface is not new, but the way it is done here and in [GP2] in the context of higher syzygies is different from the previous works of other authors. Also, the last section of this paper shows that the property N_p of a line bundle L on the surface is closely related to property N_p of L restricted to some curves lying on it. These are the so-called "extremal" curves introduced in [GP3]. Unlike in the case of surfaces with Kodaira dimension zero dealt with in [GP2], the choice of the divisor to which one reduces the problem is not always clear. In particular in Theorem 4.5 on higher syzygies, to make the induction work in a reasonable way, one has to make some non-canonical choices of these divisors.

1 Preliminaries and Notation

Some Notation and Conventions Unless otherwise stated, all surfaces in this paper have at worst canonical singularities and are all minimal. But it must be mentioned that all of the results go through almost word-for-word but with weaker bounds even for normal surfaces with log terminal singularities. But we will stick to the former to get cleaner statements.

If L is a line bundle on a surface S, L^* denotes the dual of L.

Since we are working over singular surfaces as well, all divisors that appear in this article are assumed to be Cartier divisors.

If C is an effective divisor and E any vector bundle on S,

$$H^0(E \otimes \mathcal{O}_C) = H^0(C, E \otimes \mathcal{O}_C).$$

We will not make a notational distinction between divisors and line bundles when dealing with inequalities.

Throughout this article we work over an algebraically closed field of *characteristic zero*.

Green [G2] interpreted the Betti numbers of the minimal free resolution of the coordinate ring of an embedded projective variety in terms of Koszul cohomology.

Concretely, let X be a projective variety, and let L be a globally generated vector bundle on X. We define the bundle M_L as follows:

$$(*) \qquad 0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0.$$

This sequence will be used repeatedly in this article. If L is an ample and globally generated line bundle on X and all its positive powers are non-special one has the following characterization of the property N_p:

Theorem 1.1 *Let L be an ample, globally generated line bundle on a variety X. If the group $H^1(\bigwedge^{p'+1} M_L \otimes L^{\otimes s})$ vanishes for all $0 \leq p' \leq p$ and all $s \geq 1$, then L satisfies the property N_p. If in addition $H^1(L^{\otimes r}) = 0$, for all $r \geq 1$, then the above is a necessary and sufficient condition for L to satisfy property N_p.*

We use this theorem as a definition for property N_p. We obtain higher syzygy results by proving the above vanishing. We will always prove in this article, except in section 5, the vanishing of $H^1(M_L^{\otimes p'+1} \otimes L^{\otimes s})$. This in turn implies the vanishing of the Koszul cohomology group $H^1(\bigwedge^{p'+1} M_L \otimes L^{\otimes s})$ as we will be working over an algebraically closed field of *characteristic* 0.

2 Some Technical Lemmas and Propositions

In this section we will recall some lemmas that were proved in [GP2]. These lemmas together with a lemma in the next section are necessary to obtain results on higher syzygies.

Lemma 2.1 *Let S be a surface of general type. Let B be an ample and base-point-free line bundle with $H^1(B) = 0$ and $B^2 \geq B \cdot K_S$. Then $H^1(B^{\otimes m}) = 0$ for all $m \geq 1$.*

Proof Let C be a smooth curve in $|B|$. Since $\deg(B^{\otimes m} \otimes \mathcal{O}_C) > 2g(C) - 2$ when $m \geq 3$, we only have to prove $H^1(B^{\otimes 2}) = 0$. If $B^{\otimes 2} \otimes \mathcal{O}_C \neq K_C$, then $H^1(B^{\otimes 2} \otimes \mathcal{O}_C) = 0$, hence $H^1(B^{\otimes 2}) = 0$ because $H^1(B) = 0$. If $B^{\otimes 2} \otimes \mathcal{O}_C = K_C$, then $B \otimes \mathcal{O}_C = K_S \otimes \mathcal{O}_C$. Consider the sequence

$$0 \to H^0(K_S^*) \to H^0(B \otimes K_S^*) \to H^0(B \otimes K_S^* \otimes \mathcal{O}_C) \to H^1(K_S^*).$$

Since in this case S is a surface of general type, $H^0(K_S^*) = H^1(K_S^*) = 0$, therefore $B \otimes K_S^*$ is effective and since B is ample, it must be $B \otimes K_S^* = \mathcal{O}_S$. Hence $H^1(B^{\otimes 2}) = H^1(K_S^{\otimes 2}) = 0$. ∎

Lemma 2.2 *Let S be an algebraic surface with nonnegative Kodaira dimension and let B be an ample line bundle. Let $m \geq 1$. If $B^2 \geq mK_S \cdot B$, then $K_S \cdot B \geq mK_S^2$.*

Proof We assume the contrary, *i.e.*, that $K_S \cdot B < mK_S^2$, and get a contradiction. Let $L = B \otimes K_S^{-m}$. We have that $L^2 > 0$. By Riemann–Roch,

$$h^0(L^{\otimes n}) \geq \frac{n^2 L^2 - nK_S \cdot L}{2} + \chi(\mathcal{O}_S) - h^0(K_S \otimes L^{-n}).$$

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If $B^2 > mK_S \cdot B$, $(K_S \otimes L^{\otimes -n}) \cdot B < 0$, for n large enough, and since B is ample, $K_S \otimes L^{\otimes -n}$ is not effective, so finally $L^{\otimes n}$ is effective for n large enough. But in that case $nK_S \cdot L \geq 0$, because K_S is nef, contradicting our assumption.

Now if $B^2 = mK_S \cdot B$, we have that $L^2 > 0$, $B^2 > 0$ (because B is ample), and $L \cdot B = 0$, but this is impossible by the Hodge index theorem. ■

The following is a very useful observation and will be used repeatedly:

Observation 2.3 Let E and $L_1, \dots, L_r$ be coherent sheaves on a variety X. Consider the map $H^0(E) \otimes H^0(L_1 \otimes \cdots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \cdots \otimes L_r)$ and the maps

$$H^0(E) \otimes H^0(L_1) \xrightarrow{\alpha_1} H^0(E \otimes L_1),$$
$$H^0(E \otimes L_1) \otimes H^0(L_2) \xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2),$$
$$\dots,$$
$$H^0(E \otimes L_1 \otimes \cdots \otimes L_{r-1}) \otimes H^0(L_r) \xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \cdots \otimes L_r).$$

If $\alpha_1, \dots, \alpha_r$ are surjective then ψ is also surjective.

The following from [GP2] is an elementary observation relating the surjectivity of multiplication maps on a variety to the surjectivity of its restrictions to divisors.

Lemma 2.4 *Let X be a regular variety* (i.e., *a variety such that $H^1(\mathcal{O}_X) = 0$*). *Let E be a vector bundle on X, and let C be a divisor such that $L = \mathcal{O}_X(C)$ is globally generated and $H^1(E \otimes L^{-1}) = 0$. If the multiplication map $H^0(E \otimes \mathcal{O}_C) \otimes H^0(L \otimes \mathcal{O}_C) \to H^0(E \otimes L \otimes \mathcal{O}_C)$ is surjective, then the map $H^0(E) \otimes H^0(L) \to H^0(E \otimes L)$ is also surjective.*

The following result is from [Bu]. This technical result deals with multiplication maps of global sections of semistable vector bundles on curves. In the proposition below, μ will denote the slope of a vector bundle. That is, for a vector bundle E on C of rank r and degree d, $\mu(E) = d/r$.

Proposition 2.5 (Proposition 2.2, [Bu]) *Let E and F be semistable vector bundles over a curve C such that E is generated by its global sections. If*

(1) $\mu(F) \geq 2g$, *and*
(2) $\mu(F) > 2g + \operatorname{rank}(E)\big(2g - \mu(E)\big) - 2h^1(E)$,

then the multiplication map $H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$ is surjective.

The following lemma from [GP2] is frequently used in Section 4.

Lemma 2.6 ([GP2], Lemma 2.9) *Let X be a projective variety, let q be a non-negative integer and let F be a base-point-free line bundle on X. Let Q be an effective line bundle on X and let $\mathfrak{q}$ be a reduced and irreducible member of $|Q|$. Let R be a line bundle and G a sheaf on X such that*

1. $H^1(F \otimes Q^*) = 0$
2. $H^0(M_{(F\otimes\mathcal{O}_\mathfrak{q})}^{\otimes q'} \otimes R \otimes \mathcal{O}_\mathfrak{q}) \otimes H^0(G) \to H^0(M_{(F\otimes\mathcal{O}_\mathfrak{q})}^{\otimes q'} \otimes R \otimes G \otimes \mathcal{O}_\mathfrak{q})$ *is surjective for all* $0 \leq q' \leq q$.

Then, for all $0 \leq q'' \leq q$ *and for all* $0 \leq k \leq q''$,

$$H^0(M_F^{\otimes k} \otimes M_{(F\otimes \mathcal{O}_{\mathfrak{q}})}^{\otimes q''-k} \otimes R \otimes \mathcal{O}_{\mathfrak{q}}) \otimes H^0(G) \to H^0(M_F^{\otimes k} \otimes M_{(F\otimes \mathcal{O}_{\mathfrak{q}})}^{\otimes q''-k} \otimes G \otimes R \otimes \mathcal{O}_{\mathfrak{q}})$$

is surjective.

The lemma below, a generalization of the base point-free pencil trick, is due to Green and is used in Section 3.

Lemma 2.7 (H^0 Lemma [G2], Theorem (4.e.1)) *Let* C *be a smooth and irreducible curve. Let* L *and* M *be line bundles over* C. *Let* W *be a base-point free linear subsystem of* $H^0(C, L)$. *Then the multiplication map* $W \otimes H^0(M) \to H^0(L \otimes M)$ *is surjective if* $h^1(M \otimes L^{-1}) \leq \dim W - 2$.

The following lemma called the *Castelnuovo-Mumford lemma* (see [Mu]) will be sometimes used in this article.

Lemma 2.8 (CM Lemma, [Mu]) *Let* L *be a base-point free line bundle on a variety* X *and let* $\mathcal{F}$ *be a coherent sheaf on* X. *If* $H^i(\mathcal{F} \otimes L^{-i}) = 0$ *for all* $i \geq 1$, *then the multiplication map*

$$H^0(\mathcal{F} \otimes L^{\otimes i}) \otimes H^0(L) \to H^0(\mathcal{F} \otimes L^{\otimes i+1})$$

is surjective for all $i \geq 0$.

3 Cohomology Vanishings, Projective Normality and Normal Presentation

In this section we prove theorems on projective normality and normal presentation of adjunction bundles associated to globally generated line bundles. These yield corollaries for pluricanonical linear systems and effective bounds on adjunction bundles associated to ample line bundles.

We will first prove a lemma that is needed for the theorems in Section 3 and 4.

Lemma 3.1 *Let* S *be an algebraic surface of general type with an ample divisor* B *such that* $B \not\equiv K_S$, $(B \otimes K_S^*)^2 \geq 0$ *and* $B^2 \geq B \cdot K_S$. *If either* $K_S^2 > 1$, *or* $K_S^2 = 1$ *but* $B \not\equiv 2K_S$, *then* $K_S \cdot B + B^2 \geq 2K_S^2 + 6$. *In particular, if* $|B|$ *has an irreducible member* C, *then its genus* $g(C) \geq K_S^2 + 4$.

Proof Denote $A = B \otimes K_S^*$. Note that

(3.1.1) $$A \cdot (K_S \otimes B) = 2K_S \cdot A + A^2.$$

By Lemma 2.2, we have $K_S \cdot A \geq 0$. We will break the proof into two cases: (i) $A^2 = 0$. By hypothesis $B \not\equiv K_S$, so by Hodge index theorem $K_S \cdot A$ cannot be zero and by Riemann–Roch, $K_S \cdot A$ cannot be 1. So $K_S \cdot A \geq 2$. But this implies $B \cdot K_S \geq K_S^2 + 2$ and by (3.1.1) we have $B^2 \geq K_S^2 + 4$. Hence we get the desired inequality $K_S \cdot B + B^2 \geq 2K_S^2 + 6$.

We now deal with case (ii), where $A^2 > 0$. By hypothesis we have $B^2 \geq K_S \cdot B$. In the light of Lemma 2.2, it is enough to prove that $B^2 \geq K^2+6$. By (3.1.1), if $K_S \cdot A \geq 3$ or $K_S \cdot A \geq 2$ and $A^2 \geq 2$, we are done. By hypothesis $K_S^2 \geq 2$ and $A^2 > 0$ and since $K_S \cdot A$ is (an integer) greater than 1 by Hodge Index Theorem, we have $K_S \cdot A \geq 2$. The only possibility left is when $K_S \cdot A = 2$ and $A^2 = 1$. This cannot happen by Riemann–Roch.

Note that we did not use the fact that $K_S^2 \geq 2$ when $A^2 = 0$, we used it only when $A^2 > 0$. So let $K_S^2 = 1$ and $A^2 \geq 1$. Hodge Index shows that $B \cdot A > 0$ (that is $B^2 > B \cdot K_S$) and $K_S \cdot A > 0$ (that is $B \cdot K_S > K_S^2 = 1$.) So it is enough to show that $B^2 \geq K_S^2 + 5$. In view of (3.1.1), this holds if $K_S \cdot A > 1$ or $A^2 > 2$. The possibilities $K_S \cdot A = 1$ and $A^2 = 2$ cannot happen simultaneously. The only possibility that we need to consider is $K_S \cdot A = 1$ and $A^2 = 1$. This implies by Hodge Index Theorem that $2K_S \equiv B$ as asserted. ■

Remark 3.1.1 Let $K_S^2 \geq 2$ or $K_S^2 = 1$ but $B \not\equiv 2K_S$. Then any ample B with $B \otimes K_S^*$ *nef* satisfies the inequality in Lemma 3.1 since $(B \otimes K_S^*)^2 \geq 0$. This is a geometrically interesting assumption that occurs in various contexts including in the later part of this article.

We will now prove a theorem that is new for irregular surfaces. The theorem also recovers and improves Theorem 5.1 in [GP2] for regular surfaces as well. The proof below is a uniform proof covering the case of regular as well as irregular surfaces. In Section 5, we give examples to show that the theorem is optimal. Before stating the theorem, we make the following necessary remark.

Remark 3.2 As already noted after Theorem 1.1, we need to prove the vanishing of $H^1(M_L^{\otimes p} \otimes L^{\otimes s})$ for all $s \geq 1$. Throughout this article we will prove this vanishing for $s = 1$. We point out that using Observation 2.3 repeatedly, the proof for $s \geq 2$ follows in exactly the same way as the case $s = 1$ (due to algorithmic nature of the proofs.)

In the theorem below, let $E = K_S \otimes B^{\otimes n}$ and $L = K_S \otimes B^{\otimes l}$ with $n \geq 2$ and $l \geq 2$.

Theorem 3.3 *Let S be a surface of general type. Let B be an ample and base-point-free line bundle such that $H^1(B) = 0$ and $B^2 \geq B \cdot K_S$ with $B \not\equiv K_S$. Assume that $K_S \otimes B$ is base-point free.*

(1) *If S is regular, let $p_g \geq 3$ or $h^0(B) \geq 4$, $K_S^2 \geq 2$ and $p_g \geq 1$.*
(2) *If S is irregular, let $p_g \geq 2$ and $h^0(B) \geq 4$.*

Then $H^1(M_L \otimes E^{\otimes k}) = 0$ for all $k \geq 1$.

Proof We will prove the theorem for $k = 1$ as noted in Remark 3.2. By tensoring $(*)$ on page 4 (which will recall for the benefit of the reader)

$$(*) \qquad 0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0$$

with E and taking long exact sequence one sees by the Kawamata–Viehweg (K–V) vanishing theorem that $H^1(M_L \otimes E)$ is the cokernel of the following multiplication

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map of global sections:

$$H^0(K_S \otimes B^{\otimes n}) \otimes H^0(K_S \otimes B^{\otimes l}) \to H^0(K_S^{\otimes 2} \otimes B^{\otimes n+l}). \tag{3.3.1}$$

We will prove the theorem for $l = 2$, the cases $l \geq 3$ are similar to the proof given here for $l = 2$. Applying Observation 2.3, we will first show that

$$H^0(K_S \otimes B^{\otimes n}) \otimes H^0(B) \to H^0(K_S \otimes B^{\otimes (n+1)}) \tag{3.3.2}$$

is surjective for all $n \geq 2$. Let $C \in |B|$ be a smooth and irreducible curve in its linear system. We construct the following commutative diagram:

$$\begin{array}{ccccc} H^0(E) \otimes H^0(\mathcal{O}_S) & \hookrightarrow & H^0(E) \otimes H^0(B) & \twoheadrightarrow & H^0(E) \otimes W \\ \downarrow & & \downarrow & & \downarrow \\ H^0(E) & \hookrightarrow & H^0(E \otimes B) & \twoheadrightarrow & H^0(E \otimes B \otimes \mathcal{O}_C). \end{array} \tag{3.3.3}$$

Here W denotes the cokernel of the inclusion map $H^0(\mathcal{O}_S) \to H^0(B)$. The surjectivity of the left hand vertical map is obvious. We will show that the right hand vertical map is also surjective. Note that $H^0(E) \to H^0(E \otimes \mathcal{O}_C) \to 0$. The right hand map is surjective if the following map is surjective for all $n \geq 2$:

$$H^0(K_S \otimes B^{\otimes n} \otimes \mathcal{O}_C) \otimes W \to H^0(K_S \otimes B^{\otimes n+1} \otimes \mathcal{O}_C). \tag{3.3.4}$$

By Lemma 2.7, the map (3.3.4) is surjective if $h^1(K_S \otimes B^{\otimes n-1} \otimes \mathcal{O}_C) \leq \dim W - 2$. This is obvious if $n \geq 3$. If $n = 2$, then $h^1(K_S \otimes B^{\otimes (n-1)} \otimes \mathcal{O}_C) = 1$, and the needed inequality follows provided $h^0(B) \geq 4$. For regular surfaces this follows from Riemann–Roch and hypothesis (1) (that is $p_g \geq 3$), as $h^2(B) = 0$. If S is irregular, this follows from hypothesis (2). Next step is to show that

$$H^0(K_S \otimes B^{\otimes n+1}) \otimes H^0(K_S \otimes B) \to H^0(K_S^{\otimes 2} \otimes B^{\otimes n+2}) \tag{3.3.5}$$

is surjective for all $n \geq 2$. Let $C' \in |K_S \otimes B|$ be a smooth and irreducible curve. Let W' be the linear subseries of $H^0(K_S \otimes B \otimes \mathcal{O}_{C'})$ defined as below:

$$0 \to H^0(\mathcal{O}_S) \to H^0(K_S \otimes B) \to W' \to 0.$$

Note that W' is without base points. By a process similar to the one used above in (3.3.3), we can reduce the multiplication map (3.3.5) on the surface to C'. It is enough to show that

$$H^0(K_S \otimes B^{\otimes n} \otimes \mathcal{O}_{C'}) \otimes W' \to H^0(K_S^{\otimes 2} \otimes B^{\otimes (n+1)} \otimes \mathcal{O}_{C'}) \tag{3.3.6}$$

is surjective for all $n \geq 3$. The map in (3.3.6) is surjective by Lemma 2.7 provided $h^1(B^{\otimes n-1} \otimes \mathcal{O}_{C'}) \leq \dim W' - 2$ for all $n \geq 3$. If $n > 4$, it is easy to see that $h^1(B^{\otimes n-1} \otimes \mathcal{O}_{C'}) = 0$. If $n = 4$, it is not hard to see this vanishes; the only troubling case would be if $B^{\otimes n-1} \otimes \mathcal{O}_{C'} = K_{C'}$. But in view of $B^2 \geq B \cdot K_S$ and Lemma 2.2, this can happen only if $B^2 = B \cdot K_S = K_S^2$. But the Hodge Index theorem rules out this possibility since $B \not\equiv K_S$. The case $n = 3$ needs some work, and we show it below. We assume from now on that $n = 3$. Since $H^1(B) = 0$ by hypothesis and $B^2 \geq B \cdot K_S$, $H^1(B^{\otimes l}) = 0$ for all $l \geq 2$ by Lemma 2.1, we have the following short exact sequence of vector spaces:

$$0 \to H^0\big(K_S \otimes (B^{\otimes 2})^*\big) \to H^0(K_S^{\otimes 2} \otimes B^*) \to H^0(K_S^{\otimes 2} \otimes B^* \otimes \mathcal{O}_{C'}) \to 0.$$

Since $h^0\big(K_S \otimes (B^{\otimes 2})^*\big) = 0$, we have $h^0(K_S^{\otimes 2} \otimes B^*) = h^0(K_S^{\otimes 2} \otimes B^* \otimes \mathcal{O}_{C'})$. By adjunction on C' and duality, we see that $h^1(B^{\otimes 2} \otimes \mathcal{O}_{C'}) = h^0(K_S^{\otimes 2} \otimes B^* \otimes \mathcal{O}_{C'})$. In view of all these equalities, it is enough to show that

$$h^0(K_S^{\otimes 2} \otimes B^*) \leq h^0(K_S \otimes B) - 3. \tag{3.3.7}$$

Under the hypotheses in (1) and (2) (note that if S is irregular then $K_S^2 \geq 2$, since all minimal surfaces of general type with $K_S^2 = 1$ are regular by Noether's inequality), $K_S^{\otimes 2}$ is base point free by [Ca], Theorem 1.11(i) and since B is ample, $B \cdot K_S > 0$ so we have $h^0(K_S^{\otimes 2} \otimes B^*) \leq h^0(K_S^{\otimes 2}) - 2$. It is not hard to see that $h^0(K_S \otimes B) \geq h^0(K_S^{\otimes 2})$ by Riemann–Roch, Lemma 2.2 and the fact that $B^2 \geq B \cdot K_S$. It is an equality if and only if $B^2 = B \cdot K_S = K_S^2$. By the Hodge Index Theorem this can happen only if $B \equiv K_S$ thus contradicting our assumption on B, hence $h^0(K_S \otimes B) \geq h^0(K_S^{\otimes 2}) + 1$, so the needed inequality (3.3.7) follows. So the map (3.3.6) is surjective which in turn implies the surjectivity of (3.3.1). This completes the proof of the theorem. ∎

We now state an addendum to the above theorem. We state it separately so that special cases do not get lost in the generalities.

Remark 3.3.8 If S is regular with $p_g \geq 4$, Theorem 3.3 holds dropping the hypothesis $B \not\equiv K_S$. The case $B \equiv K_S$ can be proved along the same lines as Theorem 3.3. This recovers the case $B = K_S$ proved in [GP2] for regular surfaces.

If the genus of the general member in $|B|$ is big enough, one proves the following stronger vanishing theorem. We need this result for Section 4 dealing with results on higher syzygies.

In the following theorem, $C \in |B|$ will denote a smooth and irreducible curve of genus $g(C)$. Also, let $E = K_S \otimes B^{\otimes n}$ and $L = K_S \otimes B^{\otimes l}$ with $l \geq 1$ and $n \geq 2$.

Theorem 3.4 *Let S be a surface of general type. Let B be an ample and base-point-free line bundle such that $H^1(B) = 0$ and $B^2 \geq B \cdot K_S$. Assume that*

(1) *$K_S \otimes B$ is base-point free, and*
(2) *$g(C) \geq K_S^2 + 4$.*

Then $H^1(M_L \otimes E^{\otimes k}) = 0$ for all $k \geq 1$.

Proof We prove it for $k = 1$ as noted in Remark 3.2. The group $H^1(M_L \otimes E)$ is the cokernel of the following multiplication map of global sections as seen in Theorem 3.3:

$$H^0(K_S \otimes B^{\otimes n}) \otimes H^0(K_S \otimes B^{\otimes l}) \to H^0(K_S^{\otimes 2} \otimes B^{\otimes n+l}). \tag{3.4.1}$$

We will first prove the surjectivity of (3.4.1) for the case $l = 1$. Let us denote $L' = K_S \otimes B$.

Unlike in Theorem 3.3, reduction to a smooth member in $|B|$ will not work this time. The reduction to curves has to start with a smooth $C' \in |K_S \otimes B| = |L'|$. Such a curve exists by Bertini. By constructing a commutative diagram like (3.3.3) and using the fact that $H^1(B^{\otimes n-1}) = 0$ for all $n \geq 2$ by Lemma 2.1, it is enough to show that the following multiplication map is surjective for all $n \geq 2$:

$$W \otimes H^0(K_S \otimes B^{\otimes n} \otimes \mathcal{O}_{C'}) \to H^0(K_S^{\otimes 2} \otimes B^{\otimes n+1} \otimes \mathcal{O}_{C'}).$$

Here W denotes the cokernel of the inclusion map $H^0(\mathcal{O}_S) \to H^0(L')$.

We will apply Lemma 2.7 to prove this. In order to apply Lemma 2.7, we need to show that $h^1(B^{\otimes n-1} \otimes \mathcal{O}_{C'}) \leq \dim W - 2$.

To show this, consider the following sequence:

$$0 \to \mathcal{O}_S(-C') \to \mathcal{O}_S \to \mathcal{O}_{C'} \to 0. \tag{3.4.2}$$

Tensoring this sequence with $B^{\otimes n-1}$ and taking long exact sequence of cohomology, we have

$$H^1(B^{\otimes n-1}) \to H^1(B^{\otimes n-1} \otimes \mathcal{O}_{C'}) \to H^2(B^{\otimes n-2} \otimes K_S^*) \to H^2(B^{\otimes n-1}).$$

The term on the extreme left is zero by Lemma 2.1. Hence, $h^1(B^{\otimes n-1} \otimes \mathcal{O}_{C'}) \leq h^2(B^{\otimes n-2} \otimes K_S^*) = h^0(K_S^{\otimes 2} \otimes B^{\otimes 2-n})$ for all $n \geq 2$. But $h^0(K_S^{\otimes 2} \otimes B^{\otimes 2-n}) \leq h^0(K_S^{\otimes 2})$ for all $n \geq 2$, as B is an effective divisor. In the light of the above, it would be enough to show that $h^0(K^{\otimes 2}) \leq h^0(K_S \otimes B) - 3$. By Riemann–Roch for surfaces, this is equivalent to the inequality $2K_S^2 + 6 \leq K_S \cdot B + B^2$. But this is assumption (2) in the statement of the theorem. The proof can be completed for $l \geq 2$, either by applying Observation 2.3 and going through the above process but taking into account that reduction to curves this time will be to a smooth general member $C \in |B|$, or by the CM Lemma, that is Lemma 2.8. ∎

Imitating the proof in the above theorems, one can prove the following result with the hypothesis as in Theorem 3.3, for multiples of base-point free and ample bundles B with $h^0(B) \geq p_g + 3$. Note that this hypothesis is a mild one, especially for regular surfaces since in that case $h^0(B) \geq p_g + 1$. This result has some nice applications to pluricanonical bundles that will be derived in Corollary 3.8. We now state the result for the multiples of B and leave the proof to the reader:

Proposition 3.5 *Let S be a surface of general type. Let B be as in Theorem 3.3 and also satisfying $h^0(B) \geq p_g + 3$. Let $L = B^{\otimes n}$. Then, $H^0(M_L \otimes L^{\otimes k}) = 0$ for all $n \geq 2$ and $k \geq 1$.*

The following lemma regarding base point freeness (see [GP2]) will be used frequently from now onwards:

Lemma 3.6 *Let S be a surface with nonnegative Kodaira dimension and let B be an ample and base-point-free line bundle such that $B^2 \geq 5$. If $B' \equiv B$, then $K_S \otimes B'$ is ample and base-point-free.*

Proof Assume S smooth (and by our convention minimal) first. The line bundle B' is ample because ampleness is a numerical condition and has self-intersection greater than or equal to 5. If $K_S \otimes B'$ has base points, by Reider's theorem (see [R]) there is an effective divisor E such that:

(a) $B' \cdot E = 0$ and $E^2 = -1$ or
(b) $B' \cdot E = 1$ and $E^2 = 0$.

The former cannot happen because B' is ample. We will also rule out (b). The divisor E must be irreducible and reduced because B' is ample and $B' \cdot E = 1$. On the other hand, the arithmetic genus of E is greater than or equal to 1. Now $B \cdot E = B' \cdot E = 1$ so $h^0(B \otimes \mathcal{O}_E) \leq 1$. Since B is base-point-free, E should be a smooth rational curve and this is a contradiction.

If S is singular with canonical singularities, then arguing as above but now applying results from Theorem 1, [KM] shows that $K_S \otimes B$ is base point free. ■

In view of Lemma 3.6 the following remark shows that the assumption, $K_S \otimes B$ is base-point free in Theorems 3.3 and 3.4, is a very mild one.

Remark 3.6.1 Let B be a base-point free and ample divisor on S and let $C \in |B|$ be a smooth curve of genus $g(C)$. Then $B^2 \geq 5$ if one of the following holds:

(a) $B^2 \geq B \cdot K_S, g(C) \geq K_S^2 + 4$ and $K_S^2 \geq 2$, or
(b) $B^2 > B \cdot K_S$ and $K_S^2 \geq 2$ or S is irregular with $h^0(B) \geq 4, h^1(B) = 0$, or
(c) $B \not\equiv K_S, B \otimes K_S^*$ is *nef* and $K_S^2 \geq 2$.

Proof The case (a), namely $B^2 \geq B \cdot K_S$ and $B^2 + B \cdot K_S \geq 2K_S^2 + 6$ (equivalently $g(C) \geq K_S^2 + 4$), shows that, as long as $K_S^2 \geq 2$, $B^2 \geq 5$ as claimed. Similarly, (c) follows immediately from Remark 3.1.1 and (a). Now to see (b) implies $B^2 \geq 5$, we make some observations. The inequality $B^2 > B \cdot K_S$ implies that $B^2 \geq B \cdot K_S + 2$ since $B \cdot (B - K_S)$ has to be even by Riemann–Roch. But the Hodge Index implies that $B \cdot K_S \geq 2$ as $K_S^2 \geq 2$. Hence $B^2 \geq 4$. Applying the Hodge Index again to $B \cdot K_S$ shows that $B \cdot K_S \geq 3$. This together with $B^2 \geq B \cdot K_S + 2$ implies that $B^2 \geq 5$. If S is irregular with $h^0(B) \geq 4$ as in the second part of (b) the needed inequality $B^2 \geq 5$ follows from the Clifford inequality applied to $B \otimes \mathcal{O}_C$ and the hypothesis $h^1(B) = 0$. ■

The following remark shows that under the geometrically interesting assumption of $B \otimes K_S^*$ being *nef* for a non-special divisor B, all the assumptions of Theorem 3.4 and most of the assumptions in Theorem 3.3 are automatically satisfied.

Remark 3.6.2 Let $B \not\equiv K_S$ be an ample and base-point free divisor with $B \otimes K_S^*$ *nef* and $H^1(B) = 0$. Then the hypotheses (1) and (2) in Theorem 3.4 follows easily for all surfaces with $K_S^2 \geq 2$. If $K_S^2 = 1$, then the same holds in Theorem 3.4 except when $B \equiv 2K_S$. If $K_S^2 \geq 2$ and $p_g \geq 2$, then the hypotheses $K_S \otimes B$ free and $H^0(B) \geq 4$ in (1) of Theorem 3.3 holds.

Proof Since $B \not\equiv K_S$ and $B \otimes K_S^*$ is *nef*, Hodge Index implies $B^2 > B \cdot K_S$. Also, the assumptions on B in the remark imply that $H^2(B) = 0$. So the remark for Theorem 3.3

and Theorem 3.4 follows directly from Lemma 3.1, Remark 3.6.1, Lemma 3.6 and Riemann–Roch. ■

We will now state and prove some of the corollaries of the theorems proved so far. The above remark together with Theorem 3.4 yields the following corollary:

Corollary 3.7 *Let S be a surface of general type with $K_S^2 \geq 2$. Let $B \not\equiv K_S$ be a base-point free and ample line bundle with $H^1(B) = 0$ and $B \otimes K_S^*$ nef. Let $L = K_S \otimes B^{\otimes l}$ with $l \geq 2$. Then L is very ample and embeds S as a projectively normal variety (i.e. L satisfies property N_0.)*

One can get very precise results for pluricanonical divisors. The following corollary proves new cases and recovers and improves results from [GP2] (see also [Ci, Bo].) Examples in Section 5 show that these results are optimal.

Corollary 3.8 *Let S be a surface of general type with K_S ample. Let one of the following conditions hold:*

(1) *$K_S^2 \geq 5$, or*
(2) *$K_S^2 \geq 2$ and $p_g \geq 1$ if S is regular, or*
(3) *$p_g \geq 2$ if S is irregular.*

Then the following hold:

(a) *$L = K_S^{\otimes n}$ satisfies property N_0 for all $n \geq 5$.*
(b) *If S is regular, $K_S^{\otimes n}$ satisfies N_0 for all $n \geq 4$.*
(c) *If S is irregular and $p_g \geq 6$, then $K_S^{\otimes n}$ satisfies N_0 for all $n \geq 4$.*

Proof This is a corollary of either Theorem 3.3 or Theorem 3.4. We first remark that the numerical hypothesis (1), (2), or (3) is assumed to ensure the base-point freeness of $B = K_S^{\otimes l}$ for all $l \geq 2$. This follows from [Ca] and Lemma 3.6 for smooth surfaces. Note that both these results can be used even if S has canonical singularities. The reason is that one can consider the crepant minimal resolution of S and assume for the purpose of applying [Ca] that S is smooth. But $B \otimes K_S^*$ is *nef* (and even *big*), so by Remark 3.6.2 and Theorem 3.4 (or Corollary 3.7) the corollary is proved for all odd $n \geq 5$. For even powers $n \geq 6$ the result follows easily by Observation 2.3 taking $L_i = K_S^{\otimes 2}$ and $E = K_S^{\otimes n}$, Lemma 2.8 and the K–V vanishing theorem. To prove (b) and (c), note that we need to show property N_0 of $K_S^{\otimes n}$ only for $n = 4$ since $n \geq 5$ is proved in (a). The statement (b) follows directly from Riemann–Roch and Proposition 3.5.

To prove (c), we again apply Proposition 3.5. Let $B = K_S^{\otimes 2}$. This is base-point free by Lemma 3.6 as $K_S^2 \geq 8$. We need only to show that $h^0(B) \geq p_g + 3$. Note that $h^0(B) = K_S^2 + \chi(\mathcal{O}_S)$ by Riemann–Roch and K–V vanishing. But $\chi(\mathcal{O}_S) > 0$ for a surface of general type. This follows from the Noether's formula $12\chi(O_S) = K_S^2 + c_2$, where c_2 is the second Chern class of the tangent sheaf of S and is positive for a surface of general type. So the needed inequality $h^0(B) \geq p_g + 3$ follows from Noether's inequality $K_S^2 \geq 2p_g - 4$ and the hypothesis $p_g \geq 6$. ■

We will prove a cohomology vanishing which will be used in the inductive arguments to prove higher syzygy results. This theorem refines and improves Theorem 5.1

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in [GP2]. Without this essential technical improvement we cannot proceed towards the higher syzygy results. Note that in view of Remark 3.1.1, the hypotheses (1) and (2) in the theorem below are automatic if $B \otimes K_S^*$ is *nef* with $B \not\equiv K_S$. But we prove it in greater generality with a view towards some applications.

In the theorem below, $C \in |B|$ will denote a smooth curve of genus $g(C)$ in the linear system associated to B and $L = K_S \otimes B^{\otimes n}$ and $L' = K_S \otimes B^{\otimes l}$ with $n, l \geq 2$.

Theorem 3.9 *Let S be a regular surface of general type with $p_g \geq 4$. Let B be an ample and base-point-free line bundle such that $H^1(B) = 0$. Assume that*

(1) $B^2 \geq B \cdot K_S$.
(2) $g(C) \geq K_S^2 + 4$.

Then $H^1(M_L^{\otimes 2} \otimes L'^{\otimes k}) = 0$ for all $k \geq 1$.

Proof As usual we will prove it for $k = 1$ in view of Remark 3.2. Since $p_g \geq 4$, Noether's inequality implies $K_S^2 \geq 4$, so by Remark 3.6.1 and Lemma 3.6, $K_S \otimes B$ is base-point free. By Theorem 3.4 we have $H^1(M_L \otimes L'^{\otimes k}) = 0$ for all $k \geq 1$. Tensoring $(*)$ by $M_L \otimes L'$ and taking long exact sequence, one sees that $H^1(M_L^{\otimes 2} \otimes L')$ is the cokernel of the following multiplication map:

$$H^0(M_L \otimes L') \otimes H^0(L) \to H^0(M_L \otimes L \otimes L').$$

We will prove this for $n = 2$. The rest follows in the same way. By Observation 2.3, it is enough to prove

$$H^0(M_L \otimes L') \otimes H^0(B) \xrightarrow{\alpha_1} H^0(M_L \otimes L' \otimes B)$$
$$H^0(M_L \otimes L' \otimes B) \otimes H^0(K_S \otimes B) \xrightarrow{\alpha_2} H^0(M_L \otimes L' \otimes K_S \otimes B^{\otimes 2})$$

are surjective.

We will prove in detail the surjectivity of α_1, the proof of surjectivity of α_2 is analogous. Note that by Theorem 3.4 we have the vanishing of $H^1(M_{K_S \otimes B^{\otimes n}} \otimes K_S \otimes B^{\otimes l}) = 0$ for all $n \geq 2$ and $l \geq 1$. So we can apply Lemma 2.4. Let $C \in |B|$ be a smooth curve. The idea now is to apply Lemma 2.6. One needs to show that (1) and (2) of Lemma 2.6 holds. Since $H^1(L \otimes B^*) = 0$, (1) of Lemma 2.6 holds. We will now show that (2) also holds. For this it is enough to check (see (2) of Lemma 2.6) the surjectivity of

$$H^0(M_{L \otimes \mathcal{O}_C} \otimes L' \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \to H^0(M_{L \otimes \mathcal{O}_C} \otimes L' \otimes B \otimes \mathcal{O}_C).$$

What Lemma 2.6 has done is to help us pass from an unstable vector bundle $M_L \otimes \mathcal{O}_C$, an object difficult to handle, to $M_{L \otimes \mathcal{O}_C}$, which we will show is semistable. In questions like this, these are easier objects to handle. We want to apply Proposition 2.5, but in order to do this we need to check various things. First we point out that hypothesis (2) implies that $B \not\equiv K_S$. Since $p_g \geq 3$, $K_S^2 \geq 2p_g - 4 \geq 2$ by Noether's inequality, hence hypotheses (1) and (2) show that $B^2 \geq 5$. Now we proceed to show that the required inequalities in Proposition 2.5 hold. Let $E = B \otimes \mathcal{O}_C$ and $F = M_{L\otimes\mathcal{O}_C} \otimes L' \otimes \mathcal{O}_C$. Note that since $\deg(L\otimes\mathcal{O}_C) = (K_S \cdot B + 2B^2) \geq 2g(C)$, hence $M_{L\otimes O_C}$ is semistable by Theorem 1.2 [Bu]. So one needs to show that the following hold:

(i) $\mu(F) \geq 2g(C)$;
(ii) $\mu(F) > 2g + \operatorname{rank}(E)\big(2g(C) - \mu(E)\big) - 2h^1(E)$.

For the former inequality, $\mu(F) \geq \mu(M_{L\otimes\mathcal{O}_C}) + K_S \cdot B + 2B^2$ and this is bigger than $2g(C)$ since $2g(C) = K_S \cdot B + B^2 + 2$, $B^2 \geq 5$ and $\mu(M_{L\otimes O_C}) \geq -2$ by [Bu, Theorem 1.2]. Inequality (ii) is equivalent to $\mu(M_{L\otimes\mathcal{O}_C}) + \deg E + \deg(L' \otimes \mathcal{O}_C) > 4g(C) - 2h^1(E)$. Since $\deg E + \deg(L' \otimes O_C) \geq 4g(C) - 4$, by hypothesis (1) and $h^1(E) = p_g \geq 4$, inequality (ii) holds. Since $B^2 \geq 5$, it follows that $K_S \otimes B$ is base-point free by Lemma 3.6. Since we are on a surface of general type, it is also *big*. So there is a smooth and irreducible member C' in the linear system $|K_S \otimes B|$. The surjectivity of α_2 can be proved using the same method as α_1. There are no surprises except to note that in order to start the process of reducing to curve C', one needs the vanishing of $H^1(M_L \otimes B^{\otimes l}) = 0$ for all $l \geq 2$. This follows from Observation 2.3 and the surjectivity of (3.3.2) seen in the proof of Theorem 3.3 or follows from pursuing exactly the same path as that in Theorem 3.4. ■

Remark 3.9.1 The assumption on p_g, in Theorem 3.9 and in the subsequent theorems are made to ensure that the inequality (ii) (more specifically $B^2 - B\cdot K_S > 6 - 2p_g$ in Theorem 3.9) holds. So if the difference $B^2 - B \cdot K_S$ is large enough, no assumption on p_g is required.

Corollary 3.10 *Let S be a regular surface of general type with $p_g \geq 3$. Let $B \not\equiv K_S$ be a base-point free and ample line bundle with $H^1(B) = 0$ and $B\otimes K_S^*$ nef. Let $L = K_S \otimes B^{\otimes l}$ with $l \geq 2$. Then L satisfies property N_1 (*i.e., *the ideal I_S defining S in the embedding given by L is generated by forms of degree 2).*

Proof Since $B \otimes K_S^*$ is *nef*, Lemma 3.1 together with Remark 3.1.1 shows that the inequalities (1) and (2) in Theorem 3.9 are satisfied. So Theorem 3.9 holds. Since $B^2 - B \cdot K_S > 0$ by Hodge Index (note $B \not\equiv K_S$) and is greater than or equal to 2 by Riemann–Roch, the hypothesis $p_g \geq 4$ in Theorem 3.9 can be relaxed in view of Remark 3.9.1, to $p_g \geq 3$. As we are working over a field of characteristic zero, the vanishing in Theorem 3.9 implies the vanishing of $H^1(\bigwedge^2 M_L \otimes L^{\otimes n}) = 0$ for all $n \geq 1$. So L satisfies property N_1 by Theorem 1.1. ■

We will now prove an analogous theorem for irregular surfaces of general type. This theorem is slightly weaker than the one proved for regular surfaces, but it has very similar corollaries as Theorem 3.9 on normal generation of pluricanonical bundles. The proof is different and more involved than Theorem 3.9. The proof uses the fact that irregular surfaces have a "continuous" Picard group, the technique used above of going to curves and a trick of reducing the "negativity" of M_L.

Let us denote $L = K_S \otimes B^{\otimes n}$ and $L' = K_S \otimes B^{\otimes l}$ with $B \not\equiv K_S$ in the theorem below.

Theorem 3.11 *Let S be an irregular surface of general type. Let B be a base-point free and ample divisor such that $B^2 \geq 5$ and B' is free for all $B' \equiv B$ and $H^1(B') = 0$. Assume $B \otimes K_S^*$ is nef, big and effective. Then $H^1(M_L^{\otimes 2} \otimes L'^{\otimes k}) = 0$ for all $n, l \geq 2$, $k \geq 1$. In particular, $L = K_S \otimes B^{\otimes n}$ satisfies property N_1 for all $n \geq 2$.*

Proof We will prove the theorem for $k = 1$. The rest are similar in view of Remark 3.2. Let $E \in \mathrm{Pic}^0(S)$ be such that $E^{\otimes 2} \neq \mathcal{O}_S$. Let $B_1 = B \otimes E$ and $B_2 = B \otimes E^*$. Note that, $K_S \otimes B_1$ and B_2 are base-point free, by Lemma 3.6 and by hypothesis respectively. For the sake of simplicity of notation, we will prove the theorem for $n = l = 2$. The proof for cases $n > 2$ and $l > 2$ are exactly the same (after applying Observation 2.3 repeatedly.) So we have $L = L'$. Since $H^1(M_L \otimes L') = 0$ by Theorem 3.4, the needed vanishing follows if

$$H^0(M_L \otimes L') \otimes H^0(L) \to H^0(M_L \otimes L \otimes L')$$

is surjective for all $n = l = 2$. We will decompose $L = K_S \otimes B^{\otimes 2} = K_S \otimes B_2 \otimes B_1$. We will use Observation 2.3. We will first prove the surjectivity of

$$H^0(M_L \otimes L) \otimes H^0(B_1) \to H^0(M_L \otimes L \otimes B_1). \tag{3.11.1}$$

By Lemma 2.8, we need the vanishings of $H^1(M_L \otimes K_S \otimes B_2)$ and $H^2\big(M_L \otimes K_S \otimes (E^{\otimes 2})^*\big)$. To see the first vanishing, use the path followed in Theorem 3.4 by reducing to a curve $C \in |K_S \otimes B_2|$. The needed inequalities, after reducing to the curve, follow from $B \otimes K_S^*$ being *nef* and Lemma 2.2. The second follows from diagram chase, K–V vanishing and the fact that $H^2\big(K_S \otimes (E^{\otimes 2})^*\big) = H^0(E^{\otimes 2}) = 0$.

We next need to show the surjectivity of the following multiplication map,

$$H^0(M_L \otimes L \otimes B_1) \otimes H^0(K_S \otimes B_2) \to H^0(M_L \otimes L^{\otimes 2}). \tag{3.11.2}$$

Let $N = K_S \otimes B_2$. Note that N is base-point free by Lemma 3.6 and Remark 3.6.2. We remark that the surjectivity of (3.11.2) follows if $H^1(M_L \otimes M_N \otimes L \otimes B_1) = 0$. To see this replace L in $(*)$ on page 4 by N as above, and tensor the corresponding sequence with $M_L \otimes L \otimes B_1$ and take long exact sequence of cohomology. The proof of Theorem 3.4 shows that $H^1(M_N \otimes L \otimes B_1) = 0$. So in view of this the above cohomology group vanishes if

$$H^0(M_N \otimes L \otimes B_1) \otimes H^0(L) \to H^0(M_N \otimes L^{\otimes 2} \otimes B_1) \tag{3.11.3}$$

is surjective. Note that $L = K_S \otimes B \otimes B$. We use the methods previously developed in Theorem 3.4 together with the Observation 2.3 to absorb a B in L. To complete the proof we need to show that $H^0(M_N \otimes L \otimes B_1 \otimes B) \otimes H^0(K_S \otimes B) \to H^0(M_N \otimes L^{\otimes 2} \otimes B_1)$ is surjective. Invoking Lemma 2.8, the above multiplication map is surjective if the following vanishings hold; $H^1(M_N \otimes L \otimes B_1 \otimes K_S^*) = 0$ and $H^2(M_N \otimes B_1 \otimes B \otimes K_S^*) = 0$. Since this kind of argument has been used several times by now, we will give below only the essential and vital points needed for the proof. The first of these vanishings (*i.e.* the H^1 vanishing which is needed above) follows if the multiplication map

$H^0(B^{\otimes 3} \otimes E) \otimes H^0(N) \to H^0(N \otimes B^{\otimes 2} \otimes B_1)$ is surjective. This can be accomplished by the methods of Theorem 3.4 by reducing to a smooth curve $C \in |N|$ as N is base-point free. The needed vanishing, $H^1(B_1^{\otimes 2} \otimes K_S^*) = 0$, to reduce the multiplication map to the curve C can be verified using the hypothesis $B \otimes K_S^*$ is *nef* and *big* and applying K–V vanishing. After this reduction of the multiplication map to C, the inequality $h^1(B_1^{\otimes 2} \otimes K_S^* \otimes \mathcal{O}_C) \leq h^0(K_S \otimes B_2) - 3$ needs to be verified to apply Lemma 2.7 so that the multiplication map on C is surjective. This needs some work which we will outline below. We first observe that $h^1(B_1^{\otimes 2} \otimes K_S^* \otimes \mathcal{O}_C) = h^0(K_S^{\otimes 3} \otimes B^* \otimes E^{\otimes 3^*})$. This follows from tensoring

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

with $B_1^{\otimes 2} \otimes K_S^*$ and taking long exact sequence of cohomology together with K–V vanishing and Serre duality. The second point is to note that $h^0(K_S^{\otimes 3} \otimes B^* \otimes E^{\otimes 3^*}) \leq h^0(K_S^{\otimes 2})$. This follows from the fact that $B \otimes K_S^*$ is effective. So we are reduced to checking that $h^0(K_S^{\otimes 2}) \leq h^0(K_S \otimes B_2) - 3$. This inequality holds because of the K–V vanishing theorem, Lemma 3.1 and Remark 3.1.1 unless $B \equiv 2K_S$ and $K_S^2 = 1$, but note that this latter possibility is not tenable since $B^2 \geq 5$ by hypothesis. So we are done. The vanishing of $H^2(M_N \otimes B_1 \otimes B \otimes K_S^*)$ follows from K–V vanishing and a simple diagram chase using $(*)$. The proof now follows easily by putting together all of the above. ∎

Remark 3.12 A careful analysis of the proofs of Theorems 3.3, 3.4, 3.8 and 3.10 show that the proofs depend on the numerical class of the line bundle. As a consequence one can prove a slightly more general result. Namely the conclusion drawn for $K_S \otimes B^{\otimes n}$ in the above theorems can also be drawn for $K_S \otimes B^{\otimes n} \otimes A$ where A is a *nef* divisor with the property that $B \otimes A$ is base-point free.

The following corollary recovers and improves results in [GP2] and generalizes to higher syzygies the results in [Ci] on pluricanonical linear systems.

Corollary 3.13 *Let S be a surface of general type with K_S ample. Let*

(1) $K_S^2 \geq 5$ *and* $p_g \geq 1$ *or* $p_g \geq 2$ *if S is irregular,*
(2) $K_S^2 \geq 3$ *and* $p_g \geq 1$ *or* $K_S^2 \geq 2$ *and* $p_g \geq 2$, *if S is regular.*

Then $L = K_S^{\otimes n}$ satisfies N_1 for all $n \geq 5$.

Proof The proof follows from Theorem 3.11 and Theorem 3.9 by taking $B = K_S^{\otimes n}$ with $n \geq 2$. As noted in Corollary 3.8, we can assume S to be a smooth surface by taking the minimal crepant resolution. So B is base-point free by [Ca] and Lemma 3.6. In view of Remark 3.9.1, the hypothesis on p_g in Theorem 3.9 can be relaxed, as all we need to check the surjectivity of α_1 and α_2 of Theorem 3.9 is a suitable bound on $p_g + K_S^2$. This takes care of odd powers of K_S. For even powers the result follows either from the above remark or by following the method of proof in Theorem 3.11. ∎

Another corollary of the theorems of this section is the following result giving effective bounds towards Mukai's conjecture thereby answering (Q2). This is a slight improvement of Corollary 5.10 in [GP2].

Corollary 3.14 *Let S be a surface of general type, let A be an ample line bundle and let $m = [\frac{(A\cdot(K_S+4A)+1)^2}{2A^2}]$. Let $L = K_S \otimes A^{\otimes n}$. If $n \geq 2m$, then L satisfies property N_0 and even N_1.*

Proof Denote $B = A^{\otimes m}$, then by [D] or [BS] B is base-point free with $H^1(B) = 0$. One can easily verify that the numerical condition (2) in Theorem 3.4 and the numerical condition to apply Lemma 3.6 (for the base-point freeness of $K_S \otimes B$) are easily satisfied due to m being so large. So L satisfies N_0. The statement for N_1 follows from Theorem 3.9 for X regular and Theorem 3.11 for X irregular. It would help to keep the following in mind: conditions on p_g in Theorem 3.9 is not necessary for this corollary in view of Remark 3.9.1 and the condition $B^2 \geq 5$ in Theorem 3.11 is automatic and since $\mathrm{Pic}^0(S)$, for an irregular S, is divisible [D] applies also to B' in the statement of Theorem 3.11. Also, it follows from the proof of the main theorem in [D] or [BS] that $B \otimes K_S^*$ is *nef*, *big* and effective. Hence the corollary follows as claimed. ■

4 Higher Syzygies of Surfaces of General Type

In the previous section we proved results on projective normality and normal presentation. In this section we are going to prove higher syzygy results associated to adjunction bundles $K_S \otimes B^{\otimes n}$. We carry out the proof in two steps. First we prove a technical result, which together with a cohomology vanishing result implies property N_2 for the adjunction bundle. This will serve as the first step in the inductive process towards property N_p associated to adjunction bundle. The proofs are different for N_2 and N_p. We first need the following technical (and essential) result. This is used in Theorem 4.2.

In the theorem below we shall denote $L = K_S \otimes B^{\otimes n}$ with $B \not\equiv K_S$.

Proposition 4.1 *Let S be a regular surface of general type with $p_g \geq 3$. Let B be a base-point free and ample line bundle such that $B \otimes K_S^*$ is nef and $H^1(B) = 0$. Then $H^1(M_L^{\otimes 2} \otimes B^{\otimes m}) = 0$ for all $n \geq 2$ and $m \geq 4$.*

Proof In the process of proving Theorem 3.3 we had also proved (see (3.3.2) and use Observation 2.3) that $H^1(M_L \otimes B^{\otimes m}) = 0$ for all $n \geq 2$ and $m \geq 1$. So the cohomology group $H^1(M_L^{\otimes 2} \otimes B^{\otimes m})$ is the cokernel of the multiplication map

(4.1.1) $$H^0(M_L \otimes B^{\otimes m}) \otimes H^0(L) \to H^0(M_L \otimes K_S \otimes B^{\otimes n+m}).$$

Hence it is enough to show this map is surjective. We will do so for the case $n = 2$. The rest are easier and can be proved in exactly the same manner. Note that $K_S \otimes B$ is base-point free by Remark 3.6.1 and Lemma 3.6. We will break the proof into several steps to facilitate a better exposition.

Step 1 We will apply Observation 2.3 to prove the theorem. The idea is to gather all the B's first to make the vector bundles involved in the multiplication maps sufficiently positive and then deal with $K_S \otimes B$. The proof will make this idea precise. We will first show the surjectivity of the map β_1 below:

$$H^0(M_L \otimes B^{\otimes m}) \otimes H^0(B) \to H^0(M_L \otimes B^{\otimes m+1}).$$

Note we have assumed $n = 2$ and also by hypothesis of the theorem $m \geq 4$. Since $H^1(M_L \otimes B^{\otimes l}) = 0$ for all $l \geq 2$, we can apply Lemma 2.4. We want to apply Lemma 2.6 and since $H^1(K_S \otimes B) = 0$, (1) of Lemma 2.6 holds. For condition (2) of Lemma 2.6 to hold, it is enough to show that the following multiplication map on the smooth and irreducible curve $C \in |B|$ is surjective:

$$H^0(M_{L\otimes \mathcal{O}_C} \otimes B^{\otimes m} \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \to H^0(M_{L\otimes \mathcal{O}_C} \otimes B^{\otimes m+1} \otimes \mathcal{O}_C).$$

This surjection follows from the methods used in Theorems N_1 and the inequalities needed to apply Proposition 2.5 are checked in the same way as in Theorem 3.9. There are no new twists. The next surjection that we require has some new twist, so we will explain in some detail. This includes comparing the positivity of B with respect to K_S and here is the first time that $B \otimes K_S^*$ is needed. We need to show that:

$$H^0(M_L \otimes B^{\otimes m+1}) \otimes H^0(K_S \otimes B) \to H^0(M_L \otimes K_S \otimes B^{\otimes m+2}). \tag{4.1.2}$$

The idea is to apply Lemma 2.4 and restrict to a smooth curve $C' \in |K_S \otimes B|$ and for this we need $H^1(M_L \otimes B^{\otimes m} \otimes K_S^*) = 0$. To prove this vanishing will be Step 2.

Step 2 Vanishing of $H^1(M_L \otimes B^{\otimes m} \otimes K_S^*) = 0$ for all $m \geq 4$. Since by hypothesis $B \otimes K_S^* = A$ is *nef*, we have $H^1(B^{\otimes m} \otimes K_S^*) = H^1(K_S \otimes B^{\otimes m-2} \otimes A^{\otimes 2}) = 0$ by the K–V vanishing theorem. So to prove the vanishing of $H^1(M_L \otimes B^{\otimes m} \otimes K_S^*)$ it is enough to show that

$$H^0(K_S \otimes B^{\otimes 2}) \otimes H^0(B^{\otimes m} \otimes K_S^*) \to H^0(B^{\otimes m+2})$$

is surjective for all $m \geq 4$. We have $p_g \geq 3$ and $B \otimes K_S^*$ *nef*, hence $B^2 \geq 5$ by Remark 3.6.1(c). Note that $B^{\otimes m} \otimes K_S^* = K_S \otimes B^{\otimes m-2} \otimes A^{\otimes 2}$ is base-point free.

We will perform the by now familiar "gathering B trick". First "B to be gathered" follows from proving the surjectivity of

$$H^0(B^{\otimes m} \otimes K_S^*) \otimes H^0(B) \to H^0(B^{\otimes m+1} \otimes K_S^*).$$

This follows from Lemma 2.8 as $H^1(B^{\otimes m-1} \otimes K_S^*)$ and $H^2(B^{\otimes m-2} \otimes K_S^*)$ both vanish. The first vanishing follows from K–V Vanishing and the second follows from the fact that $B \not\equiv K_S$. Next we need to show that,

$$H^0(B^{\otimes m+1} \otimes K_S^*) \otimes H^0(K_S \otimes B) \to H^0(B^{\otimes m+2}) \tag{4.1.3}$$

is surjective. Lemma 2.8 does not work for this case. This would require the method of reducing to a smooth irreducible curve $C' \in |K \otimes B|$. The multiplication map

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(4.1.3) can be reduced to C' using Lemma 2.4 because $H^1(B^{\otimes m} \otimes (K_S^*)^{\otimes 2}) = 0$. We therefore need the surjectivity of

$$(4.1.4) \qquad H^0(B^{\otimes m+1} \otimes K_S^* \otimes \mathcal{O}_{C'}) \otimes H^0(K_S \otimes B \otimes \mathcal{O}_{C'}) \to H^0(B^{\otimes m+2} \otimes \mathcal{O}_{C'}).$$

We will use Proposition 2.5 to prove the above surjection. For this we need to show various inequalities. First note that $2g(C') = (2K_S + B) \cdot (K_S + B) + 2$. We need these two inequalities to be satisfied:

(i) $\big((m+1)B - K_S\big) \cdot (K_S + B) \geq (2K_S + B) \cdot (K_S + B) + 2$. This is equivalent to $(mB - 3K_S) \cdot (K_S + B) \geq 2$. This is true for all $m \geq 4$. Next we need,

(ii) $\big((m+1)B - K_S\big) \cdot (K_S + B)$
$$> 2(2K_S + B) \cdot (K_S + B) + 4 - (K_S + B)^2 - 2h^1(K_S \otimes B \otimes O_{C'}).$$

The inequality (ii) is equivalent to $(mB - 4K_S) \cdot (K_S + B) > 4 - 2h^1(K_S \otimes B \otimes O_{C'})$ for all $m \geq 4$. This inequality is true because $B \otimes K_S^*$ is nef, $K_S \otimes B$ is base-point free and $h^1(K_S \otimes B \otimes O_{C'}) \geq 3$. The later inequality holds because $p_g \geq 3$. So (4.1.4) is surjective.

The above arguments prove that $H^1(M_L \otimes B^{\otimes m} \otimes K_S^*) = 0$ for all $m \geq 4$. This completes Step 2.

Step 3 We are now ready to prove the surjectivity of (4.1.2). Recall that $L = K_S \otimes B^{\otimes 2}$. In view of $H^1(\mathcal{O}_X) = 0$ and the vanishing proved in Step 2, we can apply Lemma 2.4 and Lemma 2.6 to reduce multiplication map (4.1.2) to the following multiplication map on curves:

$$H^0(M_{L \otimes \mathcal{O}_{C'}} \otimes B^{\otimes(m+1)} \otimes \mathcal{O}_{C'}) \otimes H^0(K_S \otimes B \otimes \mathcal{O}_{C'}) \to H^0(M_L \otimes K_S \otimes B^{\otimes(m+2)} \otimes \mathcal{O}_{C'}).$$

We want to apply Proposition 2.5 again. Note that $H^1(B) = 0$ and as a result of Lemma 2.2 and Lemma 3.1, we have $(K_S + 2B) \cdot (K_S + B) > 2g(C')$, so $M_{L \otimes O_{C'}}$ is semistable and has slope bigger than or equal to -2. Let $E = K_S \otimes B \otimes \mathcal{O}_{C'}$ and $F = M_{L \otimes \mathcal{O}_{C'}} \otimes B^{\otimes(m+1)} \otimes \mathcal{O}_{C'}$. So we need to check only that $\mu(F) > 2g(C')$ and that $\mu(F) > 4g(C') - \deg(E) - 2h^1(E)$. Both the inequalities follows as $B^2 \geq B \cdot K_S$ since $B \otimes K_S^*$ is nef. So (2) is surjective.

The upshot of all of these arguments is that (4.1.1) is surjective. So we have the needed vanishing. ■

We will now prove a Koszul cohomology vanishing which will show that $K_S \otimes B^{\otimes n}$ satisfies property N_2 for $n \geq 3$ when $B \otimes K_S^*$ is *nef*. The vanishing we now prove will be the first inductive step in proving the claimed higher syzygy result of adjunction bundle. The techniques of the proof has already been vetted well in the proofs of the above technical theorems and lemmas proved above, so we will not give all details but only sketch it and leave the details to the reader.

Let $L = K_S \otimes B^{\otimes n}$ and $L' = K_S \otimes B^{\otimes l}$ with $n, l \geq 3$ with $B \not\equiv K_S$.

Theorem 4.2 *Let S be a regular surface of general type with $p_g \geq 3$. Let B be a base point free and ample line bundle such that $B \otimes K_S^*$ is nef and $H^1(B) = 0$. Then $H^1(M_L^{\otimes p'+1} \otimes L^{\otimes s})$ vanishes for all $0 \leq p' \leq 2$ and all $s \geq 1$.*

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Proof Theorem 3.4 and Theorem 3.9 show the vanishing for $p' = 0, 1$. We will now prove it for $p' = 2$. We will indicate the proof for the case $n = 3$ and $s = 1$. The case $n > 3$ is similar. Since the result is true by Theorem 3.9 for $p' = 1$, it is enough to check that the following multiplication map of vector bundles is surjective:

$$H^0(M_L^{\otimes 2} \otimes L') \otimes H^0(L) \to H^0(M_L^{\otimes 2} \otimes L \otimes L').$$

Note we are proving a slightly stronger multiplication than necessary involving L and L'. This is done to facilitate smoothly the induction argument to be adopted later on to prove higher syzygy results. Since $n = 3$, by applying Observation 2.3, we can "gather two of the B's" in $L = K_S \otimes B^{\otimes 3}$ to make the the vector bundle involved in the multiplication maps as positive as possible before we deal with the final multiplication map involving $K_S \otimes B$. More precisely we want to indicate the surjections needed in the pecking order:

$$H^0(M_L^{\otimes 2} \otimes L') \otimes H^0(B) \xrightarrow{\alpha_1} H^0(M_L^{\otimes 2} \otimes L' \otimes B),$$

$$H^0(M_L^{\otimes 2} \otimes L' \otimes B) \otimes H^0(B) \xrightarrow{\alpha_2} H^0(M_L^{\otimes 2} \otimes L' \otimes B^{\otimes 2}),$$

$$H^0(M_L^{\otimes 2} \otimes L' \otimes B^{\otimes 2}) \otimes H^0(K_S \otimes B) \xrightarrow{\alpha_3} H^0(M_L^{\otimes 2} \otimes L' \otimes L).$$

We will indicate the proof for the α_1 and α_3. The idea is to reduce the multiplication on the surface to those on curves and lift them back to surfaces. This can be done using Lemma 2.4 and Lemma 2.6 provided we are able to fulfill the necessary hypothesis. Note that in order to reduce α_1 to a smooth member in $C \in |B|$, we need that $H^1(M_L^{\otimes 2} \otimes L' \otimes B^*) = 0$. But this is true by Theorem 3.9. Also, $\deg(L \otimes \mathcal{O}_C) > 2g(C)$ so $M_{L \otimes \mathcal{O}_C}$ is semistable. So we can reduce the map to a multiplication map on C and follow the path as in the above results. To check that α_3 is surjective, we follow the path to curves. Note that by now we have "gathered two B's". This time we want to reduce the multiplication map to a smooth curve $C' \in |K_S \otimes B|$. To do this we need $H^1(M_L^{\otimes 2} \otimes L' \otimes B^{\otimes 2} \otimes K_S^* \otimes B^*) = 0$. This follows from Proposition 4.1. The necessary inequalities follow and one can check easily that they are comfortably satisfied. ∎

Following the methods of Theorem 3.11, one can prove the analogue of the above theorem for irregular surfaces. We leave the proof to the reader.

Theorem 4.3 *Let S be an irregular surface of general type. Let $B \not\equiv K_S$ be a base-point free and ample divisor such that $B^2 \geq 5$ and B' is free for all $B' \equiv B$ and $H^1(B') = 0$. Assume $B \otimes K_S^*$ is* nef *and* big *and* effective*. Then $H^1(M_L^{\otimes 2} \otimes L') = 0$ for all $n, l \geq 3$. In particular, $L = K_S \otimes B^{\otimes n}$ satisfies property N_2 for all $n \geq 3$.*

As a corollary of the above theorems and arguing as in Corollary 3.13 we obtain the following:

Corollary 4.4 *Let S be a surface of general type with ample K_S. Let*

(1) *$K_S^2 \geq 5$ and $p_g \geq 1$ or $p_g \geq 2$ if S is irregular,*
(2) *$K_S^2 \geq 3$ and $p_g \geq 1$ or $K_S^2 \geq 2$ and $p_g \geq 2$, if S is regular.*

Then $L = K_S^{\otimes n}$ satisfies N_2 for all $n \geq 7$.

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We will now prove a cohomology vanishing theorem, one of whose corollaries gives a higher syzygy result for adjunction bundles.

Let $L = K_S \otimes B^{\otimes n}$ and $L' = K_S \otimes B^{\otimes m}$ with $B \not\equiv K_S$ in the theorem below.

Theorem 4.5 *Let S be a regular surface of general type with $p_g \geq 3$. Let B be a base point free and ample line bundle such that $B \otimes K_S^*$ is nef and $H^1(B) = 0$. Then $H^1(M_L^{\otimes p'+1} \otimes L'^{\otimes s}) = 0$ for all $0 \leq p' \leq p$, $n, m \geq p+1$ and all $s \geq 1$. In particular $L = K_S \otimes B^{\otimes n}$ satisfies property N_p for all $n \geq p+1$.*

Proof The proof rests on an inductive argument. The result is true for $p' = 0, 1, 2$ by Theorem 3.4, Theorem 3.9 and Theorem 4.2. So we may assume that $p' \geq 3$. Note that in view of Remark 3.9.1 (or Corollary 3.10), we can relax the hypothesis $p_g \geq 4$ in Theorem 3.9 to $p_g \geq 3$ for this theorem. As usual, we will prove it only for $s = 1$ as noted in Remark 3.2. Let us assume the theorem to be true for $p' = p - 1$ and prove it to be true for $p' = p$. By the induction assumption, we have $H^1(M_L^{\otimes p} \otimes L') = 0$. So tensoring $(*)$ with $M_L^{\otimes p} \otimes L'$ and taking the long exact sequence of cohomology, it is enough to show that the following multiplication map of global sections of vector bundles on S is surjective:

$$H^0(M_L^{\otimes p} \otimes L') \otimes H^0(L) \to H^0(M_L^{\otimes p} \otimes L \otimes L'). \tag{4.5.1}$$

We use the by now familiar "gathering B's" trick to show the first step in the surjection. Note that $L = K_S \otimes B^{\otimes n}$ with $n \geq p+1$ has p such B's to spare before we deal with the multiplication map involving $K_S \otimes B$. All this will be made precise below. We will show the surjectivity for the following multiplication map first:

$$H^0(M_L^{\otimes p} \otimes L') \otimes H^0(B) \to H^0(M_L^{\otimes p} \otimes L' \otimes B).$$

The way we proceed is to reduce this map to a smooth curve $C \in |B|$. In order to accomplish this we need to apply Lemma 2.4. For this we need to check $H^1(M_L^{\otimes p} \otimes L' \otimes B^*) = 0$. This follows from induction hypothesis. So we can restrict the map to C. Since $H^1(K_S \otimes B^{\otimes r}) = 0$ for all $r \geq 1$, we can apply Lemma 2.6. It is enough to show that the multiplication map

$$H^0(M_L^{\otimes p} \otimes L' \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \to H^0(M_L^{\otimes p} \otimes L' \otimes B \otimes \mathcal{O}_C)$$

is surjective. Denote $F = M_{L\otimes \mathcal{O}_C}^{\otimes i}$, $E = L' \otimes \mathcal{O}_C$. Using Lemma 2.6 we can reduce the above multiplication map to the following multiplication map of semistable vector bundle over C:

$$H^0(F \otimes L' \otimes \mathcal{O}_C) \otimes H^0(B \otimes \mathcal{O}_C) \to H^0(F \otimes L' \otimes B \otimes \mathcal{O}_C).$$

We need to prove this for all $0 \leq i \leq p$. We will prove it for $i = p$, the other cases are easier and follows similarly. This we will prove by applying Proposition 2.5 as before. Since $\deg(L \otimes \mathcal{O}_C) \geq 2g(C)$, F is a semistable vector bundle. In order to apply it we need to check two inequalities: (i) $\mu(F \otimes E) \geq 2g(C) = (K_S + B) \cdot B + 2$. Since $(K_S +$

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$nB)\cdot B \geq 2g(C)$, we have that $\mu(M_{L\otimes \mathcal{O}_C}) \geq -2$. So $\mu(F\otimes E) \geq -2p+(K_S+mB)\cdot B$. To check the above inequality, it is enough to check that that $(m-1)B^2 \geq 2p+2$ for all $m \geq p+1$. This follows as $p \geq 1$ and $B^2 \geq 5$. Note that $B^2 \geq 5$ follows from Lemma 3.1 and Remark 3.6.1. (ii) We need to check that $\mu(F\otimes E) > 4g(C)+4-B^2-2h^1(B\otimes \mathcal{O}_C)$, or equivalently $\mu(F\otimes E) > 2(K_S+B)\cdot B+4-B^2-2h^1(B\otimes \mathcal{O}_C)$. In view of the observations made in (i) this follows if $(pB-K_S)\cdot B > 2p+4-2h^1(B\otimes \mathcal{O}_C)$. This follows from the following argument; $B \not\equiv K_S$, $p_g \geq 3$ and $B^2 \geq 5$. Now the required inequality follows from the fact that $p \geq 3$. So what we have done above is to absorb a B into $M_L^{\otimes p} \otimes L'$. We follow the above procedure to absorb $(p-1)$ such B's. To complete the proof of surjectivity of multiplication map (4.5.1), we finally need to confront the multiplication map involving $K_S \otimes B$. That is;

$$H^0(M_L^{\otimes p} \otimes L' \otimes B^{\otimes p}) \otimes H^0(K_S \otimes B) \to H^0(M_L^{\otimes p} \otimes L \otimes L').$$

This multiplication map is surjective and that can be proved either by going to a smooth curve in the linear system $|K_S \otimes B|$ or by CM-Lemma together with Observation 2.3. In either case one needs to prove the vanishing of the cohomology group $H^1(M_L^{\otimes p} \otimes L' \otimes B^{\otimes p-1} \otimes K_S^*) = H^1(M_L^{\otimes p} \otimes B^{\otimes m+p-1})$ for all $m \geq p+1$ and $p \geq 3$. We will prove this below. So far the choices of curves that we made to reduce the multiplication maps on surfaces to a curve have been quite natural, here is where the proof slightly deviates from the theme of the paper. The choice below is not very natural in some sense, but this choice gives a quick proof of the vanishing.

Note that $D = B^{\otimes 2} \otimes A$ is base-point free. This is so because $A = B \otimes K_S^*$ is *nef*. So $D = K_S \otimes B \otimes A^{\otimes 2}$ is base-point free by Remark 3.6.1. Let $\mathfrak{d}$ denote a smooth irreducible member in the linear system $|D|$. Consider the exact sequence

$$0 \to \mathcal{O}_S(D^*) \to \mathcal{O}_S \to \mathcal{O}_{\mathfrak{d}} \to 0.$$

Tensor this sequence with $M_L^{\otimes p}\otimes K_S \otimes B^{\otimes(m+p-2)} \otimes A$ and take the long exact sequence of cohomology. Let us denote $E = K_S \otimes B^{\otimes(m+p-2)} \otimes A$. We have

$$\cdots \to H^1(M_L^{\otimes p} \otimes E \otimes D^*) \to H^1(M_L^{\otimes p} \otimes E) \to H^1(M_L^{\otimes p} \otimes E \otimes \mathcal{O}_{\mathfrak{d}}) \to \cdots.$$

Note that the left hand side of the above sequence $H^1(M_L^{\otimes p} \otimes E \otimes D^*) = H^1(M_L^{\otimes p} \otimes K_S \otimes B^{\otimes(m+p-4)})$. This cohomology is zero by induction provided $m+p-4 \geq p$. But this holds as $m \geq p+1$ and $p \geq 3$. The right side $H^1(M_L^{\otimes p} \otimes E \otimes \mathcal{O}_{\mathfrak{d}})$ also vanishes. To see this, we first make the observation that the vector bundle $M_L^{\otimes p} \otimes E \otimes \mathcal{O}_{\mathfrak{d}}$ is unstable, so a direct computation of the slope of vector bundle is of little use. In order to show the vanishing, we will construct a semistable filtration of this unstable vector bundle and then show the needed vanishing. We have $H^1(L \otimes D^*) = 0$, it is not hard to argue that the following sequence is exact:

$$0 \to H^0(L \otimes D^*) \otimes \mathcal{O}_{\mathfrak{d}} \to M_L \otimes \mathcal{O}_{\mathfrak{d}} \to M_{L\otimes \mathcal{O}_{\mathfrak{d}}} \to 0. \tag{4.5.2}$$

The $\deg(L \otimes \mathcal{O}_{\mathfrak{d}}) \geq 2g(\mathfrak{d})$, hence $M_{L\otimes \mathcal{O}_{\mathfrak{d}}}$ is semistable. The above sequence is the required semistable filtration that we are looking for. Since we are working over a field of characteristic zero, tensor product of semistable bundles is semistable, so $M^{\otimes p}_{L\otimes\mathcal{O}_\mathfrak{d}}$ is semistable. Tensoring sequence (4.5.2) by $M_L^{\otimes i} \otimes E \otimes \mathcal{O}_\mathfrak{d}$ and by repeated iteration, the vanishing of $H^1(M_L^{\otimes p} \otimes E \otimes \mathcal{O}_\mathfrak{d})$ holds if $H^1(M^{\otimes i}_{L\otimes\mathcal{O}_\mathfrak{d}} \otimes E \otimes \mathcal{O}_\mathfrak{d})$ vanishes for all $0 \leq i \leq p$. Since $n, m \geq p + 1$, $p \geq 3$ and $B^2 \geq 5$, it follows that the bundles $M^{\otimes i}_{L\otimes\mathcal{O}_\mathfrak{d}} \otimes E \otimes \mathcal{O}_\mathfrak{d}$ have slope bigger than $2g(\mathfrak{d}) - 1$ for all $0 \leq i \leq p$. Hence we have the required vanishing. All of this proves that $H^1(M_L^{\otimes p} \otimes B^{\otimes m+p-1}) = 0$ for all $m \geq p + 1$ and $p \geq 3$.

We will complete the proof by applying Lemma 2.8. The only thing left to be checked is the vanishing of $H^2(M_L^{\otimes p} \otimes B^{\otimes m+p-2} \otimes K_S^*)$. This follows by tensoring

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_S \to L \to 0$$

with $M_L^{\otimes p-1} \otimes B^{\otimes m+p-2} \otimes K_S^*$ and taking long exact sequence of cohomology and using induction. Ultimately it comes to proving $H^2(B^{\otimes m+p-2} \otimes K_S^*)$ and $H^1(M_L^{\otimes p-1} \otimes B^{\otimes m+n+p-1})$ for all $m \geq p + 1$ and $p \geq 3$. The vanishing of the former follows from the fact that $B \otimes K_S^*$ is *nef* and the K–V vanishing theorem. The latter follows by the process used to prove the vanishing of $H^1(M_L^{\otimes p} \otimes B^{\otimes m+p-1})$ above. ∎

Remark 4.6

(1) A slightly weaker statement holds for an irregular surface of general type S with the hypothesis on B and S as in Theorem 4.3. Combining the techniques of this paper together with Theorem 1.3, [GP2] yields $K_S \otimes B^{\otimes n}$ satisfies N_p for all $n \geq (p + 3)$.

(2) A corollary analogous to Corollary 3.7 on N_p property of pluricanonical linear systems can be deduced from the above theorem. This recovers some results in [GP2] on pluricanonical linear systems.

We close this section with the following corollary giving effective bounds towards property N_p for adjoint linear series associated to ample line bundles.

Corollary 4.7 *Let S be a regular surface of general type, and A be an ample line bundle on S. Let $m = [\frac{(A\cdot(K_S+4A)+1)^2}{2A^2}]$ and $L = K_S \otimes A^{\otimes n}$. If $n \geq mp + m$, then L satisfies property N_p.*

Proof Follow the same line of reasoning as Corollary 3.14 and apply Theorem 4.5. ∎

5 Boundary Examples and Remarks

Given a surface of general type, there is a "large" class of base point free and ample line bundles B for which $B \otimes K_S^*$ *nef* since the condition $B \cdot C \geq K_S \cdot C$ for all curves C lying on S is an open condition in the ample cone. So there is an "open set" of examples satisfying the conditions in theorems of the preceding section.

In this section we construct some examples to show that the results in previous sections are optimal for various reasons.

Example 5.1 Let $\varphi\colon S \to \mathbf{P}^2$ be the double cover of $\mathbf{P}^2$ branched along a smooth curve in $|\mathcal{O}_{\mathbf{P}^2}(2r)|$ with $r \geq 4$.

(a) Let $r = 4$. So we have, $\varphi_*(\mathcal{O}_S) = \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-4)$. Also, $K_S = \varphi^*\big(\mathcal{O}_{\mathbf{P}^2}(1)\big)$. So, S is a regular surface of general type with $p_g(S) = 3$. Note that K_S is base-point free and ample divisor. Let $L = K_S^{\otimes 3}$. Then $H^1(M_L \otimes L)$ doesn't vanish, hence $H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2})$ is not surjective. So $K_S^{\otimes 3}$ does not embed X as a projectively normal variety. This example shows that the condition $B \not\equiv K_S$ in Theorem 3.3 is necessary. Also shows that the condition $p_g \geq 4$ mentioned in the addendum to Theorem 3.3, namely Remark 3.3.8 is necessary. This example also shows that the bound on n in Corollary 3.8(b) is sharp.

Next we show that the condition (2) in Theorem 3.4 or the inequality $B^2 \geq B \cdot K_S$ in Theorem 3.3 is necessary. The examples also illustrate that the condition $h^0(B) \geq 4$ in Theorem 3.3 cannot be relaxed.

(b) Let $r \geq 5$. We have $\varphi_*(\mathcal{O}_S) = \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-r)$. Also, $K_S = \varphi^*\big(\mathcal{O}_{\mathbf{P}^2}(r-3)\big)$. Denote $B = \varphi^*\big(\mathcal{O}_{\mathbf{P}^2}(1)\big)$. So, S is a regular surface of general type with $p_g(S)$ as large as we wish it to be. Note that B is a base-point free and ample divisor which is not homologous to K_S and $H^1(B) = 0$. Also, $B^2 < B \cdot K_S$.

Denote $L = K_S \otimes B^{\otimes 2}$. So $L = \varphi^*\big(\mathcal{O}_{\mathbf{P}^2}(r-1)\big)$. We claim that L does not satisfy property N_0.

The multiplication map,

$$\alpha\colon H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2})$$

is not surjective. Indeed we have,

$$H^0(L) = H^0\big(\mathcal{O}_{\mathbf{P}^2}(r-1)\big) \oplus H^0\big(\mathcal{O}_{\mathbf{P}^2}(-1)\big) \text{ and}$$
$$H^0(L^{\otimes 2}) = H^0\big(\mathcal{O}_{\mathbf{P}^2}(2r-2)\big) \oplus H^0\big(\mathcal{O}_{\mathbf{P}^2}(r-2)\big).$$

So it follows that the image of α is $H^0\big(\mathcal{O}_{\mathbf{P}^2}(2r-2)\big)$. Hence it is not surjective for all $r \geq 2$. So L doesn't satisfy Property N_0. ■

We show below an example where the failure of B to satisfy the inequalities in Theorem 3.9 and the failure of $B \otimes K_S^*$ to be *nef* in Corollary 3.10, leads to the failure of property N_1 for the associated adjunction bundle even though all the other hypotheses are satisfied.

Example 5.2 Let S be a cyclic triple cover of $\mathbf{P}^2$ ramified along a smooth curve of degree 9. Let B be the pullback of $\mathcal{O}_{\mathbf{P}^2}(1)$ to S. The surface X is a regular surface of general type with $p_g = 11$. Also, $H^1(B) = 0$. But $L = K_S \otimes B^{\otimes 2}$ satisfies N_0 but not the property N_1. Note again that $B^2 < 9 = B \cdot K_S$, which violates the necessary hypothesis of Theorem 3.9.

Proof Note that $B \otimes K_S^*$ is not *nef*. In fact $B^2 < 9 = B \cdot K_S$. One can check as in the above cases that L satisfies property N_0 by pushing it down to $\mathbf{P}^2$. Assume

$L = K_S \otimes B^{\otimes 2}$ satisfies N_1. By Theorem 1.1, the assumption implies

$$H^1\Big(\bigwedge^2 M_L \otimes L^{\otimes n}\Big) = 0 \tag{5.2.1}$$

for all $n \geq 1$. Let $C \in |B|$ be a smooth curve. Using repeatedly the sequence $(*)$ it is easy to see that $H^2(M_L^{\otimes 2} \otimes L^{\otimes n} \otimes B^*)$ vanishes; in fact it follows since $H^1(M_L \otimes L^{\otimes n+1} \otimes B^*) = 0$ (use the fact that L satisfies N_0 and Observation 2.3) and $H^i(L^{\otimes n} \otimes B^*) = 0$ for $i = 1, 2$ and $n \geq 1$. This in turn implies that $H^2(\bigwedge^2 M_L \otimes L^{\otimes n} \otimes B^*)$ vanishes. These vanishings together with (5.2.1) imply that $H^1(\bigwedge^2 M_L \otimes L^{\otimes n} \otimes \mathcal{O}_C) = 0$. On the other hand there is an epimorphism between the vector bundles $M_L \otimes \mathcal{O}_C$ and $M_{L \otimes \mathcal{O}_C}$ on C as shown by the semistable filtration in the proof of Theorem 4.5. Therefore we have

$$H^1\Big(\bigwedge^2 M_{L \otimes \mathcal{O}_C} \otimes L^{\otimes n}\Big) = 0 \tag{5.2.2}$$

for all $n \geq 1$. Note that $L \otimes \mathcal{O}_C = K_C \otimes B$ and $\deg(B \otimes \mathcal{O}_C) = 3$. It is a well known result of Castelnuovo that a line bundle of degree greater than or equal to $2g + 1$ on a smooth curve satisfies property N_0. The curve C has genus 7 and $L \otimes \mathcal{O}_C$ has degree 15, hence $L \otimes \mathcal{O}_C$ satisfies N_0. Thus it would follow from (5.2.2) that $L \otimes \mathcal{O}_C$ satisfies also property N_1 by Theorem 1.1. But a line bundle that is the tensor product of the canonical bundle of C and an effective line bundle of degree 3, cannot satisfy property N_1 ([GL].) Such is the case with $L \otimes \mathcal{O}_C$. Therefore the original assumption (5.2.1) is false and L does not satisfy property N_1. ∎

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