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THE FIXED POINT PROBLEM FOR GENERALISED NONEXPANSIVE MAPS

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This paper is concerned with extending the theory of the existence of fixed points for generalised nonexpansive maps as far as possible. This can be seen as a continuation of the work of Maurey on the extension of the fixed point theory for nonexpansive maps beyond the requirement of normal structure type conditions.

1. INTRODUCTION

A (real) Banach space X is said to have the weak fixed point property (w–fpp) if, when C is a nonempty weak compact convex subset of X that is self mapped by a nonexpansive map T, then T has a fixed point. If X is a dual space we can define the weak star fixed point property (w*–fpp) by requiring the set C of the above definition to be weak star compact. In [**1**] Alspach showed that not every Banach space enjoys the w–fpp and it is well known that failure of the w*–fpp occurs. In [**11**] the notion of a generalised nonexpansive mapping was introduced. We study fixed point problems for this class of maps, and introduce the w–fpp (w*–fpp) for generalised nonexpansive maps which is defined in the obvious way. For basic material on nonexpansive mappings we refer the reader to [**10**].

Section 2 is concerned with some metric properties of generalised nonexpansive maps, further refining the class meriting investigation.

In Section 3 we extend some of the basic techniques from the nonexpansive to the generalised nonexpansive case. Included are results on approximate fixed point sequences that will be needed in the sequel.

In Section 4 we give fixed point theorems for generalised nonexpansive mappings. These results go beyond the requirement of normal structure criteria and generalise results from the nonexpansive case.

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2. Metric properties of generalised nonexpansive mappings

Suppose that C is a nonempty metric space with metric d. Then a self map T of C is said to be *generalised nonexpansive if there exist nonnegative* a_i, $i = 1, 2, \ldots, 5$, so that $\sum a_i \leqslant 1$ and, for, $x, y \in C$,

$$d(Tx, Ty) \leqslant a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

Since the distance function is symmetric we can replace a_2, a_3 with $(a_2 + a_3)/2$ and a_4, a_5 with $(a_4 + a_5)/2$. Thus the above definition is equivalent to the existence of nonnegative a, b, c satisfying $a + 2b + 2c \leqslant 1$ with

$$d(Tx, Ty) \leqslant a d(x, y) + b(d(x, Tx) + d(y, Ty)) + c(d(x, Ty) + d(y, Tx))$$

for $x, y \in C$. If $b = c = 0$ then T is of course nonexpansive.

We now give inequalities that will be useful. Recall that an approximate fixed point sequence (afps) for a mapping T is a sequence (x_n) satisfying $d(Tx_n, x_n) \to 0$.

Proposition 2.1.

(1) *If (x_n) is an afps for T and (y_n) is a sequence in C then (assuming that the limit supremums exist),*

$$\limsup d(Ty_n, x_n) \leqslant \frac{a + b + c}{1 - b - c} \limsup d(y_n, x_n).$$

(2) *If $a + 2c < 1$ and $x, y \in C$ then*

$$d(x, y) \leqslant \frac{1 + b + c}{1 - a - 2c}(d(Tx, x) + d(Ty, y)).$$

Proof:

(1)

$$\begin{aligned} \limsup d(Ty_n, x_n) &= \limsup d(Ty_n, Tx_n) \\ &\leqslant \limsup (a\, d(y_n, x_n) + b\, d(Ty_n, y_n)) \\ &\quad + c(d(Ty_n, x_n) + d(Tx_n, y_n)) \\ &\leqslant (a + b + c) \limsup d(x_n, y_n) + (b + c) \limsup d(Ty_n, x_n). \end{aligned}$$

Thus

$$\limsup d(Ty_n, x_n) \leqslant \frac{a + b + c}{1 - b - c} \limsup d(y_n, x_n)$$

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as required.

(2)
$$\begin{aligned} d(x,y) &\leqslant d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\ &\leqslant a\,d(x,y) + (1+b)(d(Tx,x) + d(Ty,y)) + c(d(Tx,y) + d(Ty,x)) \\ &\leqslant a\,d(x,y) + (1+b)(d(Tx,x) + d(Ty,y)) \\ &\quad + c(d(Tx,x) + 2d(x,y) + d(Ty,y)). \end{aligned}$$

Thus
$$d(x,y) \leqslant \frac{1+b+c}{1-a-2c}(d(Tx,x) + d(Ty,y))$$
as required. ∎

If X is a Banach space $d(x,y) = \|x-y\|$. In [**2**] it is noted that if $b > 0$ then the w–fpp for the associated generalised nonexpansive maps has been completely determined. Indeed, if $0 < b < 1/2$ then fixed points exist and if $b = 1/2$ then the w–fpp for such maps is equivalent to close–to–weak normal structure (see [**23**] for definition). Thus we shall be concerned with mappings for which $b = 0$. These mappings are of two types: the nonexpansive maps, and those for which $b = 0$, $c > 0$ (which we call mappings of type (c)).

It is well known that if C is a nonempty bounded convex subset of a Banach space X that is selfmapped by a nonexpansive T then T has an afps. Bae [**2**] established that if C is a bounded metric space, T a generalised nonexpansive self map of C with $b = 0$, $c > 0$, then T is asymptotically regular. That is, for any $x \in C$, $d\big(T^{n+1}x, T^n x\big) \to 0$, that is, any orbit is an afp.

We note that in [**2**] it was also shown that if T satisfies the generalised nonexpansive condition with $c = 1/2$ then, for all $x \in C$,
$$d\big(T^{n+1}x, T^n x\big) \leqslant \frac{1}{2}\frac{3}{4}\cdots\frac{2n-1}{2n}\operatorname{diam} C.$$

Thus T is uniformly asymptotically regular (the sequence of terms from the left hand side of the above inequality converges to 0 uniformly on C). If $\mathcal{F}$ denotes the collection of all self mappings of C satisfying the generalised nonexpansive inequality with $c = 1/2$ then, since the right hand side of the above inequality does not depend on T, the uniformity is over $\mathcal{F}$ and C. In fact, a similar result is true for any fixed $c \in (0, 1/2]$. One uses the nonuniform result and a technique used on page 100 of [**10**].

3. Further properties

Suppose that C is a nonempty weak (weak star) compact convex subset of a (dual) Banach space X that is self mapped by T. A subset K of C is said to be a minimal invariant if it is minimal with respect to set inclusion in the class of nonempty weak (weak star) compact convex T invariant subsets of C. Such minimal sets exist courtesy of Zorn's Lemma and the compactness condition. In the sequel we use the terms w–minimal invariant or w*–minimal invariant when any ambiguity could arise and may further specify the type of mapping under consideration.

It is obvious that X (X^*) has the w–fpp (w*–fpp) for generalised nonexpansive maps if and only if every (generalised nonexpansive) minimal invariant is a singleton. Thus, if X fails the w–fpp (w*–fpp) for generalised nonexpansive maps we can assume that it contains a minimal invariant K (associated with a generalised nonexpansive mapping) of positive diameter. By translation and dilation we can also assume for any particular $x \in X$ that $x \in K$ and $\operatorname{diam} K = 1$. Also $K = \overline{\mathrm{co}}\, T(K)$ ($K = \overline{\mathrm{co}}^{w^*} T(K)$). The following proposition has been useful in the nonexpansive theory. Recall that a real valued map f from a topological space is lower semicontinuous if $f^{-1}(r, \infty)$ is open for any $r \in \mathbb{R}$. It is called weak (weak star) lower semicontinuous if the space is a Banach space with the weak (weak star) topology. We shall find the proposition useful for generalised nonexpansive mappings.

Proposition 3.1. *Suppose that K is a minimal invariant and $f : K \to \mathbb{R}$ is weak (weak star) lower semicontinuous and convex with $f(Tx) \leqslant f(x)$ for any $x \in K$. Then f is constant on K.*

Proof: Let $\alpha := \min\{f(x) : x \in K\}$. α exists because of the weak (weak star) compactness of K and the weak (weak star) lower semicontinuity of f. Let $M = \{x \in K : f(x) = \alpha\}$. M is then weak (weak star) compact and also convex due to the convexity of f. Since $f(Tx) \leqslant f(x)$ for any $x \in K$, M is also T invariant. Thus $M = K$ by minimality of K, giving f constant. ∎

The following proposition is due to Bogin [4]. Our proof is an adaptation of one for the analogous nonexpansive result and uses a similar proof technique to that used in Proposition 3.1. We recall that if D is a nonempty closed bounded convex subset of a Banach space X then D is said to be diametral if $\operatorname{rad} D = \operatorname{diam} D$. That is, for all $x \in D$, $\sup\{\|x - y\| : y \in D\} = \operatorname{diam} D$.

Proposition 3.2. *Suppose K is a w–minimal invariant associated with a generalised nonexpansive mapping T satisfying $b = 0$. Then K is diametral.*

Proof: Consider the function $f : K \to \mathbb{R}$, $f(x) = \sup\{\|x - y\| : y \in K\}$. f is weak lower semicontinuous. Put $\alpha := \min\{f(x) : x \in K\}$ and $M := \{x \in K : f(x) =$

$\alpha\}$. Then M is T invariant. Indeed, if $x \in M$ then

$$\begin{aligned} f(Tx) &= \sup_{y \in K} \|Tx - y\| \\ &= \sup_{y \in K} \|Tx - Ty\| \ (\text{ since } K = \overline{\text{co}}\, T(K)) \\ &\leqslant \sup_{y \in K} \{a \|x - y\| + c \|x - Ty\| + c \|y - Tx\|\} \\ &\leqslant a\alpha + c\alpha + c \sup_{y \in K} \|y - Tx\| \\ &= \alpha(a + c) + cf(Tx). \end{aligned}$$

Thus $f(Tx) \leqslant \alpha(a+c)/(1-c) \leqslant \alpha$ since $a + 2c \leqslant 1$. Thus by the minimality of K we get that K is diametral. ∎

The above proposition has a w* version. Indeed, in this case the analogous f is the supremum of w* lower semicontinuous functions and is thus also w* lower semicontinuous. Also, $\sup\{\|Tx - y\| : y \in K\} = \sup\{\|Tx - Ty\| : y \in K\}$ since if β is the second quantity then $K \subseteq B_\beta(Tx)$ by the fact that the last set contains $T(K)$, is convex and w* closed.

A nonempty bounded subset of X is said to have normal structure if its only diametral subsets are singletons. X is said to have normal structure (weak normal structure) if any nonempty (weak compact) bounded subset of X has normal structure. Weak star normal structure is defined in a similar way for dual spaces. Together with the stated results in Section 2, Proposition 3.2 (and the remark after it) gives that (dual) Banach spaces with weak (weak star) normal structure have the weak (weak star) fixed point property for generalised nonexpansive maps.

The following is a variation of the Lemma of Karlovitz [**9, 14**].

Lemma 3.3. *Suppose that K is a w–minimal invariant for a generalised nonexpansive map T with $b = 0$. Then if (x_n) is an afps, $\|x_n - x\| \to \operatorname{diam} K$ for any $x \in K$. That is, (x_n) is a diameterising sequence for K.*

Proof: Define $f(x) := \limsup \|x - x_n\|$ for $x \in K$. Since f is continuous and convex it is weak lower semicontinuous. Also, if $x \in K$ then, by Proposition 2.1,

$$\begin{aligned} f(Tx) = \limsup \|Tx - x_n\| &\leqslant \limsup \|x - x_n\| \\ &= f(x). \end{aligned}$$

Thus by Proposition 3.1 f is constant on K, with value k say. Let (x_{n_i}) be a subsequence of (x_n) with $x_{n_i} \xrightarrow{w} x_0$ for some $x_0 \in K$. Then for any $x \in K$

$$k \geqslant \limsup_i \|x - x_{n_i}\| \geqslant \liminf_i \|x - x_{n_i}\| \geqslant \|x - x_0\|$$

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by weak lower semicontinuity of the norm. Thus

$$k \geqslant \operatorname{rad}(x_0, K) = \operatorname{diam} K$$

by Proposition 3.2, giving $k = \operatorname{diam} K$. Since any subsequence of (x_n) is also an afps we can repeat the above argument if necessary to deduce the required result. ∎

The following proposition is a variation of a result of Maurey [**17**]. The proof we use is an adaptation of the proof used in [**5**] for the nonexpansive case.

Proposition 3.4. *Suppose that C is a closed bounded convex subset of a Banach space X. Then, if $T : C \to C$ is a generalised nonexpansive mapping, (x_n) and (y_n) two afps's for T and $\lambda \in [0,1]$, there exists an afps (z_n) so that*

$$\limsup \|z_n - x_n\| = (1 - \lambda) \limsup \|x_n - y_n\|$$

and

$$\limsup \|z_n - y_n\| = \lambda \limsup \|x_n - y_n\|.$$

Also, if $\lim \|x_n - y_n\|$ exists then we can rewrite the above using $\lim$ instead of $\limsup$.

Proof: Write $l = \limsup \|x_n - y_n\|$. If $b > 0$ then, by the second inequality of Proposition 2.1, $l = 0$ and the first inequality implies that $z_n := \lambda x_n + (1 - \lambda) y_n$ suffices. Otherwise put

$$\widetilde{C} := \{(z_n) \in \ell_\infty(X) : z_n \in C\}.$$

Define $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ by

$$\widetilde{T}(z_n) = (Tz_n)$$

for $(z_n) \in \widetilde{C}$. Since T was generalised nonexpansive with $c > 0$, $\widetilde{T}$ has the same properties. Now define D by

$$D := \{(x_n) \in \widetilde{C} : \limsup \|z_n - x_n\| \leqslant (1 - \lambda) l, \ \limsup \|z_n - y_n\| \leqslant \lambda l\}.$$

Then D is nonempty, closed, convex and bounded. Since (x_n) and (y_n) are afps's D is also $\widetilde{T}$ invariant by Propoisition 2.1. Thus D contains an afps by Section 2. That is, for all $p \in \mathbb{N}$ there exists $(z_n^p)_{n=1}^\infty \in D$ so that $\sup \|Tz_n^p - z_n^p\| \leqslant 1/p$ for all n.

Choose $\varepsilon_n > 0$ so that $\varepsilon_n \to 0$. Then for any p there exists $n_p \in \mathbb{N}$ so that for $n \geqslant n_p$

$$\|z_n^p - x_n\| \leqslant (1 - \lambda) l + \varepsilon_p,$$

$$\|z_n^p - y_n\| \leqslant \lambda l + \varepsilon_p$$

and

$$\|Tz_n^p - z_n\| \leqslant 1/p.$$

We can also assume that $(n_p)_{p=1}^{\infty}$ is a strictly increasing sequence. Now define $(z_n) \in \widetilde{C}$ as follows.

For $n < n_1$ put $z_n = z_n^1$. If $n \in [n_p, n_{p+1})$, put $z_n = z_n^p$.

Then it is clear that $(z_n) \in D$ and is an afps.

To obtain the main statement of the proposition we note that, for $(z_n) \in D$, the inequalities used in the definition of D are actually equalities.

Now suppose that $\lim \|x_n - y_n\|$ exists. Then we can rewrite D by replacing $\limsup$ by $\lim$. Indeed, suppose that $(z_n) \in D$ and that $\liminf \|z_n - x_n\| < (1-\lambda)l$. Then

$$\begin{aligned}\liminf \|x_n - y_n\| &\leqslant \liminf \|x_n - z_n\| + \limsup \|y_n - z_n\| \\ &< (1-\lambda)l + \lambda l \\ &= l,\end{aligned}$$

a contradiction, giving that $\lim \|z_n - x_n\|$ exists. Likewise it can also be shown that $\lim \|z_n - y_n\|$ exists. ∎

The above proposition can be generalised to the following proposition, similar to a result from [7].

Proposition 3.5. *Suppose that C and T are as in the above proposition. With $m > 1$ suppose also that we are given m afps's, written (x_n^i) for $i = 1, 2, \ldots, m$. Then there exists an afps (z_n) of T so that for every i,*

$$\limsup_n \left\|z_n - x_n^i\right\| \leqslant ((m-1)/m)\operatorname{diam} C.$$

Proof: If $b > 0$ then $z_n := (1/m)\sum_i x_n^i$ suffices. Otherwise we again consider $\widetilde{C}$ and $\widetilde{T}$. This time put

$$D = \{(z_n) \in \widetilde{C} : \limsup \left\|z_n - x_n^i\right\| \leqslant ((m-1)/m)\operatorname{diam} C \text{ for all } i\}.$$

Note that $\left((1/m)(x_n^1 + x_n^2 + \cdots + x_n^m)\right) \in D \neq \emptyset$. As above, D contains an afps. Thus for any $p \in \mathbb{N}$ there exists $(y_n^p)_{n=1}^{\infty} \in D$ so that $\sup_n \|Ty_n^p - y_n^p\| \leqslant 1/p$. Choose $\varepsilon_n > 0$ so that $\varepsilon_n \to 0$. Then for any p there exists $n_p \in \mathbb{N}$ so that for $n \geqslant n_p$

$$\left\|y_n^p - x_n^i\right\| \leqslant ((m-1)/m)\operatorname{diam} C + \varepsilon_p \quad \text{for all } i.$$

We can also assume that (n_p) is a strictly increasing sequence. We now define $(z_n) \in \widetilde{C}$ as follows.

For $n < n_1$ put $z_n = y_n^1$. If $n \in [n_p, n_{p+1})$ put $z_n = y_n^p$. It is then readily seen that $(z_n) \in D$ and that (z_n) is an afps, giving the proposition. ∎

4. Fixed point results

In this section we give fixed point results that don't require normal structure type criteria for generalised nonexpansive mappings of type (c). We consider the (defined in the obvious way) w–fpp (w*–fpp) for mappings of type (c). In the sequel we only consider the w case. It should perhaps be noted that so far no space has been shown to fail the w–fpp (or even the w*–fpp) for mappings of type (c).

Suppose that K is a nontrivial (c) minimal invariant. By Section 2 it has an afps (x_n) (any orbit sufficing). Since K is weakly compact we can assume that (x_n) is weakly convergent. By translation and dilation we can assume that the weak limit is 0 and that $\operatorname{diam} K = 1$. Of course Lemma 3.3 implies that $\lim \|x - x_n\| = 1$ for all $x \in K$ and in particular $\|x_n\| \to 1$. We shall assume this state of affairs for the remainder of this section.

In [**17**] Maurey proved that reflexive subspaces of $L_1[0,1]$ have the w–fpp. The proof essentially starts from where we just left off and uses the nonexpansive versions of Lemma 3.3 and Proposition 3.4 to obtain a contradiction; that no further properties of the mapping are required can be seen by following the proof given in Part III of [**7**], a proof which also does not require the use of an ultrapower. Thus these spaces have the w–fpp for type (c) mappings. Maurey's original proof could also have been adapted to work in this case as well. We also have the following versions of results in [**7**]: The Tsirelson spaces T_s and T_s^* have the w–fpp for mappings of type (c), as do B–convex subspaces of uniformly monotone Banach lattices.

Maurey also showed that c_0 has the w–fpp. Recently (in [**12**] and [**13**]) a property was defined sufficient for the w–fpp and which is possessed by c_0. We recall its definition.

Firstly if X is a Banach space, $x, y \in X$ and λ is positive put

$$M_\lambda(x,y) := \{z \in X : \max\{\|z - x\|, \|z - y\|\} \leqslant (1/2)(1+\lambda)\|x - y\|\}.$$

If A is a bounded subset of X

$$|A| := \sup\{\|z\| : z \in A\}.$$

If (x_n) is a bounded sequence in X and λ positive we also define

$$D[(x_n)] := \limsup_n \limsup_m \|x_n - x_m\|$$

and

$$A_\lambda[(x_n)] := \limsup_n \limsup_m |M_\lambda(x_n, x_m)|.$$

Finally we say that X is Orthogonally Convex (OC) if for each sequence (x_n) in X that is weakly convergent to 0 and satisfies $D[(x_n)] > 0$ there exists $\lambda > 0$ such that $A_\lambda[(x_n)] < D[(x_n)]$.

It is shown in [**12**] that OC spaces have the w–fpp. This is easily extended. We firstly give the following.

Proposition 4.1. *The Banach space property defined by only requiring that*

$$\liminf_n \liminf_m |M_\lambda(x_n, x_m)| < D[(x_n)]$$

in the definition of OC is equivalent to the following:

Given a weak null sequence (x_n) with $D[(x_n)] > 0$ there exists a subsequence (y_n) of (x_n) and $\lambda > 0$ so that

$$\sup_{n,m \in \mathbb{N}, n \neq m} |M_\lambda(y_n, y_m)| < D[(x_n)].$$

Proof: Suppose that X satisfies the altered definition of OC, $x_n \xrightarrow{w} 0$ and $D[(x_n)] > 0$. Obviously (x_n) is an infinite sequence. We can assume that its terms are distinct. Denoting by $\mathcal{P}_2(A)$ the set of two element subsets of a set A, put

$$A^1 := \left\{\{x_n, x_m\} \in \mathcal{P}_2(\{x_n\}_{n=1}^\infty) : \left|M_{1/2}(x_n, x_m)\right| < D[(x_n)] - 1/2\right\}$$
$$B^1 := \left\{\{x_n, x_m\} \in \mathcal{P}_2(\{x_n\}_{n=1}^\infty) : \left|M_{1/2}(x_n, x_m)\right| \geqslant D[(x_n)] - 1/2\right\}.$$

Ramsey's Theorem now gives a subsequence (x_n^1) of (x_n) so that either $\mathcal{P}_2\{x_n^1\}_{n=1}^\infty \subseteq A^1$ or $\mathcal{P}_2\{x_n^1\}_{n=1}^\infty \subseteq B^1$. The first eventuality gives the required result. We shall thus assume the second. Now put

$$A^2 := \left\{\{x_n^1, x_m^1\} \in \mathcal{P}_2\left(\{x_n^1\}_{n=1}^\infty\right) : \left|M_{1/2^2}(x_n, x_m)\right| < D[(x_n)] - 1/2^2\right\}$$
$$B^2 := \left\{\{x_n^1, x_m^1\} \in \mathcal{P}_2\left(\{x_n^1\}_{n=1}^\infty\right) : \left|M_{1/2^2}(x_n, x_m)\right| \geqslant D[(x_n)] - 1/2^2\right\}.$$

Again we shall assume the existence of a subsequence (x_n^2) of (x_n^1) satisfying $\mathcal{P}_2\{x_n^2\}_{n=1}^\infty \subseteq B^2$. We can continue in this fashion. A Cantor diagonalisation will then produce a subsequence (y_n) of (x_n) so that

$$\liminf_n \liminf_m |M_\lambda(y_n, y_m)| \geqslant D[(x_n)] \geqslant D[(y_n)]$$

for any $\lambda > 0$. Since this property of (y_n) also implies that $D[(y_n)] > 0$, we have a contradiction. The converse is obvious. ∎

We now extend the result from [**12**].

Proposition 4.2. *If X is OC then it has the w–fpp for mappings of type (c).*

Proof: Back to our (c) minimal invariant K with its weak null afps (x_n). Lemma 3.3 easily gives that $D[(x_n)] = 1$. Now let (y_n) be a subsequence of it given by the above proposition, with associated $\lambda > 0$. Put $M = \sup_{n,m\in\mathbb{N}, n\neq m} |M_\lambda(y_n, y_m)| < 1$. Choose two subsequences of it, (z_n) and (w_n) say, with $\|z_n - w_n\| \to 1$. (This is possible by Lemma 3.3.) Proposition 3.4 now gives an afps (t_n) in K with $\lim \|t_n - z_n\| = 1/2$ and $\lim \|t_n - w_n\| = 1/2$. But then, for sufficiently large n, $\|t_n\| \leqslant M$, contradicting the fact (by Lemma 3.3) that $\|t_n\| \to 1$. ∎

The next results are lattice theoretic. For background material we refer the reader to [**20**]. We recall some definitions from [**5**]. A Banach lattice is said to be weakly orthogonal if, given a weak null sequence (x_n) in it,

(1) $$\liminf_n \liminf_m \| |x_n| \wedge |x_m| \| = 0.$$

This definition can be rewritten as follows.

PROPOSITION 4.3. *A Banach lattice X is weakly orthogonal if and only if given a weak null sequence (x_n) in X and $\varepsilon > 0$ there exist $n, m \in \mathbb{N}$ so that $\| |x_n| \wedge |x_m| \| < \varepsilon$. It is also equivalent to the condition requiring that for every $\varepsilon > 0$ there exists a subsequence (y_n) of (x_n) so that $\| |y_n| \wedge |y_m| \| < \varepsilon$ if $n \neq m$.*

PROOF: Clearly weak orthogonality implies the first stated condition. Also, the second condition easily implies weak orthogonality. Now suppose that X satisfies the first condition and $x_n \xrightarrow{w} 0$. In establishing the second condition we can assume that $\liminf \|x_n\| > 0$. Thus $\liminf \| |x_n| \wedge |x_n| \| > 0$. But now an application of Ramsey's Theorem easily gives the second condition. ∎

We define, after [**5**], the Riesz angle of a Banach lattice X to be

$$\alpha(X) := \sup \{ \| |x| \vee |y| \| : \|x\| \leqslant 1, \ \|y\| \leqslant 1 \}.$$

It is noted in [**5**] that X is an abstract M space if and only if $\alpha(X) = 1$. The following Lemma is [**5**, Theorem 4.2].

LEMMA 4.4. *If x, y, z are in a Banach lattice X then*

$$\|z\| \leqslant \alpha(X)(\|x - z\| \vee \|y - z\|) + \| |x| \wedge |y| \|.$$

In [**5**] it is shown that if X is a weakly orthogonal Banach lattice with $\alpha(X) < 2$ then X has the w–fpp. Lemma 4.4 gives that for two points x, y and $\lambda > 0$,

$$|M_\lambda(x, y)| \leqslant \alpha(X)(1/2)(1 + \lambda) \|x - y\| + \| |x| \wedge |y| \|.$$

Of course if $\alpha(X) < 2$ then λ can be chosen so that $\beta := \alpha(X)(1/2)(1 + \lambda) < 1$. We now see that if $\limsup_n \limsup_m$ was used instead of $\liminf_n \liminf_m$ in the definition of

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weak orthogonality then we would have arrived at orthogonal convexity. Irrespective, we obtain the condition of Proposition 4.3. One can then use the method of proof of Proposition 4.2 to establish the following proposition. [5] contains examples of spaces satisfying the conditions given. The spaces c_0 and c both satisfy the weak orthogonal condition using the double limit supremum and are thus also OC.

PROPOSITION 4.5. *If X is a weakly orthogonal Banach lattice with $\alpha(X) < 2$ then it has the w–fpp for mappings of type (c).*

We now combine the approach of [5] with a technique from [7]. Firstly, suppose X is a Banach lattice and $p \in \mathbb{N}$, $p > 1$. Then X is called p weakly orthogonal if, given a weak null sequence (x_n) in X and $\varepsilon > 0$, there exist p natural numbers $n_1, n_2, \ldots, n_p$ (not necessarily distinct) so that $\left\| \bigwedge_{i=1}^{p} |x_{n_i}| \right\| < \varepsilon$. Of course weak orthogonality is the same as being 2 weakly orthogonal. The following is essentially established in [7, Corollary 3].

EXAMPLE 4.6. Let w denote the first infinite ordinal. Given an ordinal α we denote by $C(\alpha)$ the Banach space of continuous real valued functions on the compact Hausdorff order interval $[1, \alpha]$. This space has the usual supremum norm and pointwise order. Then, for $n \in \mathbb{N}$, $\alpha < w^n$, $C(\alpha)$ is n weakly orthogonal.

It follows directly from the definition of Riesz angle that if x and y are elements of a Banach lattice X then

$$\| |x| \vee |y| \| \leqslant \alpha(X) (\|x\| \vee \|y\|).$$

This now easily implies that if $x_1, x_2, \ldots, x_p$ are p points in X (not necessarily distinct) then

$$\left\| \bigvee_{i=1}^{p} |x_i| \right\| \leqslant \alpha(X)^{p-1} \left(\bigvee_{i=1}^{p} \|x_i\| \right). \tag{2}$$

Now suppose also that $z \in X$. Then for any i

$$|z| \leqslant |x_i| + |x_i - z|.$$

Thus for any j

$$|z| \leqslant \bigvee_{i=1}^{p} |z - x_i| + |x_j|.$$

Thus

$$\begin{aligned} |z| &\leqslant \bigwedge_{j=1}^{p} \left(\bigvee_{i=1}^{p} |z - x_i| + |x_j| \right) \\ &= \bigvee_{i=1}^{p} |z - x_i| + \bigwedge_{i=1}^{p} |x_i|. \end{aligned}$$

This combined with (2) gives the following.

Lemma 4.7. *Suppose that $x_1, x_2, \dots, x_p$ (not necessarily distinct) are p ($p \geqslant 2$) points in a Banach lattice X and that $z \in X$. Then*

$$\|z\| \leqslant \alpha(X)^{p-1}\left(\bigvee_{i=1}^{p}\|z - x_i\|\right) + \left\|\bigwedge_{i=1}^{p}|x_i|\right\|.$$

We are now ready for the main fixed point result.

Theorem 4.8. *If X is a p weakly orthogonal Banach lattice satisfying*

$$\frac{p-1}{p}\alpha(X)^{p-1} < 1$$

then X has the w–fpp for both nonexpansive mappings and mappings of type (c).

Proof: Suppose K is our usual minimal invariant satisfying $\operatorname{diam} K = 1$ with a weak null afps (x_n). From the definition of p weakly orthogonal there exist p subsequences of (x_n), (x_n^1), (x_n^2), $\dots, (x_n^p)$ say, so that

$$\left\|\bigwedge_{i=1}^{p}|x_n^i|\right\| \to_n 0.$$

Now Proposition 3.5 gives an afps (z_n) satisfying $\limsup_n \|z_n - x_n^i\| \leqslant (p-1)/p$ for all i. But

$$\begin{aligned}\limsup \|z_n\| &\leqslant \alpha(X)^{p-1}\limsup_n\left(\bigvee_{i=1}^{p}\|z_n - x_n^i\|\right) + \limsup_n\left\|\bigwedge_{i=1}^{p}|x_n^i|\right\| \\ &\leqslant \alpha(X)^{p-1}\left(\bigvee_{i=1}^{p}\limsup_n\|z_n - x_n^i\|\right) \\ &\leqslant \alpha(X)^{p-1}\frac{p-1}{p} \\ &< 1,\end{aligned}$$

contradicting $\|z_n\| \to 1$ (due to Lemma 3.3). □

Since the $C(\alpha)$ mentioned in Example 4.6 are M spaces, Example 4.6 and Theorem 4.8 give the following corollary which extends the result for nonexpansive mappings given in [7].

Corollary 4.9. *If $\alpha < w^w$ then $C(\alpha)$ has the w–fpp for mappings of type (c).*

It is easily seen that the technique of the proof of Theorem 4.8 would work if X was a Banach lattice satisfying the following: Given a weak null sequence (x_n) in X

there exists $p \in \mathbb{N}$ $(p > 1)$ so that $((p-1)/p)\alpha(X)^{p-1} < 1$ and that for any $\varepsilon > 0$ there exist $n_1, n_2, \ldots, n_p \in \mathbb{N}$ such that

$$\left\| \bigwedge_{i=1}^{p} |x_{n_i}| \right\| < \varepsilon.$$

The following gives an explicit pathological sequence that violates the above. The weak null sequence used is of Schreier type and is connected with the failure of the weak Banach Saks property (see below).

Proposition 4.10. *$C(w^w)$ contains a weak null sequence of indicator functions (x_n) so that given any m natural numbers $n_1, n_2, \ldots, n_m$ with $n_i \geqslant m$ there exists $\alpha \in [1, w^w)$ so that*

$$x_{n_1}(\alpha) = x_{n_2}(\alpha) = \cdots = x_{n_m}(\alpha) = 1.$$

Proof: We firstly define sequences $(x_n^m)_{n=1}^{\infty}$ in $C(w^n)$ in the following recursive manner:

For $m = 1$ put $x_n^1 = \chi_n$. Now suppose that $(x_n^m)_{n=1}^{\infty}$ has been defined. We shall use the fact that $[1, w^m]$ is order isomorphic to $[w^m.n+1, w^m(n+1)]$ for any $m, n \in \mathbb{N}$, via f_n^m say. Also, if $x \in C[1, w^m]$ then $f_n^m(x)$ is the member of $C[w^m.n+1, w^m(n+1)]$ defined by $f_n^m(x)(\alpha) \doteq x\Big((f_n^m)^{-1}(\alpha)\Big)$ for $\alpha \in [w^m.n+1, w^m(n+1)]$. Now put

$$\begin{aligned}
x_1^{m+1} &= \chi_{[1,w^m]} \\
x_2^{m+1} &= x_1^m + \chi_{[w^m+1, w^m.2]} \quad \text{and} \\
x_n^{m+1} &= x_{n-1}^m + f_1^m\left(x_{n-1}^m\right) + \cdots + f_{n-2}^m\left(x_{n-1}^m\right) + \chi_{[w^m(n-1)+1, w^m.n]}.
\end{aligned}$$

It can then be verified, again by recursion, that the $(x_n^m)_{n=1}^{\infty}$ have the following properties.

1. $x_n^m \in C(w^m)$ for $n, m \in \mathbb{N}$.
2. $x_n^m \xrightarrow{w}{}_n 0$ for all m.
3. $x_n^m(\alpha) \in \{0, 1\}$ for $n, m \in \mathbb{N}$ and $\alpha \in [1, w^m]$.
4. If $n_1, n_2, \ldots, n_m \in \mathbb{N}$ then there exists $\alpha \in [1, w^m)$ so that $x_{n_i}^m(\alpha) = 1$ for any i.

We now define a sequence in $\mathbb{R}^{[1,w^w]}$. We use the fact that $[1, w^{n+1}]$ is order isomorphic to $[w^n + 1, w^{n+1}]$ for any n, via f^n say. Now put

$$\begin{aligned}
x_1 &= x_1^1 \\
x_2 &= x_2^1 + f^2\left(x_1^2\right) \\
x_3 &= x_3^1 + f^2\left(x_2^2\right) + f^3\left(x_1^3\right) \quad \text{and} \\
x_n &= x_n^1 + f^2\left(x_{n-1}^2\right) + \cdots + f^n(x_1^n).
\end{aligned}$$

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Then clearly $x_n \in C(w^w)$ for all n and 2, 3 and 4 give, respectively, $2'$, $3'$ and $4'$ below.

- $2'$. $x_n \xrightarrow{w} 0$.
- $3'$. $x_m(\alpha) \in \{0, 1\}$ for $n \in \mathbb{N}$ and $\alpha \in [1, w^w]$.
- $4'$. If $n_1, n_2, \ldots, n_m \in \mathbb{N}$ with $n_i \geqslant m$ then there exists $\alpha \in [1, w^w)$ so that $x_{n_i}(\alpha) = 1$ for any i.

But these are the required properties. $\square$

The sequence of the proposition can be used to show that $C(w^w)$ fails the weak Banach Saks property, first shown by Schreier [**21**]. Recall that X possesses the weak Banach Saks property if, given a weak null sequence (x_n) in X, there exists a subsequence (y_n) of (x_n) so that

$$\sum_{k=1}^{n} \frac{y_k}{n} \to_n 0.$$

Let Ω denote a compact Hausdorff space and $C(\Omega)$ the space of real valued continuous functions on Ω, given the usual supremum norm. Now suppose that $C(\Omega)$ contains a weak null sequence (x_n) so that $x_n \geqslant 0$ for all n and, if $n_1, n_2, \ldots, n_m \in \mathbb{N}$ with $n_i \geqslant m$, there exists $x \in \Omega$ such that $x_{n_i}(x) = 1$ for all i. Obviously any subsequence of (x_n) also has these properties. Assume that we are given a subsequence and denote it by (x_n). Now for $n \in \mathbb{N}$ there exists $x \in \Omega$ so that

$$\frac{1}{2n-1}(x_1 + x_2 + \cdots + x_n + \cdots + x_{2n-1})(x) \geqslant \frac{n}{2n-1} > \frac{1}{2},$$

contradicting the weak Banach Saks property.

We now recall some definitions. If X is a topological space then it is said to be dispersed if it does not contain any nonempty perfect subsets. A perfect set is one with no isolated points and it is easily seen that a space is dispersed exactly when every nonempty subset of it contains an isolated point. We denote by X' the set of limit (nonisolated) points of a topological space X. This is also referred to as the derived set of X. If α is an ordinal then the derivative of order α, denoted by $X^{(\alpha)}$, is defined by transfinite recursion as follows. $X^{(0)} = X$, $X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$, and $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ if α is a limit ordinal.

If Ω is a dispersed compact Hausdorff space there exists a smallest successor ordinal $\alpha = \beta + 1$ so that $\Omega^{(\alpha)} = \emptyset$. The topological height of Ω, $H(\Omega)$, is then defined to be β. It is well known that $H([1, w^\alpha]) = \alpha$, and that if Ω is a compact Hausdorff space then $C(\Omega)$ fails the weak Banach Saks property if and only if $\Omega^{(w)} \neq \emptyset$. (See for example [**6**, p.85].) We use similar techniques to obtain the following.

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THEOREM 4.11. *$\Omega^{(w)} \neq \emptyset$ if and only if $C(\Omega)$ contains a weak null sequence (x_n) satisfying $x_n \geqslant 0$ for all n and for which, given $n_1, n_2, \ldots, n_m \in \mathbb{N}$ with $n_i \geqslant m$, there exists $x \in \Omega$ so that for any i, $x_{n_i}(x) = 1$. $\Omega^{(w)} = \emptyset$ if and only if $C(\Omega)$ is p weakly orthogonal for some $p \in \mathbb{N}$. In fact, $C(\Omega)$ is p weakly orthogonal if and only if $H(\Omega) < p$.*

PROOF: Suppose that $\Omega^{(w)} = \emptyset$. Then obviously Ω is dispersed and $H(\Omega) = n \in \mathbb{N}$. We show that $C(\Omega)$ is $n+1$ weakly orthogonal. Suppose that (x_n) is a weak null sequence in $C(\Omega)$. As in [**6**] define an equivalence relation on Ω by putting $x \sim y$ if $x_n(x) = x_n(y)$ for every $n \in \mathbb{N}$. Now the quotient topology on the set $\widetilde{\Omega}$ of equivalence classes is compact Hausdorff and metrisable. The x_n lift in a natural way to $\widetilde{x}_n$ in $C\left(\widetilde{\Omega}\right)$. Since $\widetilde{\Omega}$ is a continuous image of Ω, $H\left(\widetilde{\Omega}\right) \leqslant H(\Omega)$ by [**19**, Lemma 1], noting that the subset notation of the lemma should be reversed. Thus $H\left(\widetilde{\Omega}\right) \leqslant n$. Since $\widetilde{\Omega}$ is also a compact dispersed metric space it is then homeomorphic to $[1, \alpha]$ for an $\alpha < w^{n+1}$ by a result of Mazurkiewicz and Sierpinski [**18**] (or [**15**, Section 5]). Example 4.6 now gives the result.

Now suppose that $\Omega^{(w)} \neq \emptyset$. First assume that Ω is not dispersed. Then by [**19**], $[0,1]$ is a continuous image of Ω, via f say. Now $[0,1]$ contains a copy M of $[1, w^w]$. The complement of this copy is a countable union of disjoint open intervals. Any $h \in C(M)$ can be extended to a member of $C[0,1]$ in the obvious way by making it affine on each of the intervals. This gives a simultaneous extension operator from $C(w^w)$ into $C[0,1]$ which will preserve the germane properties of the sequence from Proposition 4.10. Now we can apply the pullback $g \mapsto g \circ f$ from $C[0,1]$ into $C(\Omega)$ that also preserves the required properties of (x_n).

Now suppose that Ω is dispersed. Then Ω is totally disconnected and, since it is compact, Hausdorff, it is zero dimensional (see [**8**]). By [**3**, Theorem 1.1] there exists a continuous map f from Ω onto $[1, w^w]$. We can now apply the pullback directly, again obtaining the pathological sequence.

Only the last sentence of the theorem needs to be verified. This can be done using the same techniques. One needs to use the pathological sequences (x_n^m) of Proposition 4.10 and [**3**, Theorem 1.1], which implies that if X is a zero dimensional space, α is countable and $X^{(\alpha)} \neq \emptyset$ then $[1, w^\alpha]$ is a continuous image of X. ∎

Regarding the w–fpp for $C(\Omega)$, if Ω is not dispersed then $C(\Omega)$ contains an isometric copy of $C[0,1]$ by [**19**], thus failing the w–fpp by isometric embedding of Alspach's example. Now suppose that Ω is dispersed. By [**19**] any separable subspace of $C(\Omega)$ is isometric to a subspace of $C(\alpha)$ for some countable ordinal α. Thus, by the well known fact that the w–fpp is separably determined (see for example the end of [**10**, Chapter 3], and use the fact that closed convex sets are weak closed), failure of the w–fpp for

$C(\Omega)$ (Ω dispersed) would imply that $C(\alpha)$, for some countable ordinal α, fails the w–fpp. The question of whether such spaces have the w–fpp was previously raised in [**7**].

We should finally mention a result from [**22**]. In it a more restrictive notion of weak orthogonality was introduced. This required that, given a weak null sequence (x_n) in X and $x \in X$, $\||x_n| \wedge |x|\| \to 0$. It is shown that such a space has the w–fpp. The method of proof is a Banach lattice extension of that used in [**16**] to show that a space with a 1-unconditional basis has the w–fpp. None of these proofs however appear adaptable so as to work for mappings of type (c), too much use being made of the nonexpansive nature of the maps.

References

[**1**] D.E. Alspach, 'A fixed point free nonexpansive map', *Proc. Amer. Math. Soc.* **82** (1981), 423–424.

[**2**] Jong Sook Bae, 'Fixed point theorems of generalized nonexpansive maps', *J. Korean Math. Soc.* **21** (1984), 233–248.

[**3**] J.W. Baker, 'Dispersed images of topological spaces and uncomplemented subspaces of $C(X)$', *Proc. Amer. Math. Soc.* **41** (1973), 309–314.

[**4**] J. Bogin, 'A generalization of a fixed point theorem of Goebel, Kirk and Shimi', *Canad. Math. Bull.* **19** (1976), 7–12.

[**5**] J. Borwein and B. Sims, 'Non-expansive mappings on Banach lattices and related topics', *Houston J. Math.* **10** (1983), 339–355.

[**6**] J. Diestel, *Geometry of Banach spaces–Selected topics* (Springer-Verlag, Berlin, Heidelberg, New York, 1975).

[**7**] J. Elton, P.K. Lin, E. Odell and S. Szarek, 'Remarks on the fixed point problem for nonexpansive maps', in *Fixed points and nonexpansive mappings*, (R. Sine, Editor), Contemp. Math. Amer. Math. Soc. **18** (Amer. Math. Soc., Providence RI, 1983).

[**8**] R. Engelking, *Outline of general topology* (North-Holland, Amsterdam, 1968).

[**9**] K. Goebel, 'On the structure of minimal invariant sets for nonexpansive mappings', *Ann. Univ. Mariae. Curie-Sklodowska* **29** (1975), 73–77.

[**10**] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory* (Cambridge University Press, Cambridge, 1990).

[**11**] K. Goebel, W.A. Kirk and T. Shimi, 'A fixed point theorem in uniformly convex spaces', *Boll. Un. Mat. Ital.* **7** (1973), 67–75.

[**12**] A. Jimenez-Melado and E. Llorens-Fuster, 'A geometric property of Banach spaces which implies the fixed point property for nonexpansive mappings', (preprint).

[**13**] A. Jimenez-Melado and E. Llorens-Fuster, 'Stability of orthogonal convexity', (preprint).

[**14**] L. Karlovitz, 'Existence of fixed points for nonexpansive mappings in spaces without normal structure', *Pacific J. Math.* **66** (1976), 153–156.

[**15**] H. Lacey, *The isometric theory of classical Banach spaces* (Springer-Verlag, Berlin, Hei-

delberg, New York, 1974).

[16] Pei-Kee Lin, 'Unconditional bases and fixed points of nonexpansive mappings', *Pacific J. Math.* **116** (1985), 69–76.

[17] B. Maurey, *Points fixes des contractions sur un convexe forme de* L^1, Seminarie d' Analyse Fonctionnelle 1980-81 (Expose no. 8) (Ecole Polytechnique, Palaiseau).

[18] S. Mazurkiewicz and W. Sierpinski, 'Contribution a la topologie des ensembles denombrables', *Fund. Math.* **1** (1920), 17–27.

[19] A. Pełczyński and Z. Semadeni, 'Spaces of continuous functions III', *Studia Math.* **18** (1959), 211–222.

[20] H.H. Schaefer, *Banach lattices and positive operators*, Grundlehren der Math., Band 215 (Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[21] J. Schreier, 'Ein Gegen beispiel zur Theorie der schwachen Konvergenz', *Studia Math.* **2** (1930), 58–62.

[22] B. Sims, 'Othogonality and fixed points of nonexpansive maps', in *Proc. Centre Math. Anal.* **20** (Australian National University, Canberra, ACT, 1988).

[23] C. Wong, 'Close-to-normal structure and its applications', *J. Funct. Anal.* **16** (1974), 353–358.

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