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108.03 Remarks on perfect powers

A *perfect power* is a number of the form k^n, where $k \geqslant 1$ and $n \geqslant 2$ are integers; and we say that k^n is a perfect n-th power. Now, consider the first few perfect powers:

1	4	8	9	16	25	27	32	36	49	64	81	100	…
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	
1	2^2	2^3	3^2	$2^4 = 4^2$	5^2	3^3	2^5	6^2	7^2	$2^6=4^3=8^2$	$3^4 = 9^2$	10^2	…

Then we observe that seemingly between any two consecutive perfect powers of the same exponent there exists at least one perfect power of lower exponent; also there exist at least two squares between any two successive cubes, and there exist at least two cubes between any two successive quartics. The purpose of this note is to prove such simple observations as the theorem below. Their proofs are completely elementary and straightforward from the following two facts.

1. For any real x and y with $x - y \geqslant k$ for some positive integer k, there exist at least k integers in the interval $[x, y]$.
2. Bernoulli's Inequality: If $x \geqslant 0$ and $r \geqslant 1$, then $(1 + x)^r \geqslant 1 + rx$.

Theorem

Let m and n be positive integers with $m < n$.

(i) There is at least one perfect m-th power between any two perfect n-th powers.

(ii) For $n = 3, 4$, there exist at least two perfect m-th powers between any two perfect n-th powers. But this does not always hold when $n > 4$.

(iii) Given a positive integer k, then there exists an integer $a_0 = a_0(k, m, n)$ such that for any integer $a > a_0$, there exist at least k perfect m-th powers between a^n and $(a + 1)^n$.

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Proof

(i) We should prove that for any positive integers m, n, a with $m < n$, there exists at least one integer b such that $a^n \leqslant b^m \leqslant (a + 1)^n$, or $a^{n/m} \leqslant b \leqslant (a + 1)^{n/m}$; for this, putting $r = n/m > 1$, it is sufficient to show that $(a + 1)^r - a^r \geqslant 1$. Let $f(x) = (x + 1)^r - x^r$. Then $f'(x) = r\left((x + 1)^{r-1} - x^{r-1}\right) > 0$ for any $x \geqslant 0$; i.e. f is strictly increasing, and clearly we get $f(a) \geqslant f(0) = 1$ for any integer $a \geqslant 0$.

(ii) Let $x^3 \leqslant a^2 < b^2 \leqslant (x + 1)^3$ or equivalently $x^{3/2} \leqslant a < b \leqslant (x + 1)^{3/2}$. Then it is sufficient that we prove $(x + 1)^{3/2} - x^{3/2} \geqslant 2$. Clearly, $1^2, 2^2 \in [1^3, 2^3]$; hence let $x \geqslant 2$, and we have

$$x \geqslant 2 > \frac{16}{9} = \left(\frac{4}{3}\right)^2 \qquad \text{and} \qquad x > 1$$

thus

$$3x^2 > 4x^{3/2} \qquad \text{and} \qquad 3x > 3$$

and consequently

$$\begin{aligned} & 3x^2 + 3x > 4x^{3/2} + 3 \\ \Leftrightarrow\ & x^3 + 3x^2 + 3x + 1 > x^3 + 4x^{3/2} + 4 \\ \Leftrightarrow\ & (x + 1)^3 > \left(x^{3/2} + 2\right)^2 \\ \Leftrightarrow\ & (x + 1)^{3/2} > x^{3/2} + 2 \\ \Leftrightarrow\ & (x + 1)^{3/2} - x^{3/2} > 2. \end{aligned}$$

Next, let $x^4 \leqslant a^3 < b^3 \leqslant (x + 1)^4$ or equivalently $x^{4/3} \leqslant a < b \leqslant (x + 1)^{4/3}$. Then it is sufficient that we prove $(x + 1)^{4/3} - x^{4/3} \geqslant 2$. Clearly, $1^3, 2^3 \in [1^4, 2^4]$; $3^3, 4^3 \in [2^4, 3^4]$ and $5^3, 6^3 \in [3^4, 4^4]$.

Hence let $x \geqslant 4$, and we have

$$x \geqslant 4 > \frac{27}{8} = \left(\frac{3}{2}\right)^3 \qquad \text{and} \qquad \begin{cases} 6x^2 > 12x^{4/3} \\ 4x > 7 \end{cases}$$

thus

$$4x^3 > 6x^{8/3} \qquad \text{and} \qquad 6x^2 + 4x > 12x^{4/3} + 7$$

and consequently

$$\begin{aligned} & 4x^3 + 6x^2 + 4x > 6x^{8/3} + 12x^{4/3} + 7 \\ \Leftrightarrow\ & x^4 + 4x^3 + 6x^2 + 4x + 1 > x^4 + 6x^{8/3} + 12x^{4/3} + 8 \\ \Leftrightarrow\ & (x + 1)^4 > \left(x^{4/3} + 2\right)^3 \\ \Leftrightarrow\ & (x + 1)^{4/3} > x^{4/3} + 2 \\ \Leftrightarrow\ & (x + 1)^{4/3} - x^{4/3} > 2. \end{aligned}$$

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For quintic numbers (perfect fifth powers), we have the first few

1	32	243	1024	3125	7776	16807	32768	…
↓	↓	↓	↓	↓	↓	↓	↓	
1^5	2^5	3^5	4^5	5^5	6^5	7^5	8^5	…

but clearly in the first steps, we observe that $2^4 < 2^5 < 3^4 < 3^5 < 4^4$. Actually, we have for any integer $n \geqslant 5$,

$$2^{n-1} < 2^n < 3^{n-1} < 3^n < 4^{n-1}$$

since easily and inductively we can show that

$$2 < \left(1 + \frac{1}{2}\right)^{n-1} \qquad \text{and} \qquad 3 < \left(1 + \frac{1}{3}\right)^{n-1}$$

for any integer $n \geqslant 3$ and for any integer $n \geqslant 5$, respectively, and these inequalities yield $2^n < 3^{n-1}$ and $3^n < 4^{n-1}$; thus in $[2^n, 3^n]$ there is only one perfect $(n - 1)$-th power which is 3^{n-1}, for any integer $n \geqslant 5$.

(iii) By Bernoulli's Inequality, for $a \geqslant 1$ and $r = \frac{n}{m} > 1$, we have

$$(a + 1)^r - a^r = a^r\left(\left(1 + \frac{1}{a}\right)^r - 1\right) \geqslant a^r\left(1 + \frac{r}{a} - 1\right) = ra^{r-1}$$

and now for $a \geqslant \left(\frac{k}{r}\right)^{1/(r-1)}$, we get $(a + 1)^r - a^r \geqslant k$.

Therefore, put $a_0 = a_0(k, m, n) = \left(\frac{km}{n}\right)^{m/(n-m)}$. This completes the proof.

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108.04 Digital root analysis of Smith numbers

Introduction

A composite integer N whose digit sum $S(N)$ is equal to the sum of the digits of its prime factors $S_p(N)$ is called *a Smith number*. For example 636 is a Smith number because the digit sum of 636 i.e. $S(636) = 6 + 3 + 6 = 15$, which is equal to the sum of the digits of its prime factors i.e. $S_p(636) = S_p(2 \times 2 \times 3 \times 53) = 2 + 2 + 3 + 5 + 3 = 15$.

Albert Wilansky [1] named Smith numbers from his brother-in-law Harold Smith's telephone number 4937775 with this property i.e. $4937775 = 3 \times 5 \times 5 \times 65837$, since

$$4 + 9 + 3 + 7 + 7 + 7 + 5 = 3 + 5 + 5 + (6 + 5 + 8 + 3 + 7) = 42.$$