RINGS ALL OF WHOSE TORSION QUASI-INJECTIVE MODULES ARE INJECTIVE

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1. Introduction and background. Throughout this paper it is assumed that rings are associative, have the identity element, and all modules are left unital. R will denote a ring with identity, R-Mod the category of left R-modules, and for each left R-module M, $E(M)$ (resp. $J(M)$) will represent the injective hull (resp. Jacobson radical) of M. Also, for a module M, $A \subseteq' M$ will mean that A is an essential submodule of M, and $Z(M)$ denotes the singular submodule of M. M is called *singular* if $Z(M) = M$, and it is called *non-singular* in case $Z(M) = 0$. For fundamental definitions and results related to torsion theories, we refer to [**12**] and [**14**]. In this paper we shall deal mainly with Goldie torsion theory. Recall that a pair (G, F) of classes of left R-modules is known as *Goldie torsion theory* if G is the smallest torsion class containing all modules B/A, where $A \subseteq' B$, and the torsion free class F is precisely the class of non-singular modules.

A ring R is called a *left V-ring* if each simple left R-module is injective. R is a left V-ring if and only if each left R-module has zero Jacobson radical (if and only if each left ideal of R is the intersection of maximal left ideals) (see [**13**, Theorem 2.1]). In the commutative case, it is well known that R is a V-ring if and only if R is regular (in the sense of von Neumann). A module M is *quasi-injective* if every homomorphism from a submodule of M into M can be lifted to an endomorphism of M. A ring R is called a *left QI-ring* if each quasi-injective left R-module is injective. These rings were originally introduced in [**1**], and later studied by many authors (see, for example, [**2**, **3**, **4**, **6**, **7**, **8**, **10**]). Left QI-rings are left Noetherian and left V-rings (see [**1**]). Thus commutative QI-rings are semisimple Artinian. The goal of this paper is to study rings all of whose (Goldie) torsion quasi-injective modules are injective. The results developed in the next section indicate that these rings provide an effective torsion theoretical generalization of left QI-rings.

2. Results. Let (T, F) be a hereditary torsion theory, and let $\mathcal{F}(T)$ be the associated filter of left ideals of R. A left ideal of R which is a member of $\mathcal{F}(T)$ will be called an *$\mathcal{F}$-ideal*. Relative to (T, F), a left R-module E is called *T-injective* if $\mathrm{Ext}_R(R/I, E) = 0$ for all $I \in \mathcal{F}(T)$. This definition is equivalent to the following statement: For any $I \in \mathcal{F}(T)$, each homomorphism $f: I \to E$ can be extended to a homomorphism $g: R \to E$. The following result concerning T-injective modules can be found in Golan and Teply [**11**, Lemma 2, p. 252].

LEMMA 1. *Let (T, F) be a hereditary torsion theory for R-*Mod. *Then the following are equivalent:*

(1) *R has ACC on $\mathcal{F}$-ideals;*

(2) *any direct sum of (countably many) T-injective torsion modules is T-injective.*

We now consider the Goldie torsion theory (G, F) for R-Mod, whose associated filter of left ideals is denoted by $\mathcal{F}(G)$. Since all essential left ideals of R belong to $\mathcal{F}(G)$, it follows that, relative to the Goldie torsion theory, a left R-module is T-injective if and only if it is injective. Thus the following proposition is immediate in view of the above lemma.

Proposition 1. *Let (G, F) be the Goldie torsion theory for R-Mod. Then R has ACC on $\mathcal{F}$-ideals if and only if each direct sum of torsion injective modules is injective.*

We now phrase a definition for the sake of brevity.

Definition. A ring R will be called a *left TQI-ring* if each torsion quasi-injective left R-module is injective.

Thus every left QI-ring is left TQI. The converse, however, is not true in general (see Example 1 below). We now prove some lemmas concerning left TQI-rings. Throughout the remainder of this paper, it will be assumed that we are working in the context of Goldie torsion theory.

Lemma 2. *Let R be a left TQI-ring. Then for each torsion left R-module M, $J(M)=0$.*

Proof. Let M be a torsion left R-module. Let $x \in M$, $x \neq 0$. Then, by Zorn's lemma, there is a submodule Y of M which is maximal among the submodules X of M with $x \notin X$. Let $D = Rx + Y$. Then $x \in D$, and $D/Y \neq (0)$. Also, D/Y is simple and torsion. Hence D/Y is injective. Therefore, $M/Y = D/Y \oplus K/Y$, where K is a submodule of M. Since $x \notin K$, $M/Y = D/Y$. Hence Y is a maximal submodule of M. Since $x \notin Y$, $J(M) = (0)$.

Corollary 1. *Let I be an $\mathcal{F}$-ideal of R. Then I is an intersection of maximal left ideals.*

Proof. Since I is an $\mathcal{F}$-ideal, R/I is a torsion left R-module. Therefore $J(R/I) = (0)$. Hence I is an intersection of maximal left ideals.

Corollary 2. *If each $\mathcal{F}$-ideal is the intersection of maximal left ideals then each simple torsion left R-module is injective.*

Proof. Let S be a simple torsion left R-module, and let I be an $\mathcal{F}$-ideal. Suppose $f \in \mathrm{Hom}_R(I, S)$. We claim that f is extendable to an element of $\mathrm{Hom}_R(R, S)$. Let $\mathrm{Ker}\, f = K$. Consider the sequence: $0 \to I/K \to R/K \to R/I \to 0$. Since I/K and R/I are torsion, R/K is torsion. Thus both K and I are $\mathcal{F}$-ideals and $I \supseteq K$. If $I = K$ then we are done, otherwise there is a maximal left ideal L of R such that L contains K, but does not contain I. Thus $L + I = R$ and $L \cap I = K$. Hence $R/K = L/K \oplus I/K$. Thus the above sequence splits and it can be shown that f is extendable to a map from R to S. Hence S is T-injective, and so it is injective, as we are working in the Goldie torsion theory.

The proofs of Lemma 2, Corollaries 1 and 2 are adaptions from [**13**, Theorem 2.1].

Lemma 3. *Let R be a left TQI-ring. Then R has ACC on $\mathcal{F}$-ideals.*

Proof. Let $I_1 \subset I_2 \subset \ldots \subset I_n \subset \ldots$ be an ascending chain of distinct $\mathcal{F}$-ideals of R.

Then by Corollary 1 to Lemma 2, there are maximal left ideals M_k, $k=1,2,\ldots$, such that $I_k \subset M_k$ but $I_{k+1} \not\subset M_k$. Let $\pi_k : R \to R/M_k$ be the natural projection. Also, let $I = \bigcup_{k=1}^{\infty} I_k$, and define $f : I \to \sum_{k=1}^{\infty} \oplus R/M_k$ by $f(x) = \sum_{k=1}^{\infty} \pi_k(x)$. Note that x is an element of only a finite number of the M_k, for all $x \in I$, and f is an epimorphism. Since $\sum_{k=1}^{\infty} \oplus R/M_k$ is semisimple and torsion, it is torsion quasi-injective and hence injective, as R is a left TQI-ring. Hence f extends to $g : R \to \sum_{k=1}^{\infty} \oplus R/M_k$. Since R has an identity, $g(R)$ and hence $g(I)$ is contained in $\sum_{k=1}^{n} \oplus R/M_k$, for some positive integer n. This implies that the above chain of left $\mathcal{F}$-ideals is finite.

We now prove the following.

THEOREM 1. *Let (G, F) be the Goldie torsion theory for R-*Mod. *Then the following are equivalent:*

(1) *R is left TQI;*

(2) *each direct sum of torsion quasi-injective modules is quasi-injective.*

Proof. Suppose R is a left TQI-ring. Let $M = \bigoplus_i M_i$ be a direct sum of torsion quasi-injective modules M_i. Then each M_i is injective by the hypothesis. Hence M is injective by Proposition 1. Now, assume (2). Let M be a torsion quasi-injective module. Then $M \oplus E(M)$ is quasi-injective, by the assumption. From this it follows that $M \cong E(M)$ and so M is injective.

According to a well-known result, every injective module over a left Noetherian ring is the direct sum of indecomposable injective modules. A similar decomposition property is also true for quasi-injective modules over left Noetherian rings. The next two lemmas proved below provide analogues of these results for torsion injective and torsion quasi-injective modules over rings whose $\mathcal{F}$-ideals satisfy the ascending chain condition.

LEMMA 4. *Let R be a ring whose $\mathcal{F}$-ideals have ACC. Then every torsion injective left R-module is the direct sum of indecomposable injective modules.*

Proof. Let M be a torsion injective left R-module. Let $x \in M$, $x \neq 0$. Then $Rx \cong R/I$, for some left ideal I of R. Since Rx is torsion, I is an $\mathcal{F}$-ideal. Furthermore, since R has ACC on $\mathcal{F}$-ideals, it is easy to see that Rx is a Noetherian left R-module. Hence Rx contains a uniform submodule U. Since $E(U) \subseteq M$, M has a torsion injective submodule which is the injective hull of its every non-zero submodule. Let $\{M_j\}_{j \in J}$ be a maximal independent family of submodules of M such that each M_j is the injective hull of all of its non-zero submodules, where J is an indexing set. Now let $y \in M$ be a non-zero element. Then, as above, $E(Ry)$ contains a submodule which is the injective hull of all of its non-zero submodules, and hence $\bigoplus_{j \in J} M_j \cap Ry \neq (0)$. This implies that $\bigoplus_{j \in J} M_j \subseteq' M$. On the other hand, since each M_j is torsion injective and also since R has ACC on $\mathcal{F}$-ideals, it follows from Proposition 1 that $\bigoplus_{j\in J} M_j$ is injective. Hence $\bigoplus_{j\in J} M_j$ is a direct summand of M. But $\bigoplus_{j\in J} M_j \subseteq' M$. Therefore, $\bigoplus_{j\in J} M_j = M$. Further, since the endomorphism ring of each indecomposable injective module is local, it follows from the Krull–Remak–Schmidt theorem that this decomposition is unique (up to isomorphism).

LEMMA 5. *Let R be a ring whose $\mathcal{F}$-ideals have ACC. Then each torsion quasi-injective left R-module is the direct sum of indecomposable quasi-injective modules.*

Proof. Let M be a torsion quasi-injective left R-module. Then $E(M)$ is torsion, as we are dealing with Goldie torsion theory. Hence, by Lemma 4, $E(M) = \bigoplus_i E_i$, where each E_i is an indecomposable injective module. Let $p_i : E(M) \to E_i$ be the projection map. Then since M is quasi-injective, $p_j(M) \subset M$. Now, let $x \in M$, say $x = \sum_j x_j$, $x_j \in E_j$; then $x_j \in M$. Then $M = \sum M \cap E_j$, i.e. $M = \sum \oplus (M \cap E_j)$. Hence M is the direct sum of indecomposable quasi-injective modules.

Before we state the next theorem, we note that we shall use the notation $S(M)$ for the socle of a left R-module M. In particular, $S(R)$ will denote the socle of ${}_RR$.

THEOREM 2. *Let R be a ring with $S(R) \subseteq' R$, and (G, F) be the Goldie torsion theory for R-*Mod. *Then the following are equivalent:*

(1) *R is a left TQI-ring;*

(2) *R has ACC on $\mathcal{F}$-ideals, each $\mathcal{F}$-ideal is the intersection of maximal left ideals, and $R/S(R)$ is a left QI-ring.*

Proof. (1) Let us assume that R is a left TQI-ring. Then, by Lemma 3, R has ACC on $\mathcal{F}$-ideals, and each $\mathcal{F}$-ideal is the intersection of maximal left ideals by Corollary 1 to Lemma 2. Furthermore, since $S(R) \subseteq' R$ by the hypothesis, $S(R)$ is an $\mathcal{F}$-ideal. Hence $R/S(R)$ is semiprime by Lemma 2. Also, it follows immediately from Lemma 3, that $R/S(R)$ is Noetherian. Moreover, since R is a left TQI-ring, it is straightforward to argue that $R/S(R)$ is also a left TQI-ring. Now it is a known result that every semiprime left Noetherian left TQI-ring is a left QI-ring (see [**2**, Corollary 9, p. 48]). Hence $R/S(R)$ is a left QI-ring.

(2) Conversely, we now assume that R has ACC on $\mathcal{F}$-ideals, each $\mathcal{F}$-ideal is the intersection of maximal left ideals, and $R/S(R)$ is a left QI-ring. We prove that R is a left TQI-ring. So, let M be a torsion quasi-injective left R-module. Then, by Lemma 5, $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$, where each M_α is torsion, indecomposable and quasi-injective. Consider an arbitrary but fixed M_α. If $S(M_\alpha) \neq 0$ then there exists a simple (torsion) submodule S $(\neq 0)$ of M. Since each $\mathcal{F}$-ideal is the intersection of maximal left ideals, it follows from Corollary 2 to Lemma 2 that S is injective. So, S is a direct summand of M_α. But M_α is an indecomposable module. Hence $M_\alpha = S$, i.e. M_α is injective. Let us now suppose that $S(M_\alpha) = 0$. Since M_α is torsion, M_α may be regarded as an $R/S(R)$-module. Thus M_α is a quasi-injective $R/S(R)$-module. Since $R/S(R)$ is a left QI-ring, M_α is $R/S(R)$-injective. We claim that M_α is R-injective. Let I be a left ideal of R, and let $\alpha : I \to M_\alpha$ be an (R-)

homomorphism. Let us define a map $\beta:(I+S(R))/S(R)\to M_\alpha$, by the rule $\beta(x+S(R))=\alpha(x)$, for all $x\in I$. If $x\in I\cap S(R)$ then $\alpha(x)\in\alpha(S(I))\subseteq S(M_\alpha)=0$. Hence β is a well-defined map, which is an (R-) homomorphism. Also, it is clear that β is an $R/S(R)$-homomorphism. Since M_α is $R/S(R)$-injective, there is an $m\in M_\alpha$, such that $\beta(x+S(R))=m(x+S(R))$, for all $(x+S(R))\in(I+S(R))/S(R)$. Thus, we have $\alpha(x)=mx$, for all $x\in I$. Hence M_α is (R-) injective. Therefore, $M=\bigoplus_{\alpha\in\Lambda} M_\alpha$ is a direct sum of torsion injective modules. Now, since R has ACC on $\mathcal{F}$-ideals, it follows from Proposition 1 that M is injective.

Now we recall a definition. R is called a left *T-ring* if every non-zero left R-module has non-zero socle. For the purpose of the next theorem, we need a definition which is weaker than that of left T-rings. Namely, rings for which every non-zero torsion cyclic module has non-zero socle. Such rings will be called *weakly T-rings*.

THEOREM 3. *Let R be a weakly T-ring, and (G,F) be the Goldie torsion theory for R-*Mod. *Then the following are equivalent:*

(1) *R is a left TQI-ring;*

(2) *R has ACC on $\mathcal{F}$-ideals and each $\mathcal{F}$-ideal is the intersection of maximal left ideals;*

(3) *the torsion class coincides with the class of semisimple modules.*

Proof. (1)$\Rightarrow$(2). This follows from Corollary 1 to Lemma 2 and Lemma 3, and does not depend on the assumption that R is a weakly T-ring.

(2)$\Rightarrow$(3). Let us first consider a torsion quasi-injective left R-module M. By Lemma 5, $M=\bigoplus_\alpha M_\alpha$, where each M_α is a torsion indecomposable quasi-injective left R-module. Now consider an arbitrary but fixed M_α. Let $x\in M_\alpha$, $x\neq 0$. Then Rx is a non-zero torsion cyclic submodule of M_α. Since R is a weakly T-ring, Rx has non-zero socle. So, let S be a non-zero simple torsion submodule of Rx. By Corollary 2 to Lemma 2, it follows that S is injective. Hence S is a direct summand of M_α. But M_α is indecomposable. So, $M_\alpha=S$. Thus $M=\bigoplus_\alpha M_\alpha$ is a direct sum of torsion injective simple modules. Hence M is semisimple, and also injective by Proposition 1. Since for the Goldie torsion class, every torsion module is a submodule of a torsion injective module, it follows that every torsion module is semisimple.

(3)$\Rightarrow$(1). This is immediate.

We now give an example of a left TQI-ring which is not a left QI-ring.

EXAMPLE 1. Let F be a field, and let $R=\begin{bmatrix}F&0\\F&F\end{bmatrix}$. Also, let $A=\begin{bmatrix}F&0\\F&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\F&F\end{bmatrix}$.

In the above example, $Z({}_RR)=0$, and B is the unique proper essential left ideal of R, which is also a maximal left ideal. Clearly, R is an Artinian ring. Thus R is a left T-ring whose $\mathscr{F}$-ideals have ACC and each proper essential left ideal is an intersection of maximal ideals. Now consider the Goldie torsion theory over R. Since $Z({}_RR)=0$, the associated filter $\mathscr{F}(G)$ consists of essential left ideals of R. Hence by Theorem 2 (statement (2)), R is a left TQI-ring. On the other hand, since $J(R)=A\cap B\neq(0)$, R is not a V-ring. Hence there exists a simple left R-module which is not injective. Since simple modules are quasi-injective, it follows that R is not a left QI-ring.

The purpose of the next theorem is to obtain a characterization of semilocal TQI-rings. By a *semilocal ring* we mean a ring which contains only a finite number of distinct maximal left ideals.

THEOREM 4. *Let R be a semilocal ring. Then the following are equivalent:*

(1) *R is left TQI;*

(2) *R has ACC on $\mathscr{F}$-ideals and each $\mathscr{F}$-ideal is the intersection of maximal left ideals.*

Proof. The implication (1)$\Rightarrow$(2) follows Corollary 1 to Lemma 2 and Lemma 3, and it does not depend on the hypothesis that R is semilocal. Hence we need only to prove (2)$\Rightarrow$(1). Let M be a torsion quasi-injective left R-module. Then, by Lemma 5, $M=\bigoplus_\alpha M_\alpha$, where each M_α is a (torsion) indecomposable quasi-injective left R-module. Consider an arbitrary but fixed M_α. Let $x\in M_\alpha$, $x\neq 0$. If $Rx\cong R$ then M_α contains a copy of R. Since M_α is quasi-injective, an application of Baer's criterion implies that M_α is injective. Now suppose $Rx\cong R/I$, for some left ideal I of R. Then I is an $\mathscr{F}$-ideal. Hence there exist maximal left ideals $M_1,\ldots,M_n$ of R such that $I=M_1\cap\ldots\cap M_n$. We claim that R/I is semisimple Artinian. Let us define a map $\phi: R/I\to R/M_1\oplus\ldots\oplus R/M_n$, by $\phi(\bar r)=(\bar r_1,\ldots,\bar r_n)$, where $\bar r=r+I$ and $\bar r_k=r+M_k$, $k=1,\ldots,n$. Clearly, ϕ is an R-homomorphism. Also, ϕ is an injection because for each non-zero $\bar r\in R/I$, there is some M_i such that $r\notin M_i$. Thus R/I is a submodule of a semisimple Artinian module. Hence R/I is semisimple Artinian. As shown in Corollary 2 to Lemma 2, if each $\mathscr{F}$-ideal is the intersection of maximal left ideals then each torsion simple left R-module is injective. Hence R/I is injective. This implies that Rx is a direct summand of M_α. But M_α is indecomposable. Hence M_α is injective. Consequently, $M=\bigoplus_\alpha M_\alpha$ is injective, by Proposition 1, as R has ACC on $\mathscr{F}$-ideals.

We shall now study commutative TQI-rings. Let us first recall that a ring R is said to *have SP* if every left R-module splits (i.e. if the torsion submodule of each left R-module M is a direct summand of M). The next lemma is proved for not necessarily commutative TQI-rings.

LEMMA 6. *Let R be a left TQI-ring. Then every $\mathscr{F}$-ideal of R is idempotent.*

Proof. Let I be an $\mathscr{F}$-ideal of R. If $I=I^2$ then there is nothing to prove. So, suppose $I\neq I^2$. Since, for each $r\in I$, $I(r+I^2)=0$ in I/I^2, i.e. $I(I/I^2)=0$, it follows that I/I^2 is torsion. Now consider the exact sequence: $0\to I/I^2\to R/I^2\to R/I\to 0$. Since I/I^2 and R/I are torsion, R/I^2 is torsion. Hence I^2 is an $\mathscr{F}$-ideal. Hence by Corollary 1 to Lemma 1, both I and I^2 are intersections of maximal left ideals. Thus there is a maximal left ideal M of R

such that $I^2 \subseteq M$ but $I \nsubseteq M$. Since M is a maximal ideal, $R = I + M$ and $1 = x + m$, for some $x \in I$ and $m \in M$. Then $x = x^2 + xm \in M$. Thus $1 \in M$, which is a contradiction.

LEMMA 7. *Every commutative TQI-ring is a V-ring (and hence a regular ring).*

Proof. It is enough to show that $I = I^2$, for each ideal I of R. If I is an essential ideal then I is an $\mathcal{F}$-ideal. Hence, by the above lemma, $I = I^2$. Now suppose I is not an essential ideal, then there is an ideal J of R such that $I \cap J = (0)$ and $(I + J) \subseteq' R$. Then $(I + J) = (I + J)^2 = I^2 + IJ + IJ + J^2 = I^2 + J^2$. This equation implies that $I = I^2$.

The next lemma is due to Cateforis and Sandomierski [**5**, Theorem 2.1, p. 156].

LEMMA 8. *For a commutative ring R, the following are equivalent:*

(1) *R has SP;*

(2) *$Z(R) = 0$ and for every essential ideal I of R, the ring R/I is semisimple Artinian;*

(3) *every R-module M with $Z(M) = M$ is R-injective. In particular, if R has SP then R is hereditary.*

THEOREM 5. *Let R be a commutative ring and (G, F) be the Goldie torsion theory for* R-Mod. *Then the following are equivalent:*

(1) *R is TQI;*

(2) *R has SP.*

Proof. Let us assume (1). Then, by Lemma 7, R is regular. Hence $Z(R) = 0$. Now let I be an essential ideal of R. Then R/I is a regular ring. Since R is a TQI-ring, R has ACC on $\mathcal{F}$-ideals by Lemma 3. Further, since every essential ideal of R is an $\mathcal{F}$-ideal, it follows that R/I is Noetherian, as an R-module and hence as an R/I-module. Thus R/I is a regular Noetherian ring. Hence R/I is a semisimple Artinian ring. Therefore, R has SP by Lemma 8 (statement (2)). Let us now assume that R has SP. Then R is hereditary by Lemma 8. Hence $Z(R) = 0$. Thus the class of torsion modules, relative to the Goldie torsion theory for R-Mod, are precisely those modules M for which $Z(M) = M$. But such modules are injective by Lemma 8. Hence, in particular, torsion quasi-injective modules are injective. So, R is a TQI-ring.

We now give an example of a commutative TQI-ring which is not a QI-ring.

EXAMPLE 2. [**5**, p. 161]. Let K be a field and A an infinite indexing set. Let $Q = \prod_{\alpha \in A} K^{(\alpha)}$, where $K^{(\alpha)} = K$ and $R = \sum_{\alpha \in A} \oplus K^{(\alpha)} + 1K \subseteq Q$, $1 \in Q$. Then R has only one essential ideal, namely, $I = \sum_{\alpha \in A} \oplus K^{(\alpha)}$, and I is maximal. Since R is regular, R has SP by Lemma 8. Hence R is a TQI-ring by Theorem 5. On the other hand, since R is not semisimple Artinian and since commutative QI-rings are necessarily semisimple Artinian, it follows that R is not a QI-ring.

As noted earlier, commutative TQI-rings are hereditary. Thus the next proposition is an immediate consequence of the following result, which is also due to Cateforis and Sandomierski [**5**, Theorem 4.1, p. 164].

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LEMMA 9. *Let $\{R_\alpha \mid \alpha \in \Lambda$ and $R_\alpha \neq 0$ for all $\alpha\}$ be an infinite collection of hereditary rings R_α (with identity). Then $\prod_\alpha R_\alpha$ is not a hereditary ring.*

PROPOSITION 2. *An infinite direct product of commutative TQI-rings cannot be a TQI-ring.*

We now characterize rings whose torsion quasi-injective left R-modules are Σ-quasi-injective. Recall that a quasi-injective module M is *Σ-quasi-injective* if $M^{(A)}$ (the direct sum of card A copies of M) is also quasi-injective, for any set A. First we state two needed results.

LEMMA 10. [**14**, Prop. 4.2, p. 21]. *Let (T, F) be any stable torsion theory for R-*Mod *(i.e. T is closed under injective hulls). Then every injective left R-module splits.*

LEMMA 11. [**9**, Cor. 2.2]. *If M is quasi-injective and $E(M)^{(A)}$ is injective then $M^{(A)}$ is quasi-injective for any set A.*

THEOREM 6. *Let R be any ring and (G, F) be the Goldie torsion theory for R-*Mod. *Then the following are equivalent:*

(1) *each torsion quasi-injective left R-module is Σ-quasi-injective;*

(2) *R has ACC on $\mathcal{F}$-ideals.*

Proof. (1) Suppose each torsion quasi-injective left R-module is Σ-quasi-injective. Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_k \subseteq \ldots$ be an ascending chain of $\mathcal{F}$-ideals of R. Then we get the ascending chain

$$(0) = I_1/I_1 \subseteq I_2/I_1 \subseteq \ldots \subseteq I_k/I_1 \subseteq \ldots$$

of submodules of the left R-module $\bar{R} = R/I_1$. Let $\bar{I}_k = I_k/I_1$, $k = 1, 2, \ldots$. Now consider the left R-modules $\bar{R}/\bar{I}_k$. Clearly, each $\bar{R}/\bar{I}_k$ $(\cong R/I_k)$ is a torsion R-module. Let $Q_i = E(\bar{R}/\bar{I}_i)$. Then each Q_i is a torsion left R-module. Let $Q = \bigoplus_j Q_j$ and $M = \prod_j Q_j$. Since M is injective, Lemma 10 implies that M splits. Suppose $M = T \oplus K$, where T and K are the torsion and torsion free parts of M, respectively. Since each $Q_j \subseteq T$ and each Q_j is injective, we may write $T = T_j = Q_j \oplus P_j$, for some $P_j \subseteq T$. Hence $\bigoplus_i T_j = \bigoplus_i Q_j \oplus \bigoplus_i P_j$. Thus $Q = \bigoplus_j Q_j$ is a direct summand of $\bigoplus_j T_j$. But $\bigoplus_j T_j$ is quasi-injective by the hypothesis. Hence Q is quasi-injective. Now let $\bar{I} = \bigcup_{k=1}^{\infty} \bar{I}_k$. Then the natural ($R$-) homomorphism $f_k : \bar{I} \to \bar{R}/\bar{I}_k$ maps $\bar{I}$ into Q_k. Note that if $a \in \bar{I}$ then $a \in \bar{I}_t$ for some t. Thus $f_k(a) = 0$, for all $k \geq t$. Let us now define a map $f : \bar{I} \to Q$ by $f(a) = (f_1(a), \ldots, f_i(a), \ldots)$, $a \in \bar{I}$. This definition is meaningful since only a finite number of terms on the right-hand side are non-zero. Clearly, f is an R-homomorphism. Now $\bar{I} \subseteq \bar{R} \subseteq Q_1 \subseteq Q$, and Q is quasi-injective. Hence there is a map $\lambda \in \mathrm{Hom}_R(Q, Q)$ which induces f. Thus $f(\bar{I}) \subseteq \lambda(\bar{R}) \subseteq Q$. Suppose $\lambda(1 + I_1) = m$, where 1 is the identity of R. Now

$m \in \sum_{i=1}^{t} Q_i$, for some t. Hence $f(\bar{I}) \subseteq Rm \subseteq \sum_{i=1}^{t} Q_i$. This implies that $\bar{I}_{t+1} = \bar{I}_{t+2} = \ldots = \bar{I}$. Hence $I_{t+1} = I_{t+2} = \ldots = I$. This proves that R has ACC on $\mathcal{F}$-ideals.

(2) Let us now suppose that R has ACC on $\mathcal{F}$-ideals, and let M be any torsion quasi-injective left R-module. Then $E(M)$ is torsion, as we are working in the Goldie torsion theory. Moreover, since R has ACC on $\mathcal{F}$-ideals, $E(M)^{(A)}$ is injective, for any set A, by Proposition 1. Hence, by Lemma 11, $M^{(A)}$ is quasi-injective, for any set A.

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