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FITTING CLASSES AND LATTICE FORMATIONS I

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Abstract

A lattice formation is a class of groups whose elements are the direct product of Hall subgroups corresponding to pairwise disjoint sets of primes. In this paper Fitting classes with stronger closure properties involving $\mathscr{F}$-subnormal subgroups, for a lattice formation $\mathscr{F}$ of full characteristic, are studied. For a subgroup-closed saturated formation $\mathscr{G}$, a characterisation of the $\mathscr{G}$-projectors of finite soluble groups is also obtained. It is inspired by the characterisation of the Carter subgroups as the $\mathscr{N}$-projectors, $\mathscr{N}$ being the class of nilpotent groups.

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1. Introduction

All groups considered are finite and soluble.

In this paper $\mathscr{F}$-Fitting classes, for a lattice formation $\mathscr{F}$, are defined in a natural way by closure properties involving $\mathscr{F}$-subnormal subgroups. A lattice formation is a class of groups whose elements are the direct product of Hall subgroups corresponding to fixed pairwise disjoint sets of primes. When $\mathscr{F} = \mathscr{N}$, the class of nilpotent groups, we recover the classical Fitting classes.

This study is motivated by the following concepts and facts:

In [3] an extension of normality for subgroups, called $\mathscr{F}$-Dnormality, for a saturated formation $\mathscr{F}$, was introduced (see Definition 2.2 (b) below). It is associated naturally with $\mathscr{F}$-subnormality in an obvious way. If $\mathscr{F}$ is a lattice formation, the set of all $\mathscr{F}$-subnormal subgroups is a lattice in every group. This lattice contains the set of all $\mathscr{F}$-Dnormal subgroups as a sublattice.

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In fact, the lattice properties of $\mathcal{F}$-subnormal subgroups, and also the lattice properties of $\mathcal{F}$-Dnormal subgroups, characterize the lattice formations among all the subgroup-closed saturated formations $\mathcal{F}$. (See Theorem 2.7.)

Then, given a lattice formation $\mathcal{F}$ containing $\mathcal{N}$, we define $\mathcal{F}$-Fitting classes in a natural way by closure operations involving $\mathcal{F}$-subnormal subgroups. We also see that $\mathcal{F}$-Dnormality can substitute for $\mathcal{F}$-subnormality in this definition, exactly as normality substitutes for subnormality in Fitting classes.

Theorem 2.8 states that every lattice formation $\mathcal{F}$ containing $\mathcal{N}$ is an $\mathcal{F}$-Fitting class. (In fact, this property provides a characterisation for lattice formations; see [7].) We construct a large family of Fitting formations $\mathcal{G}$ which are $\mathcal{F}$-Fitting classes, for some related lattice formations $\mathcal{F}$, in particular, whenever $\mathcal{F} \subseteq \mathcal{G}$. This family contains, in particular, lattice formations and the class of p-nilpotent groups, for every prime p. Other examples of $\mathcal{F}$-Fitting classes of a different nature are also given.

We complete the paper by providing a characterisation of the $\mathcal{H}$-projectors, for a subgroup-closed saturated formation $\mathcal{H}$, which involves the concepts of $\mathcal{H}$-subnormality and $\mathcal{H}$-Dnormality. This result generalises the characterisation of the $\mathcal{N}$-projectors as the Carter subgroups in every group. Other generalisations of this result for $\mathcal{H}$-projectors were proposed by Carter and Hawkes (see Theorem 2.14) and by Graddon in [14, Theorem 2.15].

Our characterisation of $\mathcal{H}$-projectors has interest in its own right but also finds application in the study of the injectors associated to $\mathcal{F}$-Fitting classes. In this manner, notice that an $\mathcal{F}$-Fitting class is also a Fitting class, as the lattice formation $\mathcal{F}$ contains $\mathcal{N}$. In a forthcoming paper [2], the desired behaviour of the associated injectors, with respect to $\mathcal{F}$-subnormal (and $\mathcal{F}$-Dnormal) subgroups, is obtained. In fact, this property characterizes $\mathcal{F}$-Fitting classes. This is the natural extension of the known characterisation of the Fitting classes as the injective classes of groups. A previous result is Theorem 2.8 (3).

2. Notation and preliminaries

We use standard notation and terminology taken mainly from [12]. The reader is also referred to this book for the results on saturated formations, projectors and Fitting classes.

In particular, if $\mathcal{X}$ is a class of groups, the characteristic of $\mathcal{X}$ is $\mathrm{char}(\mathcal{X}) = \{p \in \mathbb{P} : Z_p \in \mathcal{X}\}$, where $\mathbb{P}$ denotes the set of all prime numbers and Z_p the cyclic group of order p.

If π is a set of primes, $\mathcal{S}$ and $\mathcal{S}_\pi$ denote the class of all soluble groups and the class of all soluble π-groups, respectively. $\pi' = \mathbb{P} \setminus \pi$ is the complementary set of π in $\mathbb{P}$. If H is a subgroup of a group G, $\sigma(|G : H|)$ denotes the set of all prime numbers dividing $|G : H|$. $\mathcal{N}$ denotes the class of all nilpotent groups. For a group G and a prime $q \in \mathbb{P}$, V_q denotes a G-module over $\mathbb{F}_q$, the finite field of q elements, and the group $[V_q]G$ is always the semidirect product with respect to the action of G on V_q.

It is well known that a formation $\mathcal{G}$ is saturated if and only if

$$\mathcal{G} = LF(g) = \mathcal{S}_\pi \cap \left(\bigcap_{p \in \pi} \mathcal{S}_{p'} \mathcal{S}_p g(p) \right), \quad \pi = \mathrm{char}(\mathcal{G}),$$

that is, if $\mathcal{G}$ is a local formation defined by a formation function g. In this case, $\mathcal{G}$ has a uniquely determined full and integrated formation function defining $\mathcal{G}$, which is called the *canonical local definition* of $\mathcal{G}$ and will be identified by G. We write $\underline{g}$ to denote the smallest local definition of $\mathcal{G}$. (See [12, IV, Definitions 3.9].)

A lattice formation $\mathcal{F}$ of characteristic π is a saturated formation locally defined by a formation function f given by: $f(p) = \mathcal{S}_{\pi_i}$, if $p \in \pi_i \subseteq \pi$, where $\{\pi_i\}_{i \in I}$ is a partition of π, and $f(q) = \emptyset$, the empty formation, if $q \notin \pi$.

In this case, for a prime $p \in \pi$, the set of primes π_i such that $p \in \pi_i$, will be also identified by $\pi(p)$.

LEMMA 2.1 ([6, Remark 3.6], [5, Lemma 3.2]). *Let $\mathcal{F}$ be a lattice formation and $p \in \pi = \mathrm{char}(\mathcal{F})$. Then:*

(a) *The canonical local definition F and the smallest local definition $\underline{f}$ of $\mathcal{F}$ are given by setting:*

(i) *If $|\pi(p)| = 1$, then $F(p) = \mathcal{S}_p$ and $\underline{f}(p) = (1)$.*
(ii) *If $|\pi(p)| \geq 2$, then $F(p) = \underline{f}(p) = \mathcal{S}_{\pi(p)}$. In particular, for a group G, $G^{F(p)} = G^{\underline{f}(p)} = O^{\pi(p)}(G)$.*

(b) *A group G belongs to $\mathcal{F}$ if and only if G is a soluble π-group with a normal Hall π_i-subgroup, for every $i \in I$.*

Henceforth $\mathcal{F}$ will denote a lattice formation and the above notation will be assumed. $\mathcal{G}$ will always denote a saturated formation with $\mathrm{char}(\mathcal{G}) = \pi$.

The key concepts and results needed in the paper are the following:

DEFINITION 2.2. (a) [12, III, Definition 4.13] A maximal subgroup M of a group G is $\mathcal{G}$-normal in G, if $G/\mathrm{Core}_G(M) \in \mathcal{G}$; otherwise it is called $\mathcal{G}$- abnormal.

(b) [3, Definition 3.1] A subgroup H of a group G is $\mathcal{G}$-Dnormal in G if $\sigma(|G : H|) \subseteq \pi$ and $[H_G^p, H^{\underline{g}(p)}] \leq H$, for every $p \in \pi$, where $H_G^p = \langle G_p \in \mathrm{Syl}_p(G) : G_p$ reduces into H, that is, $G_p \cap H \in \mathrm{Syl}_p(H) \rangle$. We write H $\mathcal{G}$-Dn G.

REMARK 2.3. (1) If H is a maximal subgroup of G, then H $\mathcal{G}$-Dn G if and only if H is $\mathcal{G}$-normal in G.

(2) A subgroup H of a group G is $\mathcal{N}$-Dnormal in the group G if and only if H is normal in G.

(3) [3, Theorem 4.8] For a lattice formation $\mathcal{F}$, a subgroup H of a group G is $\mathcal{F}$-Dnormal in G if and only if H satisfies:

$$\begin{aligned} [O^{p'}(G), O^{\pi(p)}(H)] &\leq O^{\pi(p)}(H), \quad \text{if } |\pi(p)| \geq 2 \quad \text{or} \\ [O^{p'}(G), H] &\leq H, \quad \text{if } \quad \pi(p) = \{p\}, \end{aligned}$$

for every $p \in \sigma(|G : H|) \subseteq \pi$.

DEFINITION 2.4 ([12, IV, Definition 5.12]). A subgroup H of a group G is said to be $\mathcal{G}$-subnormal in G if either $H = G$ or there exists a chain $H = H_n < H_{n-1} < \cdots < H_0 = G$ such that H_{i+1} is a $\mathcal{G}$-normal maximal subgroup of H_i, for every $i = 0, \ldots, n-1$. We write H $\mathcal{G}$-sn G.

REMARK 2.5. (1) [3, Proposition 3.5] A subgroup H of a group G is $\mathcal{G}$-subnormal in G if and only if there exists a chain $H = T_l \leq T_{l-1} \leq \cdots \leq T_0 = G$ such that T_{i+1} is a $\mathcal{G}$-Dnormal subgroup of T_i, for every $i = 0, \ldots, l-1$. In particular, a $\mathcal{G}$-Dnormal subgroup of a group is $\mathcal{G}$-subnormal in the group.

(2) A subgroup H of a group G is $\mathcal{N}$-subnormal in the group G if and only if H is subnormal in G.

(3) If $\mathcal{N} \subseteq \mathcal{G}$, the normal and the subnormal subgroups of a group are $\mathcal{G}$-Dnormal and $\mathcal{G}$-subnormal, respectively in the group.

LEMMA 2.6 ([13, Lemma 1.1]). *Let $\mathcal{G}$ be a subgroup-closed saturated formation. If H is $\mathcal{G}$-subnormal in G and $H \leq U \leq G$, then H is $\mathcal{G}$-subnormal in U.*

THEOREM 2.7 ([5, Theorem 3.5], [3, Corollary 4.10]). *Let $\mathcal{G}$ be a subgroup-closed saturated formation. The following statements are equivalent:*

(i) *$\mathcal{G}$ is a lattice formation.*
(ii) *The set of all $\mathcal{G}$-subnormal subgroups is a lattice in every group.*
(iii) *The set of all $\mathcal{G}$-Dnormal subgroups is a lattice in every group.*

A previous result to our development of $\mathcal{F}$-Fitting classes is the following.

THEOREM 2.8 ([5, Theorem 4.1 and Theorem 4.5]). *Let $\mathcal{F}$ be a lattice formation.*

(1) *If H and K are $\mathcal{F}$-subnormal $\mathcal{F}$-subgroups of a group G, then $\langle H, K\rangle \in \mathcal{F}$.*
(2) *If $\mathcal{N} \subseteq \mathcal{F}$, then the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of G has the form*

$$G_{\mathcal{F}} = \langle X \in \mathcal{F} : X \text{ is } \mathcal{F}\text{-subnormal in } G\rangle.$$

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(3) *If* $\mathcal{N} \subseteq \mathcal{F}$, *V is an* $\mathcal{F}$*-injector of G and H is an* $\mathcal{F}$*-subnormal subgroup of G, then* $V \cap H$ *is an* $\mathcal{F}$*-injector of H. (For the description of the* $\mathcal{F}$*-injectors see* [15, Theorem 2.1.1].)

In fact, these properties characterize lattice formations (see [7, Theorem 1]). The following result will be needed in the sequel.

LEMMA 2.9 ([3, Lemma 4.1]). *Let* $\mathcal{F}$ *be a lattice formation and let H and K be* $\mathcal{F}$*-subnormal subgroups of a group* $G = \langle H, K \rangle$. *Then*

$$G^{F(p)} = \langle H^{F(p)}, K^{F(p)} \rangle, \quad \textit{for every } p \in \mathrm{char}(\mathcal{F}).$$

We introduce next some concepts and results needed in Section 4.

DEFINITION 2.10 ([14, Definition], [16, Definition 5.8]). A subgroup H of a group G is said to be $\mathcal{G}$-abnormal in G if every link in every maximal chain joining H to G is $\mathcal{G}$-abnormal; that is, H is a $\mathcal{G}$-abnormal subgroup of G if, whenever $H \leq M < L \leq G$ and M is a maximal subgroup of L, then M is a $\mathcal{G}$-abnormal subgroup of L. We write H $\mathcal{G}$-abn G.

DEFINITION 2.11 ([12, III, Definition 3.2]). Let $\mathcal{X}$ be a class of groups. A subgroup U of a group G is called an $\mathcal{X}$-projector of G if UK/K is $\mathcal{X}$-maximal in G/K, for all $K \trianglelefteq G$.

For a saturated formation $\mathcal{G}$, it is well known that $\mathcal{G}$-projectors and $\mathcal{G}$-covering subgroups coincides. In particular, if U is a $\mathcal{G}$-projector of G, then U is a $\mathcal{G}$-projector of L, for every subgroup L of G containing U.

LEMMA 2.12 ([12, IV, Theorem 5.18]). *Let G be a group whose* $\mathcal{G}$*-residual* $G^{\mathcal{G}}$ *is abelian. Then* $G^{\mathcal{G}}$ *is complemented in G and any two complements in G of* $G^{\mathcal{G}}$ *are conjugate. The complements are the* $\mathcal{G}$*-projectors of G.*

As a consequence, the following result can be easily deduced.

COROLLARY 2.13. *If H is a* $\mathcal{G}$*-projector of a group G and* $H \leq U \leq G$, *then* $H \cap U^{\mathcal{G}} \leq (U^{\mathcal{G}})'$.

THEOREM 2.14 ([11, Lemma 5.1], [16, Satz 5.22]). *Let H be a subgroup of a group G. Then H is a* $\mathcal{G}$*-projector of G if and only if* $H \in \mathcal{G}$ *and H is* $\mathcal{G}$*-abnormal in G.*

3. $\mathcal{F}$-Fitting classes

DEFINITION 3.1. Let $\mathcal{F}$ be a lattice formation containing $\mathcal{N}$. A class $\mathcal{X}$ ($\neq \emptyset$) of groups is called an $\mathcal{F}$-Fitting class if the following conditions are satisfied:

(i) If $G \in \mathcal{X}$ and H $\mathcal{F}$-sn G, then $H \in \mathcal{X}$.

(ii) If H, K $\mathcal{F}$-sn $G = \langle H, K\rangle$ with H and K in $\mathcal{X}$, then $G \in \mathcal{X}$.

REMARK 3.2. (1) $\mathcal{X}$ is a Fitting class if and only if $\mathcal{X}$ is an $\mathcal{N}$-Fitting class.
(2) Let $\mathcal{N} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are lattice formations. If $\mathcal{X}$ is an $\mathcal{F}_2$-Fitting class, then $\mathcal{X}$ is an $\mathcal{F}_1$-Fitting class. (Notice that the $\mathcal{F}_1$-subnormal subgroups of a group are $\mathcal{F}_2$-subnormal in the group.) In particular, $\mathcal{X}$ is a Fitting class.
(3) For a class $\mathcal{X}$ of groups and a lattice formation $\mathcal{F}$ containing $\mathcal{N}$, we define:

$$\mathsf{S}_{n,\mathcal{F}}(\mathcal{X}) = (G : G\,\mathcal{F}\text{-sn } H \text{ for some } H \in \mathcal{X});$$
$$\mathsf{N}_{0,\mathcal{F}}(\mathcal{X}) = (G : \exists\, K_i\ \mathcal{F}\text{-sn } G\ (i = 1, \dots, r) \text{ with } K_i \in \mathcal{X} \text{ and } G = \langle K_1, \dots, K_r\rangle).$$

A routine computation shows that $\mathsf{S}_{n,\mathcal{F}}$ and $\mathsf{N}_{0,\mathcal{F}}$ are closure operations.

Obviously the $\mathcal{F}$-Fitting classes are the classes of groups which are both $\mathsf{S}_{n,\mathcal{F}}$- and $\mathsf{N}_{0,\mathcal{F}}$-closed. Thus, $\mathcal{X}$ is an $\mathcal{F}$-Fitting class exactly if $\langle \mathsf{S}_{n,\mathcal{F}}, \mathsf{N}_{0,\mathcal{F}}\rangle\mathcal{X} = \mathcal{X}$. (For details about closure operations see [12, II].)

Henceforth we will moreover assume that the lattice formation $\mathcal{F}$ contains $\mathcal{N}$.

PROPOSITION 3.3. *A class $\mathcal{X}$ ($\neq \emptyset$) is an $\mathcal{F}$-Fitting class if and only if the following two conditions are satisfisfied*:

(i′) *If $G \in \mathcal{X}$ and H $\mathcal{F}$-Dn G, then $H \in \mathcal{X}$.*

(ii′) *If H, K $\mathcal{F}$-Dn $G = \langle H, K\rangle$ with H and K in $\mathcal{X}$, then $G \in \mathcal{X}$.*

PROOF. If $\mathcal{X}$ is an $\mathcal{F}$-Fitting class, it is clear that $\mathcal{X}$ satisfies (i′) and (ii′) because $\mathcal{F}$-Dnormal subgroups are $\mathcal{F}$-subnormal subgroups by Remark 2.5.

Assume now that $\mathcal{X}$ satisfies (i′) and (ii′).

Let $G \in \mathcal{X}$ and H $\mathcal{F}$-sn G. By Remark 2.5 there exists a chain of subgroups $H = H_n \leq H_{n-1} \leq \cdots \leq H_0 = G$ with H_{i+1} $\mathcal{F}$-Dn H_i, for every $i = 0, \dots, n-1$. Then (i′) implies that $H \in \mathcal{X}$.

Assume that condition (ii) in the definition of $\mathcal{F}$-Fitting class is not true and take a group G of minimal order among the groups which do not belong to $\mathcal{X}$ but are generated by two $\mathcal{F}$-subnormal subgroups in $\mathcal{X}$. Among the pairs (A, B) of subgroups of G such that A, B $\mathcal{F}$-sn $G = \langle A, B\rangle$ and $A, B \in \mathcal{X}$, choose a pair (H, K) with $|H| + |K|$ maximum.

If H and K are normal in G, then $G \in \mathscr{X}$ by the hypothesis. So we can assume that H is not normal in G.

Note that $G = \langle H, H^g \rangle$, for every $g \in G \setminus N_G(H)$. Otherwise there exists $g \in G \setminus N_G(H)$ such that $\langle H, H^g \rangle < G$. By the choice of G, it follows that $\langle H, H^g \rangle \in \mathscr{X}$. But this contradicts the choice of the pair (H, K) since $\langle H, H^g \rangle$ is also $\mathscr{F}$-subnormal in G.

By the hypothesis we can assume that $H < M$, for some $\mathscr{F}$-normal maximal subgroup M of G. Clearly $H \trianglelefteq M$ and so $H \leq M_{\mathscr{X}}$. Again the choice of the pair (H, K) implies that $H = M_{\mathscr{X}}$.

We claim that $H = M_{\mathscr{X}}$ is $\mathscr{F}$-Dnormal in G, which provides the final contradiction, since $G = \langle H, H^g \rangle$ with $g \in G \setminus N_G(H)$.

If $p \,|\, |G : M|$, then $G^{F(p)} \leq M$ because M is $\mathscr{F}$-normal in G. Moreover $G^{F(p)} = \langle H^{F(p)}, (H^g)^{F(p)} \rangle$ by Lemma 2.9, and so $G^{F(p)} \in \mathscr{X}$ by the choice of G, that is, $G^{F(p)} \leq M_{\mathscr{X}}$. Since $G^{F(p)} = O^{\pi(p)}(G) = \langle G_q : G_q \in \mathrm{Syl}_q(G),\ q \notin \pi(p) \rangle$, it is clear that $G^{F(p)} = (M_{\mathscr{X}})^{F(p)} = H^{F(p)}$.

In particular, $\sigma(|G : H|) \subseteq \pi(p)$ and clearly H is $\mathscr{F}$-Dnormal in G. □

PROPOSITION 3.4. *Let $\mathscr{X}$ be an $\mathscr{F}$-Fitting class and let G be a group. Then*:

(a) *$\mathscr{X}$ is a Fitting class and $G_{\mathscr{X}} = \langle H \leq G : H\ \mathscr{F}\text{-sn}\ G,\ H \in \mathscr{X} \rangle = \langle H \leq G : H\ \mathscr{F}\text{-Dn}\ G, H \in \mathscr{X} \rangle$.*

(b) *If H is an $\mathscr{F}$-subnormal subgroup of G, then $H_{\mathscr{X}} = H \cap G_{\mathscr{X}}$.*

PROOF. (a) Since $\mathscr{X}$ is an $\mathscr{F}$-Fitting class, the result is clear taking into account Remark 2.5 (3) and Remark 2.5 (1).

(b) Obviously $H_{\mathscr{X}} \leq H \cap G_{\mathscr{X}}$. But $H \cap G_{\mathscr{X}}$ is $\mathscr{F}$-subnormal in G, then $H \cap G_{\mathscr{X}}$ is also $\mathscr{F}$-subnormal in both H and $G_{\mathscr{X}}$ by Lemma 2.6. The result is now clear because $\mathscr{X}$ is an $\mathscr{F}$-Fitting class and statement (a). □

REMARK 3.5. In [6] the following stronger definition of $\mathscr{G}$-normality, for a saturated formation $\mathscr{G}$, was introduced.

DEFINITION ([6, Definition 3.1′]). A subgroup H of a group G is said to be $\mathscr{G}$-normal in G if either $H = G$ or $H/\mathrm{Core}_G(H) \in \underline{g}(p)$, for every prime $p \in \pi(|G : H|)$.

The subgroup-closed saturated formations which provide lattice properties for these $\mathscr{G}$-normal subgroups differs in general of the lattice formations (see [6]).

But some remarks should be done:

(1) The $\mathscr{G}$-normal subgroups are $\mathscr{G}$-Dnormal subgroups. The converse is not true (see [3, Remark 3.2 (6)]).

(2) Remark 2.5 (1) is also true if $\mathscr{G}$-Dnormal is changed by $\mathscr{G}$-normal. In particular, for maximal subgroups, $\mathscr{G}$-normality and $\mathscr{G}$-Dnormality coincides.

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(3) If $\mathcal{N} \subseteq \mathcal{G}$, normal subgroups are also $\mathcal{G}$-normal.
(4) Propositions 3.3 and 3.4 are also true if we change $\mathcal{G}$-Dnormality by $\mathcal{G}$-normality.

If $\mathcal{X}$ is a Fitting class with characteristic π, then $\mathcal{N}_\pi \subseteq \mathcal{X}$. The corresponding result for $\mathcal{F}$-Fitting class is the following:

PROPOSITION 3.6. *If $\mathcal{X}$ is an $\mathcal{F}$-Fitting class with* char$(\mathcal{X})=\pi$, *then* $\mathcal{F} \cap \mathcal{S}_\pi \subseteq \mathcal{X}$.

PROOF. Suppose that the result is not true and let G be a group of minimal order in $(\mathcal{F} \cap \mathcal{S}_\pi) \setminus \mathcal{X}$. Since G belongs to $\mathcal{F}$, every maximal subgroup of G is $\mathcal{F}$-normal. By the choice of G, there is a unique maximal subgroup of G. This implies that G is a cyclic p-group, for some $p \in \pi$. Then $G \in \mathcal{X}$, which contradicts the choice of G. □

REMARK 3.7. (1) In particular, if $\mathcal{X}$ is an $\mathcal{F}$-Fitting class and $\mathcal{N} \subseteq \mathcal{X}$, then $\mathcal{F} \subseteq \mathcal{X}$.
(2) There exists Fitting classes which are not $\mathcal{F}$-Fitting classes for any lattice formation $\mathcal{F}$ containing properly $\mathcal{N}$. The class of all metanilpotent groups $\mathcal{N}^2$ is an example. To see this notice that the minimal local definition of $\mathcal{N}^2$, as saturated formation, is the formation function $\underline{g}$ defined by

$$\underline{g}(p) = \mathrm{Q}(G/O_{p'p}(G) : G \in \mathcal{N}^2) = \mathcal{N}_{p'},$$

for every prime p, (see [12, IV, Proposition 3.10]). If $\mathcal{F}$ is a lattice formation such that $\mathcal{F} \subseteq \mathcal{N}^2$, then $\underline{f}(p) \subseteq \underline{g}(p)$ for every prime p, by [12, IV, Proposition 3.11]. But this implies that $\mathcal{F} = \mathcal{N}$.
A different example with a Fitting class $\mathcal{X}$, containing a lattice formation $\mathcal{F}$, such that $\mathcal{N} \subset \mathcal{F}$, is given below after (3).
(3) Let $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{G}$ be lattice formations. Note that in this case $\mathcal{F}$-subnormal subgroups are $\mathcal{G}$-subnormal subgroups. Then Theorem 2.8 tells in particular that $\mathcal{G}$ is an $\mathcal{F}$-Fitting class.

We wonder which type of formations, related to the class of nilpotent groups and to lattice formations, satisfy the property stated in Remark 3.7 (3). In [4, 9, 10] the following formations were taken into consideration:

Let $\mathcal{G} = LF(g)$ be the saturated formation locally defined by the formation function g given by $g(p) = \mathcal{S}_{\sigma(p)}$, for some $\sigma(p) \subseteq \mathbb{P}$ such that $p \in \sigma(p)$, if $p \in \pi = $ char$(\mathcal{G})$, and $g(q) = \emptyset$, if $q \notin \pi$.

If $\mathcal{N} \subseteq \mathcal{F} \subseteq \mathcal{G}$, it is not true in general that $\mathcal{G}$ is an $\mathcal{F}$-Fitting class. Take for instance $\mathcal{F} = LF(f)$ locally defined by $F(2) = F(3) = \mathcal{S}_{\{2,3\}}$ and $F(q) = \mathcal{S}_q$, for every prime $q \neq 2, 3$, and $\mathcal{G} = LF(g)$ locally defined by $g(2) = \mathcal{S}_{\{2,3\}}$, $g(3) = \mathcal{S}_{\{2,3,5\}}$, $g(5) = \mathcal{S}_{\{3,5\}}$ and $g(q) = \mathcal{S}_q$, for every prime $q \neq 2, 3, 5$. (Notice that $\mathcal{N}$

and $\mathscr{F}$ are the unique lattice formations contained in $\mathscr{G}$.) Then $\mathscr{G}$ is not an $\mathscr{F}$-Fitting class. To see this consider the primitive group $[V_2]Z_3$. By [12, B, Corollary 10.7] this group has an irreducible and faithful module V_5 over $\mathbb{F}_5$. Let $G = [V_5]([V_2]Z_3)$. Then $H = V_5Z_3$ is $\mathscr{F}$-subnormal in G and $H \in \mathscr{G}$, but $G = \langle H, H^x \rangle$, for $1 \neq x \in V_2$, and $G \notin \mathscr{G}$.

With some restrictions on the sets of primes $\sigma(p)$ which define $\mathscr{G}$, it is possible to obtain a stronger form of above-mentioned property. The formations which appear were also studied in [8] with full characteristic.

LEMMA 3.8. *Let $\mathscr{G}$ be a saturated formation with* char($\mathscr{G}$) $= \pi \subseteq \mathbb{P}$, *locally defined by the formation function g given by $g(p) = \mathscr{S}_{\sigma(p)}$, for some $\sigma(p) \subseteq \mathbb{P}$ such that $p \in \sigma(p)$, if $p \in \pi$, and $g(q) = \emptyset$, if $q \notin \pi$. (Notice that we can assume without loss of generality that $\sigma(p) \subseteq \pi$.)*

Assume also that the following property holds: if $q \in \sigma(p)$, then $\sigma(q) \subseteq \sigma(p)$, for every pair of prime numbers $p, q \in \pi$. Then $G \in \mathscr{G}$ if and only if $G \in \mathscr{S}_\pi$ and G has a normal Hall $\sigma(p)'$-subgroup for every prime number p.

PROOF. Take $\mathscr{G}_1$, the saturated formation locally defined by the formation function g_1, given by $g_1(p) = g(p) = \mathscr{S}_{\sigma(p)}$, if $p \in \pi$, and $g_1(q) = \mathscr{S}_{\pi'}$, if $q \notin \pi$.

It is clear that $G \in \mathscr{G}$ if and only if $G \in \mathscr{G}_1 \cap \mathscr{S}_\pi$. By [8, Remark] we know that $G \in \mathscr{G}_1$ if and only if G has a normal Hall $\sigma(p)'$-subgroup, for every prime number $p \in \pi$, and a normal Hall π-subgroup. Now the result is easily deduced. □

THEOREM 3.9. *Let $\mathscr{G} = LF(g)$ be a saturated formation with* char($\mathscr{G}$) $= \pi$ *as in* Lemma 3.8. *Let $\mathscr{F}$ be a lattice formation containing $\mathscr{N}$. The following statements are equivalent*:

(i) *$\mathscr{G}$ is an $\mathscr{F}$-Fitting class.*
(ii) *$F(p) \subseteq \mathscr{S}_{\sigma(p)}$, for every $p \in \pi$.*
(iii) *$F(p) \subseteq G(p)$, for every $p \in \pi$.*

If $\mathscr{N} \subseteq \mathscr{G}$, they are also equivalent to $\mathscr{F} \subseteq \mathscr{G}$.

PROOF. It is not difficult to prove that (ii) is equivalent to (iii) taking into account that $G(p) = \mathscr{S}_{\sigma(p)} \cap \mathscr{G}$, for every $p \in \pi$, (see [12, IV, Proposition 3.8]).

Assume that (i) is true and take $p \in \pi$. If $\underline{f}(p) = (1)$, then $F(p) = \mathscr{S}_p \subseteq \mathscr{S}_{\sigma(p)}$. Otherwise, $F(p) = \underline{f}(p) = \mathscr{S}_{\pi(p)}$. Let $p \neq r \in \pi(p)$ and take $G = [V_r]Z_p$, with V_r an irreducible and faithful Z_p-module over $\mathbb{F}_r$. Z_p is an $\mathscr{F}$-subnormal $\mathscr{G}$-subgroup of G. By hypothesis, $G \in \mathscr{G}$. In particular, $r \in \pi$.

Now a similar primitive group $[V_p]Z_r$ belongs also to $\mathscr{G}$, which implies that $r \in \sigma(p)$.

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We prove next that (ii) implies (i). Notice first that $\mathcal{G}$ is subgroup-closed. We claim that $\mathsf{N}_{0,\mathcal{F}}(\mathcal{G}) = \mathcal{G}$. Assume that this is not true and take a group G of minimal order among the groups which do not belong to $\mathcal{G}$ but are generated by two $\mathcal{F}$-subnormal subgroups in $\mathcal{G}$. Among the pairs (A, B) of subgroups of G such that A, B $\mathcal{F}$-sn $G = \langle A, B\rangle$ and $A, B \in \mathcal{G}$, choose a pair (H, K) with $|H| + |K|$ maximum.

Since $\mathcal{G}$ is a Fitting class, we can assume without loss of generality that H is not normal in G. By the choice of G and the choice of the pair (H, K), we can deduce that $G = \langle H, H^g\rangle$, for every $g \in G \setminus N_G(H)$. This implies that $M = N_G(H)$ is the unique maximal subgroup of G containing H. Since H is $\mathcal{F}$-subnormal in G, then M is $\mathcal{F}$-normal in G. Again the choice of H implies that $H = M_{\mathcal{G}}$. Arguing as in the proof of Proposition 3.3, we deduce that $G^{F(p)} \leq H$, if $p \in \sigma(|G : M|)$.

Since G does not belong to $\mathcal{G}$, the hypothesis implies that $1 \neq G^{F(p)}$. Then H contains a minimal normal subgroup N of G.

By the choice of G, it is clear that $G/N \in \mathcal{G}$. Since $\mathcal{G}$ is a saturated formation, G is a primitive group and N is the unique minimal normal subgroup of G.

If N is a q-group, for some prime q, then H is a $\sigma(q)$-group. Otherwise, since $H \in \mathcal{G}$, we know by Lemma 3.8 that H has a normal Hall $\sigma(q)'$-subgroup, which centralizes N, a contradiction. Consequently, $H/G^{F(p)} \in \mathcal{S}_{\sigma(q)} \cap \mathcal{S}_{\pi(p)}$.

Assume that there exists $r \in \sigma(q) \cap \pi(p) \subseteq \pi$. By the hypothesis $\pi(p) = \pi(r) \subseteq \sigma(r) \subseteq \sigma(q)$. This implies that G is a $\sigma(q)$-group. Since N is a q-group and $G/N \in \mathcal{G}$, it follows that $G \in \mathcal{G}$ a contradiction.

If $\sigma(q) \cap \pi(p)$ is empty, then $H = G^{F(p)}$, but this is not possible because H is not normal in G and we are done. □

REMARK 3.10. Lattice formations and also the class of p-nilpotent groups, for every prime p, are particular examples of the formations $\mathcal{G}$ considered in Theorem 3.9. In particular, this theorem and Proposition 3.4 (a) improve Theorem 2.8, parts (1) and (2).

We show next some more examples of $\mathcal{F}$-Fitting classes of a different nature.

EXAMPLE I. Consider the normal Fitting class

$$\mathcal{D} = \mathcal{D}(\{3\}) = \Big(G \in \mathcal{S} : \textstyle\prod_{i=1}^{n} \det(g \text{ on } M_i) = 1, \text{ for all } g \in G, \text{ where the product is taken over the 3-chief factors } M_1, \dots, M_n \text{ of a given chief series of } G\Big)$$

(see [12, IX, Example 2.14 (b)]). Let $\mathcal{F}$ be a lattice formation containing $\mathcal{N}$. Then:

(1) $\mathcal{F} \subseteq \mathcal{D}$ if and only if $\pi(2) \neq \pi(3)$.

PROOF. If $\mathcal{F} \subseteq \mathcal{D}$, it is obvious that $\pi(2) \neq \pi(3)$. The converse is also clear because of the structure of $\mathcal{F}$-groups; see Lemma 2.1. □

(2) If $\mathscr{F} \subseteq \mathscr{D}$, then $\mathscr{D}$ is an $\mathscr{F}$-Fitting class.

PROOF. $\mathsf{S}_{n,\mathscr{F}}(\mathscr{D}) = \mathscr{D}$. Let G be a group in $\mathscr{D}$ and H an $\mathscr{F}$-normal maximal subgroup of G. It is enough to prove that $H \in \mathscr{D}$. Since H is $\mathscr{F}$-normal, $H^{F(p)} \trianglelefteq G$, if $p \in \sigma(|G : H|)$, in particular $H^{F(p)} \in \mathscr{D}$. If H would not belong to $\mathscr{D}$, then $|H : H_{\mathscr{D}}| = 2$. Since $H^{F(p)} \leq H_{\mathscr{D}}$, we would have $2 \in \pi(p)$, and so $3 \notin \pi(p)$ by (1). Consequently $H^{F(p)}$ covers every 3-chief factor of G. Consider now a chief series of G through $H^{F(p)}$, take the intersection with H and refine it to a chief series of H. An easy computation shows that $H \in \mathscr{D}$.

$\mathsf{N}_{0,\mathscr{F}}(\mathscr{D}) = \mathscr{D}$. Assume that the result is not true and take a group $G \notin \mathscr{D}$ and a pair of subgroups (H, K) as in the proof of Theorem 3.9. Arguing as in that proof we deduce from this choice the following facts: we can assume, without loss of generality, that H is not normal in G, there is a unique maximal subgroup M of G containing $H = M_{\mathscr{D}}$ and $G^{F(p)} \leq H$, if $p \in \sigma(|G : M|)$. Moreover, $G^{F(p)} \leq G_{\mathscr{D}}$. Then $2 \in \pi(p)$, because $|G : G_{\mathscr{D}}| = 2$. Consequently $3 \notin \pi(p)$. This implies that $G^{F(p)}$ covers every 3-chief factor of G. But $G = HG_{\mathscr{D}}$ because otherwise $M_{\mathscr{D}} = H \leq G_{\mathscr{D}} = M$ which would imply $H \trianglelefteq G$, a contradiction. By a computation as above if follows that $G \in \mathscr{D}$, which provides the final contradiction. □

EXAMPLE II. Consider the dominant Fitting class

$$\mathscr{D}^{\pi} = (G \in \mathscr{S} : G/C_G(O_{\pi}(G)) \in \mathscr{S}_{\pi})$$

for a set of primes π (see [12, IX, Example 2.5 (b) and Theorem 4.16]). Let $\mathscr{F}$ be a lattice formation with $\mathscr{N} \subseteq \mathscr{F}$. Then:

(1) $\mathscr{F} \subseteq \mathscr{D}^{\pi}$ if and only if $\pi = \bigcup_{p\in\pi} \pi(p)$.

PROOF. Assume that $\mathscr{F} \subseteq \mathscr{D}^{\pi}$. It is clear that $\pi \subseteq \bigcup_{p\in\pi} \pi(p)$. Assume that there is $r \in \pi(p) \setminus \pi$ for some $p \in \pi$. Then the primitive group $[V_p]Z_r$ belongs to $\mathscr{F}$ but does not belong to $\mathscr{D}^{\pi}$, a contradiction. Then $\pi = \bigcup_{p\in\pi} \pi(p)$. The converse is clear taking into account the structure of $\mathscr{F}$-groups; see Lemma 2.1. □

(2) If $\mathscr{F} \subseteq \mathscr{D}^{\pi}$, then $\mathscr{D}^{\pi}$ is an $\mathscr{F}$-Fitting class.

PROOF. $\mathsf{S}_{n,\mathscr{F}}(\mathscr{D}^{\pi}) = \mathscr{D}^{\pi}$. Let H be an $\mathscr{F}$-normal maximal subgroup of a group G in $\mathscr{D}^{\pi}$. It is enough to prove that $H \in \mathscr{D}^{\pi}$ in order to obtain the result. If $\{p\} = \sigma(|G : H|)$, then $H^{F(p)} \trianglelefteq G$ because H is $\mathscr{F}$-normal. In particular, $H^{F(p)} \in \mathscr{D}^{\pi}$. Distinguish the following cases:

(a) $\pi(p) \subseteq \pi$. In this case $O^{\pi}(G) \leq H^{F(p)} \cap C_G(O_{\pi}(G))$, because $G \in \mathscr{D}^{\pi}$. Notice that $O_{\pi}(G) \cap H = O_{\pi}(H)$, because every Hall π-subgroup of G reduces in H. Then $O^{\pi}(H) \leq O^{\pi}(G) \leq C_H(O_{\pi}(G)) \leq C_H(O_{\pi}(H))$, that is, $H \in \mathscr{D}^{\pi}$.

(b) $\pi(p) \not\subseteq \pi$. In this case $O^{\pi'}(G) \leq H^{F(p)} \leq H$, which implies, $O^{\pi'}(G) = O^{\pi'}(H)$ and so $O_\pi(G) = O_\pi(H)$. Since $G \in \mathcal{D}^\pi$, we have $O^\pi(H) \leq O^\pi(G) \leq C_G(O_\pi(G)) = C_G(O_\pi(H))$. This means that $H \in \mathcal{D}^\pi$.

$\mathrm{N}_{0,\mathcal{F}}\mathcal{D}^\pi = \mathcal{D}^\pi$. Assume that the result is not true and take a group $G \notin \mathcal{D}^\pi$ and a pair of subgroups (H, K) as in the proof of Proposition 3.9. With the usual arguments of this proof, we can assume, without loss of generality, that H is not normal in G and $G = \langle H, H^g \rangle$, for every $g \in G \setminus N_G(H)$. In particular, there is a unique maximal subgroup M of G containing $H = M_{\mathcal{D}^\pi}$ and $G^{F(p)} \leq G_{\mathcal{D}^\pi}$, if $p \in \sigma(|G : M|)$.

If $H < O_\pi(G)H < G$, then $O_\pi(G)H$ is an $\mathcal{F}$-subnormal $\mathcal{D}^\pi$-subgroup of G. But this contradicts the choice of the pair (H, K).

Assume that $G = O_\pi(G)H$. In this case, $p \in \pi$ and so $\pi(p) \subseteq \pi$. Consequently, if G_π denotes a Hall π-subgroup of G, we have $G = G^{F(p)}G_\pi \leq G_{\mathcal{D}^\pi}G_\pi = C_G(O_\pi(G))G_{\pi_i} \in \mathrm{Inj}_{\mathcal{D}^\pi}(G)$ by [12, IX, Theorem 4.16], that is $G \in \mathcal{D}^\pi$, a contradiction.

Consider now the case $O_\pi(G) \leq H$. Since $H \in \mathcal{D}^\pi$, by [12, IX, Theorem 4.16] it follows that H is contained in a $\mathcal{D}^\pi$-injector I of G. But $I = C_G(O_\pi(G))G_\pi$, for some Hall π-subgroup G_π of G. By the choice of G, $I < G$. Then $p \notin \pi$ and so $\pi \subseteq \pi(p)'$. Since M is $\mathcal{F}$-normal in G, it is clear that $G^{F(p)} \leq M$. In this case, this implies that M contains every Hall π-subgroup of G. Moreover $I \leq M$. Consequently, if $g \in G \setminus M$, we have $G = \langle H, H^g \rangle \leq \langle I, I^g \rangle \leq M$, which provides the final contradiction. □

The following results are proved with the similar arguments to those used for the corresponding classical results, with obvious changes (see [12, IX, Theorem 1.12 (a) and Lemma 1.13]).

PROPOSITION 3.11. (a) *If $\mathcal{H}$ and $\mathcal{X}$ are two $\mathcal{F}$-Fitting classes, then $\mathcal{H} \diamond \mathcal{F}$ is an $\mathcal{F}$-Fitting class.*
(b) (Quasi-R_0-lemma) *Let N_1 and N_2 be normal subgroups of a group G such that $N_1 \cap N_2 = 1$ and G/N_1N_2 is $\mathcal{F}$-group, and let $\mathcal{X}$ be an $\mathcal{F}$-Fitting class containing G/N_1. Then $G \in \mathcal{X}$ if and only if $G/N_2 \in \mathcal{X}$.*

4. A characterisation of $\mathcal{G}$-projectors

Let $\mathcal{G}$ be a saturated formation, G a group and H a subgroup of G. It is obvious that the following statements are equivalent:

(i) Whenever H $\mathcal{G}$-Dn $T \leq G$, then $H = T$.
(ii) Whenever H $\mathcal{G}$-sn $T \leq G$, then $H = T$.
(iii) If H is a $\mathcal{G}$-normal maximal subgroup of $T \leq G$, then $H = T$.

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In this case, the subgroup H is said to be self-$\mathscr{G}$-normalizing in G.

We provide in Theorem 4.2 a characterisation of the $\mathscr{G}$-projectors, for a subgroup-closed saturated formation $\mathscr{G}$. It is an extension of the characterisation of the $\mathscr{N}$-projectors as the Carter subgroups. Proposition 4.1 tells that some additional condition should be satisfied by a self-$\mathscr{G}$-normalizing $\mathscr{G}$-subgroup to be a $\mathscr{G}$-projector. The proposed required condition is motivated by Corollary 2.13. Some related results were obtained by Carter and Hawkes in [11] (see Theorem 2.14) and by Graddon in [14, Theorem 2.15].

PROPOSITION 4.1. *Let $\mathscr{F}$ be a lattice formation containing $\mathscr{N}$. The following statements are equivalent*:

(i) *Either $\mathscr{F} = \mathscr{N}$ or $\mathscr{F} = \mathscr{S}$.*

(ii) *In every group G, the $\mathscr{F}$-projectors of G are exactly the self-$\mathscr{F}$-normalizing $\mathscr{F}$-subgroups of G.*

PROOF. It is clear that (i) implies (ii).

Assume that statement (ii) holds. If $\mathscr{F} \neq \mathscr{N}$, there exists a prime p such that the corresponding set of primes $\pi(p)$ defining $\mathscr{F}$ satisfies $|\pi(p)| \geq 2$. Take $p \neq q \in \pi(p)$. If $\mathscr{F} \neq \mathscr{S}$, there exists a prime $r \in \pi(p)'$. Consider the primitive group $X = [V_p]Z_q$. By [12, B, Corollary 11.7], X possesses an irreducible and faithful module V_r over $\mathbb{F}_r$ such that $[V_r, Z_q] < V_r$. Then $V_r = [V_r, Z_q] \times C_{V_r}(Z_q)$, with $1 \neq [V_r, Z_q] < V_r$, by [12, A, Proposition 12.5]. Take $G = [V_r]X$ the corresponding semidirect product. Consider the $\mathscr{F}$-subgroup $H = C_{V_r}(Z_q)Z_q$. We claim that H is self-$\mathscr{F}$-normalizing in G. Notice that the unique maximal subgroup of G containing H is V_rZ_q. If H were $\mathscr{F}$-Dnormal in some subgroup T containing H properly, then $r \in \sigma(|T : H|)$. Moreover the Sylow r-subgroup T_r of T would verify $[T_r, Z_q] \leq H \cap [V_r, Z_q] = 1$. This would imply that $T_r \leq C_{V_r}(Z_q) \leq H$. But this contradicts $r \in \sigma(|T : H|)$. Therefore, H is a self-$\mathscr{F}$-normalizing $\mathscr{F}$-subgroup of G, but H is not an $\mathscr{F}$-projector of G. This contradicts statement (ii) and concludes the proof. □

THEOREM 4.2. *Let $\mathscr{G}$ be a subgroup-closed saturated formation. For a subgroup H of a group G, the following statements are equivalent*:

(a) *H is a $\mathscr{G}$-projector of G;*

(b) *H is a self-$\mathscr{G}$-normalizing $\mathscr{G}$-subgroup of G and H satisfies the following property*:

$$(*) \qquad \text{If } H \leq K \leq G, \text{ then } H \cap K^{\mathscr{G}} \leq (K^{\mathscr{G}})'.$$

PROOF. If H is a $\mathscr{G}$-projector of G, then H is a self-$\mathscr{G}$-normalizing $\mathscr{G}$-subgroup

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of G by Theorem 2.14. Moreover, H is also a $\mathscr{G}$-projector in every subgroup K of G containing H. Then statement (2) is clear by Corollary 2.13.

Conversely, suppose that statement (2) holds. We observe first that H is a $\mathscr{G}$-maximal subgroup of G. We use induction on $|G|$. Then we may assume that H is a $\mathscr{G}$-projector of every proper subgroup of G containing H.

If H were a maximal subgroup of G, then H would be a $\mathscr{G}$-projector of G by Theorem 2.14 and we would be done.

Let M be a maximal subgroup of G containing H.

Suppose that M is $\mathscr{G}$-abnormal in G. By [12, V, Lemma 3.4] there exists a $\mathscr{G}$-normalizer D of G, and a $\mathscr{G}$-normalizer D_1 of M such that $D \leq D_1$. Since H is a $\mathscr{G}$-projector of M, we may assume by [12, V, Theorem 4.1] and by the conjugacy of the $\mathscr{G}$-normalizers, that $D \leq D_1 \leq H$. We claim that H is $\mathscr{G}$-abnormal in G. For any maximal subgroup L of G containing H, we have that H is $\mathscr{G}$-abnormal in L by Theorem 2.14 because H is a $\mathscr{G}$-projector of L. But $D \leq H \leq L$, then [12, V. Lemma 3.4] implies that L is $\mathscr{G}$-abnormal in G. This means that H is $\mathscr{G}$-abnormal in G. Then H is a $\mathscr{G}$-projector of G by Theorem 2.14.

Consequently, we can suppose that every maximal subgroup of G containing H is $\mathscr{G}$-normal in G.

We split the rest of the proof into the following steps:

Step 1. *$M = HG^{\mathscr{G}}$. In particular, M is the unique maximal subgroup of G containing H.*

Since $G^{\mathscr{G}} \leq M$, the result is clear because H is a $\mathscr{G}$-projector of M.

Step 2. *We may suppose that* $\text{Core}_G(H) = 1$.

Assume that $K = \text{Core}_G(H) \neq 1$. We have that H/K is a self-$\mathscr{G}$-normalizing $\mathscr{G}$-subgroup of G/K. Moreover, if $H/K \leq T/K \leq G/K$, then

$$\begin{aligned}(H/K) \cap (T/K)^{\mathscr{G}} &= (H \cap T^{\mathscr{G}} K)/K = (H \cap T^{\mathscr{G}})K/K \\ &\leq (T^{\mathscr{G}})'K/K = ((T/K)^{\mathscr{G}})'.\end{aligned}$$

By inductive hypothesis, H/K is a $\mathscr{G}$-projector of G/K. Thus H is a $\mathscr{G}$-projector of G. Then we may suppose that $\text{Core}_G(H) = 1$.

Step 3. *$N \leq G^{\mathscr{G}}$, for every minimal normal subgroup N of G.*

Let N be a minimal normal subgroup of G. Obviously $HN < G$. Therefore, since H is a $\mathscr{G}$-projector of M, we have that $HN = HN \cap M = HN \cap HG^{\mathscr{G}} = H(N \cap G^{\mathscr{G}})$ by [12, IV, Theorem 5.4]. Thus Step 2 implies that $N \cap G^{\mathscr{G}} \neq 1$. Then $N = N \cap G^{\mathscr{G}}$, that is, $N \leq G^{\mathscr{G}}$.

Step 4. *We may suppose that for each minimal normal subgroup N of G, there exists a subgroup T of G such that HN is a $\mathscr{G}$-normal maximal subgroup of T. Otherwise H is a $\mathscr{G}$-projector of G.*

Let N be a minimal normal subgroup of G and assume that HN/N is self-$\mathscr{G}$-normalizing in G/N. Moreover, $HN/N \in \mathscr{G}$. We claim that HN/N verifies (∗) in

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G/N. Consider $HN/N \le L/N \le G/N$.

If $L < G$, then H is a $\mathscr{G}$-projector of L and the result is clear by Lemma 2.12.

If $L = G$, then $(HN/N) \cap (G/N)^{\mathscr{G}} = (HN/N) \cap (G^{\mathscr{G}}/N) = (H \cap G^{\mathscr{G}})N/N \le (G^{\mathscr{G}})'N/N = ((G/N)^{\mathscr{G}})'$.

By inductive hypothesis, HN/N is a $\mathscr{G}$-projector of G/N. But H is $\mathscr{G}$-projector of $HN < G$. Consequently, it is well known that H is a $\mathscr{G}$-projector of G. Hence we may suppose that the statement of Step 4 holds.

Step 5. *$M = HN$, for every minimal normal subgroup N of G.*

Let N be a minimal normal subgroup N of G and take a subgroup T for N as in Step 4. If $T < G$, then H is a $\mathscr{G}$-projector of T, but this contradicts that HN is $\mathscr{G}$-normal in T by Theorem 2.14. Then $T = G$. But this implies that $HN = M$.

Step 6 *G is monolithic.*

If N_1 and N_2 are two minimal normal subgroups of G, then $M = HN_1 = HN_2$. Therefore, $M^{\mathscr{G}} \le N_1 \cap N_2 = 1$, that is $M \in \mathscr{G}$. This is not possible because H is $\mathscr{G}$-maximal in G.

Step 7. *The final conclusion.*

If $(G^{\mathscr{G}})' \neq 1$ and N is the unique minimal subgroup of G, we would have $G^{\mathscr{G}} = G^{\mathscr{G}} \cap M = G^{\mathscr{G}} \cap HN = (G^{\mathscr{G}} \cap H)N \le (G^{\mathscr{G}})'$, which is not possible because G is soluble. Hence $G^{\mathscr{G}} \cap H = 1$ and $G^{\mathscr{G}} = N$. In particular, $G = NR$ is a primitive group, with R a maximal subgroup of G such that $\mathrm{Core}_G(R) = 1$. Now, since H is $\mathscr{G}$-maximal in G, we can apply [12, III, Lemma 3.24] to obtain that $H = (H \cap N)(H \cap R^g)$ for some $g \in NH$. Since $H \cap N = 1$, we have that $H \le R^g$, but this is not possible by Step 1 and the proof is concluded. □

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