



On the greatest common divisor of n and the n th Fibonacci number, II

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Abstract. Let \mathcal{A} be the set of all integers of the form $\gcd(n, F_n)$, where n is a positive integer and F_n denotes the n th Fibonacci number. Leonetti and Sanna proved that \mathcal{A} has natural density equal to zero, and asked for a more precise upper bound. We prove that

$$\#(\mathcal{A} \cap [1, x]) \ll \frac{x \log \log \log x}{\log \log x}$$

for all sufficiently large x . In fact, we prove that a similar bound also holds when the sequence of Fibonacci numbers is replaced by a general nondegenerate Lucas sequence.

1 Introduction

Let (u_n) be a nondegenerate linear recurrence with integral values. Arithmetic relations between n and u_n have been studied by several authors. For example, the set of positive integers n such that n divides u_n has been studied by Alba González, Luca, Pomerance, and Shparlinski [2], assuming that the characteristic polynomial of (u_n) is separable, and by André-Jeannin [3], Luca and Tron [13], Sanna [22], and Somer [28], when (u_n) is a Lucas sequence. Furthermore, Sanna [24] showed that the set of natural numbers n such that $\gcd(n, u_n) = 1$ has a natural density (see [15] for a generalization). Mastrostefano and Sanna [14, 23] studied the moments of $\log(\gcd(n, u_n))$ and $\gcd(n, u_n)$ when (u_n) is a Lucas sequence, and Jha and Nath [9] performed a similar study over shifted primes. Part of the interest in studying $\gcd(n, u_n)$ resides in the fact that this task can be considered a simpler, albeit nontrivial, case of the general problem of studying the greatest common divisor (GCD) of terms of two linear recurrences, a problem that led to the famous Bugeaud–Corvaja–Zannier bound [5] and the difficult Ailon–Rudnick conjecture [1]. (See the survey of Tron [30] for an extensive treatise on GCDs of terms of linear recurrences, especially Section 3 for these considerations on $\gcd(n, u_n)$.) Furthermore, more abstractly, when (u_n) is a Lucas sequence of discriminant Δ_u , we have that $(\gcd(n, u_n))$ is the GCD sequence naturally associated, by Silverman’s correspondence [27], with the algebraic group $\mathbb{G}_a \times \mathbb{G}_m(\sqrt{\Delta_u})$, which is the product of the additive group with a twist of the multiplicative group.

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Let (F_n) be the linear recurrence of Fibonacci numbers, which is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for every positive integer n . Sanna and Tron [26] proved that, for each positive integer k , the set of positive integers n such that $\gcd(n, F_n) = k$ has a natural density, which is given by an infinite series. Kim [11] and Jha [8] obtained formally analogous results in cases of elliptic divisibility sequences and orbits of polynomial maps, respectively. Let \mathcal{A} be the set of numbers of the form $\gcd(n, F_n)$, for some positive integer n . Leonetti and Sanna [12] provided an effective method to enumerate the elements of \mathcal{A} in increasing order. In particular, the first elements of \mathcal{A} are

1, 2, 5, 7, 10, 12, 13, 17, 24, 25, 26, 29, 34, 35, ...

(see [17, A285058] for more terms). Then they proved that

$$(1.1) \quad \#\mathcal{A}(x) \gg \frac{x}{\log x}$$

for all $x \geq 2$. Their approach relied on a result of Cubre and Rouse [6], which in turn follows from Galois theory and the Chebotarev density theorem. Later, Jha and Sanna [10, Proposition 1.4] obtained an elementary proof as an application of related arithmetic problem over shifted primes. Leonetti and Sanna [12] also gave the upper bound $\#\mathcal{A}(x) = o(x)$ as $x \rightarrow +\infty$, and asked for a more precise estimate. We prove the following upper bound on $\#\mathcal{A}(x)$.

Theorem 1.1 *We have*

$$\#\mathcal{A}(x) \ll \frac{x \log \log \log x}{\log \log x}$$

for all sufficiently large x .

In fact, we prove that an upper bound similar to that of Theorem 1.1 also holds when the sequence of Fibonacci numbers is replaced by a general nondegenerate Lucas sequence (see Theorem 3.1).

In light of the gap between the upper bound of Theorem 1.1 and the lower bound (1.1), it is natural to wonder which is the true order of $\#\mathcal{A}(x)$. By performing some numerical experiments (see Section 4), we found that $\#\mathcal{A}(x)$ appears to be asymptotic to $x/(\log x)^c$, as $x \rightarrow +\infty$, for some constant $c \approx 0.63$ (see Figure 1).

1.1 Notation

For every set of positive integers \mathcal{S} and for every $x > 0$, we define $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$. Throughout, we reserve the letter p for prime numbers. We employ the Landau–Bachmann “Big Oh” and “little oh” notation O and o , as well as the associated Vinogradov symbols \ll and \gg . Moreover, we write $f \asymp g$ to denote that $f \ll g$ and $f \gg g$. Any dependence of the implied constants is explicitly stated or indicated with subscripts. In particular, O_u is a shortcut for O_{a_1, a_2} , and similarly for \ll_u , \gg_u , and \asymp_u . We let $\text{Li}(x) := \int_2^x (\log t)^{-1} dt$ denote the integral logarithm.



Figure 1: A plot of $\#A(x)/(x/(\log x)^c)$ for x up to 10^6 .

2 Preliminaries on Lucas sequences

In what follows, let (u_n) be a *Lucas sequences*, that is, a sequence of integers satisfying $u_0 = 0, u_1 = 1$, and $u_n = a_1u_{n-1} + a_2u_{n-2}$ for every integer $n \geq 2$, where a_1, a_2 are fixed nonzero relatively prime integers. Furthermore, assume that (u_n) is nondegenerate, which means that the ratio of the roots of the characteristic polynomial $X^2 - a_1X - a_2$ is not a root of unity. In particular, the discriminant $\Delta_u := a_1^2 + 4a_2$ is nonzero.

For each positive integer m that is relatively prime with a_2 , let $z_u(m)$ be the *rank of appearance* of m in the Lucas sequence, that is, the smallest positive integer n such that m divides u_n . It is well known that $z_u(m)$ exists (see, e.g., [20]). Furthermore, put $\ell_u(m) := \text{lcm}(m, z_u(m))$ and, for each positive integer n , let $g_u(n) := \text{gcd}(n, u_n)$ and $\mathcal{A}_u := \{g_u(n) : n \in \mathbb{N}\}$.

The next lemma collects some elementary properties of z_u, ℓ_u, g_u , and \mathcal{A}_u .

Lemma 2.1 For all positive integers m, n and all prime numbers p , with $p \nmid a_2$, we have:

- (i) $m \mid u_n$ if and only if $\text{gcd}(m, a_2) = 1$ and $z_u(m) \mid n$.
- (ii) $z_u(m) \mid z_u(n)$ whenever $\text{gcd}(mn, a_2) = 1$ and $m \mid n$.
- (iii) $z_u(p) \mid p - (-1)^{p-1}\eta_u(p)$, where

$$\eta_u(p) := \begin{cases} +1, & \text{if } p \nmid \Delta_u \text{ and } \Delta_u \equiv x^2 \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1, & \text{if } p \nmid \Delta_u \text{ and } \Delta_u \not\equiv x^2 \pmod{p} \text{ for all } x \in \mathbb{Z}, \\ 0, & \text{if } p \mid \Delta_u. \end{cases}$$

- (iv) $z_u(p^n) = p^{e_u(p,n)} z_u(p)$, where $e_u(p, n)$ is some nonnegative integer less than n .
- (v) $\ell_u(p^n) = p^n z_u(p)$ if $p \nmid \Delta_u$, and $\ell_u(p^n) = p^n$ if $p \mid \Delta_u$.
- (vi) $g_u(m) \mid g_u(n)$ whenever $m \mid n$.
- (vii) $n \mid g_u(m)$ if and only if $\gcd(n, a_2) = 1$ and $\ell_u(n) \mid m$.
- (viii) $n \in \mathcal{A}_u$ if and only if $\gcd(n, a_2) = 1$ and $n = g_u(\ell_u(n))$.
- (ix) $m \mid n$ whenever $\gcd(mn, a_2) = 1$, $\ell_u(m) \mid \ell_u(n)$, and $n \in \mathcal{A}_u$.

Proof (i)–(iv) are well-known properties of the rank of appearance of a Lucas sequence (see, e.g., [20], [21, Chapter 1], or [22, Section 2]), whereas (v)–(vii) follow easily from (i)–(iv) and from the definitions of ℓ_u and g_u . Let us prove (viii). If $n \in \mathcal{A}_u$, then there exists a positive integer m such that $n = g_u(m)$. In particular, $n \mid g_u(m)$ and, by (vii), we get that $\gcd(n, a_2) = 1$ and $\ell_u(n) \mid m$. Therefore, by (vi), we have that $g_u(\ell_u(n)) \mid n$. Since n divides both $\ell_u(n)$ and $u_{\ell_u(n)}$ (by (i)), it follows that $n \mid g_u(\ell_u(n))$. Hence, $n = g_u(\ell_u(n))$. The other implication is straightforward. Finally, (ix) follows quickly from (vii) and (viii). ■

For each positive integer d , let $\mathcal{P}_{u,d}$ be the set of prime numbers p not dividing a_2 and such that d divides $z_u(p)$. Cubre and Rouse [6] proved an asymptotic formula for $\#\mathcal{P}_d(x)$ in the special case in which (u_n) is the sequence of Fibonacci numbers. Sanna [25] extended this result to Lucas sequences (under some mild restrictions) and also provided an error term. In particular, as a consequence of [25, Theorem 1.1], we have the following asymptotic formula.

Lemma 2.2 *There exists a constant $B_u > 0$ such that for all $x \geq \exp(B_u d^{40})$ and for all odd positive integers d , with $3 \nmid d$ if the square-free part of Δ_u is equal to -3 , we have that*

$$(2.1) \quad \#\mathcal{P}_d(x) = \delta_u(d) \operatorname{Li}(x) + O_u\left(\frac{x}{(\log x)^{12/11}}\right),$$

where $\delta_u(d)$ is a quantity satisfying $\delta_u(d) \asymp_u 1/d$.

Proof If Δ_u is not a square, then, from [25, Theorem 1.1], we have that there exists a constant $B_u > 0$ such that

$$(2.2) \quad \#\mathcal{P}_d(x) = \delta_u(d) \operatorname{Li}(x) + O_u\left(\frac{d}{\varphi(d)} \cdot \frac{x (\log \log x)^{\omega(d)}}{(\log x)^{9/8}}\right),$$

for all odd positive integers d , with $3 \nmid d$ if the square-free part of Δ_u is equal to -3 , and for all $x \geq \exp(B_u d^{40})$, where $\delta_u(d) \asymp_u 1/d$, whereas $\varphi(d)$ and $\omega(d)$ are the Euler totient function and the number of prime factors of d , respectively.

If Δ_u is a square, then, by the Binet formula, we have that

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for every positive integer n , where $\alpha := (a_1 - \sqrt{\Delta_u})/2$ and $\beta := (a_1 + \sqrt{\Delta_u})/2$ are integers. Consequently, for every prime number p not dividing $a_2 \Delta_u$, we have that $z_u(p)$ is equal to the multiplicative order of α/β modulo p . Then (2.2) follows from a

result of Moree [16, Lemma 1]. (Moree did not make explicit the factor $d/\varphi(d)$, but this can be easily done [cf. [25, Lemma 6.3]].)

Note that we can assume that B_u (and consequently x) is sufficiently large. In particular, we have that $d \leq (\log x)^{1/40}$. Put $\varepsilon := 1/330$. By the classic lower bound for $\varphi(d)$ (see, e.g., [29, Chapter I.5, Theorem 5.6]), we have that

$$\frac{d}{\varphi(d)} \ll \log \log d \ll \log \log \log x \leq (\log x)^\varepsilon.$$

Recall that $\omega(d) \leq (1 + o(1)) \log d / \log \log d$ as $d \rightarrow +\infty$ (see, e.g., [29, Chapter I.5, Theorem 5.5]). Therefore, there exists an absolute constant $C > 0$ such that if $d > C$, then

$$\omega(d) \leq (1 + \varepsilon) \frac{\log d}{\log \log d} \leq \left(\frac{1}{40} + 2\varepsilon\right) \frac{\log \log x}{\log \log \log x},$$

and consequently $(\log \log x)^{\omega(d)} \leq (\log x)^{\frac{1}{40} + 2\varepsilon}$. Furthermore, if $d \leq C$, then

$$(\log \log x)^{\omega(d)} \leq (\log x)^\varepsilon.$$

Thus, (2.1) follows. ■

Remark 2.3 In Lemma 2.2, the exponent $12/11$ can be replaced by $11/10 + \varepsilon$, for every $\varepsilon > 0$, assuming that x is sufficiently large depending on ε .

We also need an upper bound for $\#\mathcal{P}_{u,d}(x)$.

Lemma 2.4 We have

$$\#\mathcal{P}_{u,d}(x) \ll_u \frac{x}{\varphi(d) \log(x/d)}$$

for all positive integers d and for all $x > d$.

Proof By Lemma 2.1(iii), we have that

$$\#\mathcal{P}_{u,d}(x) \leq O_u(1) + \#\{p \leq x : p \equiv \pm 1 \pmod{d}\} \ll_u \frac{x}{\varphi(d) \log(x/d)},$$

where we applied the Brun–Titchmarsh inequality [29, Chapter I.4, Theorem 4.16]. ■

Now, we give an upper bound for the sum of reciprocals of primes in $\mathcal{P}_{u,d}$.

Lemma 2.5 We have

$$\sum_{p \in \mathcal{P}_{u,d}(x)} \frac{1}{p} = \delta_u(d) \log \log x + O_u\left(\frac{\log(2d)}{\varphi(d)}\right)$$

for all odd positive integers d , with $3 \nmid d$ if the square-free part of Δ_u is equal to -3 , and for all $x \geq 3$.

Proof First, suppose that $x < \exp(B_u d^{40})$, where B_u is the constant of Lemma 2.2. Hence, we have that

$$\delta_u(d) \log \log x \ll_u \frac{\log \log x}{d} \ll_u \frac{\log(2d)}{d}.$$

Moreover, by [19, Theorem 1 and Remark 1], we have that

$$\sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} = \frac{2 \log \log x}{\varphi(d)} + O\left(\frac{\log(2d)}{\varphi(d)}\right).$$

This together with Lemma 2.1(iii) yields that

$$\sum_{p \in \mathcal{P}_{u,d}(x)} \frac{1}{p} \leq \frac{O_u(1)}{d} + \sum_{\substack{p \leq x \\ p \equiv \pm 1 \pmod{d}}} \frac{1}{p} \ll_u \frac{1}{d} + \frac{\log(2d)}{\varphi(d)}.$$

Hence, the claim follows. Now, suppose that $x \geq \exp(B_u d^{40})$. By partial summation, we have that

$$\sum_{p \in \mathcal{P}_{u,d}(x)} \frac{1}{p} = \frac{\#\mathcal{P}_{u,d}(x)}{x} + \int_1^x \frac{\#\mathcal{P}_{u,d}(t)}{t^2} dt.$$

From Lemma 2.1(iii), we get easily that $\#\mathcal{P}_{u,d}(x)/x \ll_u 1/d$. Thus, it remains to bound the integral. By Lemma 2.1(iii) again, we have that $\#\mathcal{P}_{u,d}(t) > 0$ only if $t \geq d-1$, and that $\#\mathcal{P}_{u,d}(t) \ll_u 1$ for $t \in [1, 2d]$. Hence, we have

$$\int_1^{2d} \frac{\#\mathcal{P}_{u,d}(t)}{t^2} dt \ll_u \int_{d-1}^{2d} \frac{dt}{t^2} \ll \frac{1}{d}.$$

By Lemma 2.4, we have that

$$\begin{aligned} \int_{2d}^{\exp(B_u d^{40})} \frac{\#\mathcal{P}_{u,d}(t)}{t^2} dt &\ll_u \int_{2d}^{\exp(B_u d^{40})} \frac{dt}{\varphi(d) t \log(t/d)} \\ &= \left[\frac{\log \log(t/d)}{\varphi(d)} \right]_{t=2d}^{\exp(B_u d^{40})} \ll \frac{\log(2d)}{\varphi(d)}. \end{aligned}$$

By Lemma 2.2, we have that

$$\begin{aligned} \int_{\exp(B_u d^{40})}^x \frac{\#\mathcal{P}_{u,d}(t)}{t^2} dt &= \int_{\exp(B_u d^{40})}^x \frac{\delta_u(d) \text{Li}(t)}{t^2} dt + O_u\left(\int_{\exp(B_u d^{40})}^x \frac{dt}{t(\log t)^{12/11}}\right) \\ &= \delta_u(d) \left[\log \log t - \frac{\text{Li}(t)}{t} \right]_{t=\exp(B_u d^{40})}^x + O_u\left(\frac{1}{d^{40/11}}\right) \\ &= \delta_u(d) (\log \log x + O(\log(2d))) + O_u\left(\frac{1}{d^{40/11}}\right) \\ &= \delta_u(d) \log \log x + O_u\left(\frac{\log(2d)}{d}\right). \end{aligned}$$

Putting these together, the claim follows. ■

The following sieve result is a special case of [7, Theorem 7.2] (cf. [18, Lemma 2.2]).

Lemma 2.6 We have

$$\#\{n \leq x : p \mid n \Rightarrow p \notin \mathcal{P}\} \ll x \prod_{p \in \mathcal{P}(x)} \left(1 - \frac{1}{p}\right),$$

for all $x \geq 2$ and for all sets of prime numbers \mathcal{P} .

We also need the so-called *Primitive Divisor Theorem* for Lucas sequence [4].

Theorem 2.7 For every integer $r \geq 31$, there exists a prime number p such that $z_u(p) = r$.

3 Proof of Theorem 1.1

We shall prove the following more general result, of which Theorem 1.1 is a particular case.

Theorem 3.1 We have

$$\#\mathcal{A}_u(x) \ll_u \frac{x \log \log \log x}{\log \log x}$$

for all sufficiently large x .

Proof Let $r := 30 \prod_{p \mid 6\Delta_u} z_u(p) + 1$. Since $r \geq 31$, it follows from Theorem 2.7 that there exists a prime number q such that $z_u(q) = r$. Furthermore, by Lemma 2.1(i), we have that $\gcd(6\Delta_u, u_r) = 1$ and so $q \nmid 6\Delta_u$. Note also that $\gcd(6, r) = 1$.

From Lemma 2.2, we know that $\delta_u(dr) \geq C_u/d$ for every positive integer d with $\gcd(6, d) = 1$, where $C_u > 0$ is a constant depending only on a_1, a_2 . (Note that r is completely determined by a_1, a_2 .) Suppose that $x > 0$ is sufficiently large, and put

$$k := \left\lceil \frac{1}{\log q} \log \left(\min(1, C_u) \frac{\log \log x}{\log \log \log x} \right) \right\rceil$$

and $d := q^k$. Hence, we get that

$$d \leq \min(1, C_u) \frac{\log \log x}{\log \log \log x}$$

and so

$$\frac{\log d}{\delta_u(dr)} \leq \frac{d}{C_u} \log d \leq \frac{\log \log x}{\log \log \log x} \log d \leq \log \log x.$$

Therefore, we have that

$$(3.1) \quad (\log x)^{\delta_u(dr)} \geq d \gg_u \frac{\log \log x}{\log \log \log x}.$$

We split \mathcal{A}_u into two subsets: \mathcal{A}'_u is the subset of \mathcal{A}_u consisting of integers without prime factors in $\mathcal{P}_{u, dr}$, and $\mathcal{A}''_u := \mathcal{A}_u \setminus \mathcal{A}'_u$.

First, we give an upper bound on $\#\mathcal{A}'_u(x)$. Note that $\gcd(6, dr) = 1$. By Lemmas 2.5 and 2.6, we get that

$$(3.2) \quad \#\mathcal{A}'_u(x) \ll x \prod_{p \in \mathcal{P}_{u, dr}(x)} \left(1 - \frac{1}{p}\right) \leq x \exp\left(-\sum_{p \in \mathcal{P}_{u, dr}(x)} \frac{1}{p}\right) \ll_u \frac{x}{(\log x)^{\delta_u(dr)}},$$

where we also used the inequality $1 - x \leq \exp(-x)$, which holds for $x \geq 0$.

Now, we give an upper bound on $\#\mathcal{A}''_u(x)$. If $n \in \mathcal{A}''_u$, then n has a prime factor $p \in \mathcal{P}_{u, dr}$. Hence, we have that $p \mid n$ and $dr \mid z_u(p)$. Thus, by Lemma 2.1(ii), we get that $z_u(p) \mid z_u(n)$ and so $dr \mid \ell_u(n)$. Recalling that $d = q^k$, $z_u(q) = r$, and $q \mid \Delta_u$, by Lemma 2.1(v), we have that $\ell_u(d) = dr$. Hence, we get that $\ell_u(d) \mid \ell_u(n)$ and, by Lemma 2.1(ix), it follows that $d \mid n$. Thus, all the elements of \mathcal{A}''_u are multiples of d . Consequently, we have that

$$(3.3) \quad \#\mathcal{A}''_u(x) \leq \frac{x}{d}.$$

Therefore, putting together (3.2) and (3.3), and using (3.1), we obtain that

$$\#\mathcal{A}_u(x) = \#\mathcal{A}'_u(x) + \#\mathcal{A}''_u(x) \ll_u \frac{x}{(\log x)^{\delta_u(dr)}} + \frac{x}{d} \ll_u \frac{x \log \log \log x}{\log \log x},$$

as desired. The proof is complete. \blacksquare

4 Numerical computations

We computed the elements of $\mathcal{A} \cap [1, 10^6]$ by using a program written in C that employs Lemma 2.1(viii). Note that computing $g(\ell(n))$ directly as $\gcd(\ell(n), F_{\ell(n)})$ would be prohibitive, in light of the exponential growth of Fibonacci numbers. Instead, we used the fact that

$$g(\ell(n)) = \gcd(\ell(n), F_{\ell(n)} \bmod \ell(n)),$$

and we computed Fibonacci numbers modulo an integer by efficient matrix exponentiation.

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