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On a system of Feferman

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A system of set theory which appears as an extension of Ackermann set theory is introduced. In this sytem we construct a syntactic model for a theory proposed by Feferman for the development of category theory.

Feferman introduced in [1] a system of axiomatic set theory. In this paper we consider a theory T, which can be viewed as an extension of Ackermann set theory (see, for example, [4]). Our main result is the construction in T of a syntactic model for the system of Feferman. We shall use freely the terminology and notation of Feferman [1].

1. Feferman's system

The set theory of Feferman, $\mathrm{ZF}/\underline{s}$, is formulated in a first order language, $L_{\underline{s}}$, obtained by adding to L a constant symbol $\underline{s}$.

The axioms are taken to consist of the following (in the basic symbolism of $L_{\underline{s}}$)

(1) the axioms of ZF ,

(2) $\exists x(x \in \underline{s})$,

(3) $\forall x, y(y \in x \wedge x \in \underline{s} \to y \in \underline{s})$,

(4) $\forall x, y\big(x \in \underline{s} \wedge \forall z(z \in y \to z \in x) \to y \in \underline{s}\big)$,

(5) $\forall \mathrm{x} \in \underline{s}\big(\varphi^{(\underline{s})}(\mathrm{x}) \leftrightarrow \varphi(\mathrm{x})\big)$, for each formula φ of L with free variables x .

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(5) is a reflexion principle. What it means is the following: suppose (M, E, s) is a model of ZF/$\underline{s}$; define $M_s = \{x \mid x \in M$ and $xEs\}$, and say $xE_s y$ if and only if $x, y \in M_s$ and xEy . Then (M, E) is an elementary extension of (M_s, E_s) .

2. The system T

The language L_u of T is obtained by adding to L a constant symbol u .

The axioms are the universal closures of

(T1) $(\forall t)(t \in x \leftrightarrow t \in y) \rightarrow x = y$,

(T2) $(\exists t)(t \in u)$,

(T3) $(\forall x)(\exists y)(\forall t)(t \in y \leftrightarrow t \in x \wedge \varphi)$ where φ is a formula of L_u with free variables among t, x ,

(T4) $y \in x \wedge x \in u \rightarrow y \in u$,

(T5) $x \in u \wedge (\forall t)(t \in y \rightarrow t \in x) \rightarrow y \in u$,

(T6)

$$(\forall x)\big(x \in u \wedge (\exists x)(\forall t)\big(\varphi(x, t)\big) \rightarrow t \in x\big) \rightarrow (\exists z)\big(z \in u \wedge (\forall t)\big(\varphi(x, t) \rightarrow t \in z\big)\big) ,$$

for each formula φ of L with free variables t, x .

Let $\mathrm{Rn}(x, y)$ be the formula

$$\mathrm{Ord}(x) \wedge (\exists z)\big((\forall w)\big(w \in x \rightarrow (\exists v)\mathrm{Rl}(w, v, z)\big) (\forall w, v)\big(\big(w \in x \wedge \mathrm{Rl}(w, v, x)\big) \vee (w = x \wedge v = y) \rightarrow (\forall t)\big(t \in v \leftrightarrow (\exists w', v') \big(w' \in w \wedge \mathrm{Rl}(w', v', z) \wedge t \subset v'\big)\big)\big)\big) ,$$

where $\mathrm{Rl}(x, y, z)$ is the formula

$$(\exists v)(v = \langle x, y\rangle \wedge v \in z) .$$

(T7) $(\forall x)(\exists \beta)(\exists y)\big(\mathrm{Rn}(\beta, y) \wedge x \in y\big)$.

If we interpret u as the class of all sets of Ackermann, we can easily show that in T the axioms of A hold (for the axioms of A see Reinhardt [4]).

DEFINITION 1. Let φ be a sentence of A. The u-transform of φ in T, φ_u, is obtained by replacing in φ every part of the form $x \in M$ by $x \in u$.

THEOREM 1. *If* φ *is a theorem of* A *then* φ_u *is a theorem of* T.

Proof. It is enough to show that the u-transforms of the axioms of A are theorems in T.

Trivially we have $(A1)_u$, $(A2)_u$, and $(A3)_u$.

The u-transform of (A4) is

$$(\forall x \in u)\big((\forall t)(\varphi(t) \to t \in u) \to (\exists z)(z \in u \wedge (\forall t)(t \in z \leftrightarrow \varphi(t)))\big) .$$

Taking $x = u$, by (T6) we have

$$(\exists z)\big(z \in u \wedge (\forall t)(\varphi(t) \to t \in z)\big) .$$

Let $w = \{t \mid t \in u \wedge \varphi(t)\}$. $w \subset z$ and $z \in u$, so $w \in u$ and $(\forall t)(\varphi(t) \leftrightarrow t \in w)$.

As for Ackermann set theory, u cannot be defined in T, using the language L, that is, if $\varphi(t, x)$ is a formula of L we can show that

$$\sim (\exists x)\big(x \in u \wedge (\forall t)(t \in u \leftrightarrow \varphi(t, x))\big) .$$

In fact, suppose

$$(\forall t)(t \in u \leftrightarrow \varphi(t, x)) \quad \text{and} \quad x \in u .$$

By (T6),

$$(\exists z)\big(z \in u \wedge \forall t(t \in z \leftrightarrow \varphi(t, x))\big) .$$

By (T1), $z = u$ and then $u \in u$. By (T3),

$$(\exists y)(\forall t)(t \in y \leftrightarrow t \in u \wedge t \notin t) ,$$

and we have $y \subset u$. Then $y \in u$ and we obtain a contradiction, $y \in y \leftrightarrow y \notin y$.

DEFINITION 2. Let $M = (M, U, R)$ be a model of T. We say that $A \subset M$ is definable in (M, B) if and only if there is a formula $\varphi(v, x)$ of L and elements b of B such that for every $t \in M$, $t \in A \leftrightarrow M \models \varphi(t, b)$; that is,

$$A = \{t \in M \mid M \models \varphi(t, b)\} .$$

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THEOREM 2. *If* $\mathrm{M} = (M, U, R)$ *is a model of* T, *then* U *is not definable in* (M, U).

Proof. Suppose U is definable. Then there exists a formula $\varphi(v, \mathrm{x})$ of L and elements b of U such that for every $t \in M$,

$$t \in U \leftrightarrow \mathrm{M} \models \varphi(t, \mathrm{b}) , \tag{6}$$

that is $U = \{t \in M \mid \mathrm{M} \models \varphi(t, \mathrm{b})\}$. Hence,

$$\mathrm{M} \models \mathrm{b} \in u \wedge (\forall t)\big(\varphi(t, \mathrm{b}) \to t \in u\big) .$$

Since $\mathrm{M} \models (\mathrm{T6})$,

$$\mathrm{M} \models \exists z\big(z \in u \wedge \forall t\big(t \in z \leftrightarrow \varphi(t, \mathrm{b})\big)\big) .$$

Hence there is $y \in U$ such that for every $t \in M$,

$$t \in y \leftrightarrow \mathrm{M} \models \varphi(t, \mathrm{b}) .$$

Since $\mathrm{M} \models (\mathrm{T1})$, by (6) we have $y = U$ and $U \in U$.

We construct now in T a syntactic model of $\mathrm{ZF}/\underline{s}$.

DEFINITION 3. Let φ be a sentence of $\mathrm{ZF}/\underline{s}$. The u-transform of φ in T, φ_u, is obtained replacing in φ every part of the form $x \in \underline{s}$ by $x \in u$.

THEOREM 3. *If* φ *is a theorem of* $\mathrm{ZF}/\underline{s}$, *then* φ_u *is a theorem of* T.

We need a lemma.

LEMMA 1.

(a) $\mathrm{Rn}(\alpha, y) \wedge \mathrm{Rn}(\alpha, y') \to y = y'$;

(b) $\mathrm{Rn}(\alpha, y) \to \mathrm{Sc}(y)$;

(c) $\alpha \in u \wedge y \in u \to \big(\mathrm{Rn}(\alpha, y) \leftrightarrow y = R(\alpha)\big)$;

(d) $(\forall \alpha \in u)(\exists y \in u)\mathrm{Rn}(\alpha, y)$;

(e) $(\forall \alpha)(\exists y)\mathrm{Rn}(\alpha, y)$.

All *(a)-(e)* can be proved in A (see Lévy [2] and Lévy and Vaught [3]), so, by Theorem 1, also in T.

Proof of Theorem 3. It is enough to show in T the u-transforms of the axioms of $\mathrm{ZF}/\underline{s}$.

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Trivially we have $(2)_u$, $(3)_u$, and $(4)_u$.

The u-transform of (5) is

$$(\forall x)\left(x \in u \to \left(\varphi^{(u)}(x)\right) \leftrightarrow \varphi(x)\right)\right) .$$

If φ has no quantifiers then $\varphi^{(u)}$ is simply φ.

Assume φ of the form $\exists t\psi$, where ψ is a formula of L with free variables t, x. So we have to show that

$$(\forall x)\left(x \in u \wedge (\exists t)\psi \to (\exists t \in u)\psi\right) .$$

Assume the hypothesis. By (T7) there exists β and z such that $\mathrm{Rn}(\beta, z)$, $t \in z$ and ψ. Also by (T7) there exists α and y such that $\mathrm{Rn}(\alpha, y)$ and $z \in y$. Let $a = \{x \mid \mathrm{Rn}(\alpha, y) \wedge x \in y \wedge \Phi(x)\}$, where $\Phi(x)$ is the formula $(\exists\gamma)\left(\mathrm{Rn}(\gamma, x) \wedge (\exists t \in x)\psi\right)$.

Since $z \in y$, $\mathrm{Rn}(\beta, z)$, and $\Phi(z)$, a is not empty. Therefore, there exists $b \in a$ such that

(i) $(\forall x)(x \in a \to x \notin b)$.

For b we can prove

(ii) $(\forall x)(x \in a \to b \subset x)$.

In fact, since $b \in a$, $b \subset x$ or $x \subset b$. But if $x \subset b$, we have $x \in b$ because $\Phi(x)$, which contradicts (i). b is the set of elements v such that for all x, $\Phi(x)$ implies $v \in x$; that is,

(iii) $\forall v\left(v \in b \leftrightarrow \forall x\left(\Phi(x) \to v \in x\right)\right)$.

In fact, if $\forall x\left(\Phi(x) \to v \in x\right)$, then $v \in b$, since $b \in a$, and then $\Phi(b)$.

Conversely, suppose $v \in b$ and $\Phi(x)$. We have $x \subset z$ or $z \subset x$. If $x \subset z \in y$ then $x \in y$ and $x \in a$. By (i), $b \subset x$ and then $v \in x$. If $z \subset x$, since $z \in a$, by (ii), $b \subset z \subset x$ and then $v \in x$.

Finally we apply (T6) to the formula

$$(\forall x)\left(\Phi(x) \to t \in x\right) .$$

By (iii), $t \in b$, and then

$$(\exists z)\left(z \in u \wedge (\forall t)\left((\forall x)\left(\Phi(x) \to t \in x\right) \to t \in z\right)\right) .$$

By (T4), $b \subset z \subset u$ and $t \in u$.

It remains to show that the axioms of ZF are provable in T. It is enough to show their relativizations.

The extensionality axiom follows from (T1).

The empty set axiom follows from (T2) and (T5).

The unordered pairs axiom: let φ be the formula $t = a \vee t = b$. φ implies $t \in u$. By (T3) there exists y such that $y = \{t \mid \varphi\}$. By (T6) there exists $z \in u$ such that $y \subset z$. Therefore, by (T5), $y = \{a, b\} \in u$.

The union set axiom: let φ be the formula $t \in a \wedge a \in b$. We have $\varphi \to t \in u$ and there exists y such that $y = \{t \mid \varphi\}$. By (T6) there exists $z \in u$ such that $y \subset z$. By (T5), $y = \bigcup b \in u$.

The power set axiom: let φ be the formula $t \subset a$. φ implies $t \in u$ and there exists y such that $y = \{t \mid \varphi\}$. Therefore there exists $z \in u$ such that $y \subset z$. So we have $y = \mathcal{P}a \in u$.

The axiom of infinity: let $\varphi(z)$ be the formula

$$(\forall y)\big(0 \in y \wedge \forall t(t \in y \to t \cup \{t\} \in y) \to z \in y\big) .$$

$\varphi(z)$ implies $z \in u$, since $0 \in u$ and $\forall t(t \in u \to t \cup \{t\} \in u)$. Then there exists $w \in u$ such that $w = \{z \mid \varphi(z)\}$. This w can be easily seen to be as required by the axiom of infinity.

The replacement axiom schema: the relativization of the replacement axioms is as follows, for any formula φ of L with free variables x, y, a, z in all:

$$\forall a, z \in u\Big[\forall x, y_1, y_2 \in u\big[\varphi^{(u)}(x, y_1) \wedge \varphi^{(u)}(x, y_2) \to y_1 = y_2\big]\Big] \to (\exists b \in u)(\forall y \in u)\big(y \in b \leftrightarrow \exists x\big(x \in a \wedge \varphi^{(u)}(x, y)\big)\big) .$$

Let ψ be the formula

$$\big(\forall x, y_1, y_2\big)\big(\varphi(x, y_1) \wedge \varphi(x, y_2) \to y_1 = y_2\big) \to (\exists b)(\forall y)\big(y \in b \leftrightarrow \exists x\big(x \in a \wedge \varphi(x, y)\big)\big) .$$

By $(5)_u$ we have

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(i) $z, x, y \in u \to (\varphi \leftrightarrow \varphi^{(u)})$, and

(ii) $z, x \in u \to (\exists y\varphi \leftrightarrow (\exists y \in u)\varphi^{(u)})$.

Then,

(iii) $z, x \in u \to (\exists y\varphi \leftrightarrow (\exists y \in u)\varphi)$.

Let φ be a function and $b = \{y \mid \exists x \in a\varphi(x, y)\}$. By (iii), $b \in u$ since $a \subset u$ and then $x \in u$. Therefore

$$z, a \in u \to \psi .$$

But, by $(5)_u$, $z, a \in u \to (\psi \leftrightarrow \psi^{(u)})$. Then $z, a \in u \to \psi^{(u)}$ and this completes the proof.

The foundation axiom follows from (T7) and Lemma 1. The proof is now complete.

Consider the theory T' obtained from T by replacing (T3) by

(T3)' $(\forall x)(\forall x)(\exists y)(\forall t)(t \in y \leftrightarrow t \in x \wedge \varphi(t, x))$, for each formula φ of L with free variables t, x .

In ZF/$\underline{s}$ we can construct a syntactic model of T'.

DEFINITION 4. Let φ be a sentence of T'. The $\underline{s}$-transform of φ in ZF/$\underline{s}$, $\varphi_{\underline{s}}$, is obtained replacing in φ every part of the form $x \in u$ by $x \in \underline{s}$.

THEOREM 4. *If* φ *is a theorem of* T', *then* $\varphi_{\underline{s}}$ *is a theorem of* ZF/$\underline{s}$.

Proof. It is enough to show in ZF/$\underline{s}$ the $\underline{s}$-transforms of the axioms of T'.

Evidently we have $(T1)_{\underline{s}}$, $(T2)_{\underline{s}}$, $(T3)'_{\underline{s}}$, $(T4)_{\underline{s}}$, and $(T5)_{\underline{s}}$. We show $(T6)_{\underline{s}}$; that is

$$(\forall x)\big((x \in s \wedge (\exists x)(\forall t)(\varphi(t, x)) \to t \in x)\big) \to (\exists z)\big(z \in s \wedge (\forall t)(\varphi(t, x) \to t \in z)\big) .$$

Assume the hypothesis. Let ψ be the formula

$$(\forall t)(\varphi(x, t) \to t \in x) .$$

By (5),

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$$\mathrm{x} \in \underline{s} \to \big((\exists x)(\forall t)(\varphi \to t \in x) \leftrightarrow (\exists x \in \underline{s})(\forall t \in \underline{s})\big(\varphi^{(\underline{s})} \to t \in x\big)\big) .$$

Hence

$$\text{(i)} \quad (\exists x \in \underline{s})(\forall t \in \underline{s})\big(\varphi^{(\underline{s})} \to t \in x\big) .$$

We also have

$$\text{(ii)} \quad (\forall t \in \underline{s})\big(\varphi^{(\underline{s})} \to t \in x\big) .$$

Since $\mathrm{x} \in \underline{s}$ and $x \in \underline{s}$,

$$\text{(iii)} \quad \psi \leftrightarrow \psi^{(\underline{s})} .$$

Then by (ii) we obtain ψ and this completes the proof.

Finally we show $(\mathrm{T7})_{\underline{s}}$.

Let $\sigma = \{\alpha \mid \alpha \in \underline{s} \wedge \mathrm{Ord}(\alpha)\}$. Then σ is an ordinal and is the least one not in $\underline{s}$. Moreover, σ is a limit ordinal with $\sigma > \omega$, and furthermore we have $\underline{s} = R(\sigma)$.

By (5), $(\mathrm{T7})_{\underline{s}}$ is equivalent to

$$(\mathrm{T7})_{\underline{s}}^{(\underline{s})} \quad \forall x \in \underline{s}\ \exists y \in \underline{s}\ \exists z \in \underline{s}\big(\mathrm{Rn}(y, z) \wedge x \in z\big) ,$$

and this follows from Lemma 1 and the fact that $\underline{s} = R(\sigma)$.

The proof is now complete.

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