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On profiniteness of compact totally disconnected algebras

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The paper presents a necessary and sufficient condition for a given compact totally disconnected space C to be the projective limit of a given directed cone of epimorphisms onto finite discrete quotients of C. This problem is related to the question of when a compact totally disconnected algebra is profinite and some observations in this direction are recorded.

Introduction

The notion of a pro-object is closely related to problems in duality (see Day [2] and Hofmann [4]). The usual technique is to obtain duality on the models M and then lift this duality to the pro-M-objects.

In the present paper we reverse the above mentioned procedure and use Stone duality to deduce a necessary and sufficient condition for a given compact totally disconnected space to be a pro-M-object for a given M. The resulting observations on profiniteness of algebras are closely related to the work of Choe [1] and Numakura [6]. The method we employ is a generalisation of Numakura's method for semigroups and leads to Choe-type conditions for profiniteness.

The general references for this article are Grätzer [3] and Mac Lane [5].

1. General conditions for profiniteness

Let $K = (K, 1, \times, [-, -], \ldots)$ be the cartesian closed category of

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compactly generated Hausdorff spaces and let $\mathcal{B}$ be the category of boolean rings. Then, for each $C \in \mathcal{C}$, the category of compact totally disconnected spaces, we have an isomorphism $C \cong \mathcal{B}([C, 2], 2)$ by Stone duality which asserts that $\mathcal{B}(-, 2) : \mathcal{B}^{op} \to \mathcal{C}$ is a category equivalence.

Let $U : \mathcal{B} \to \mathcal{E}ns$ be the underlying-set functor. Then U creates filtered colimits since $\mathcal{B}$ is finitary over $\mathcal{E}ns$; U also preserves and reflects regular epimorphisms.

Now let $D : \mathcal{D} \to \mathcal{C}$ be a diagram in $\mathcal{C}$ with each $D(a)$ a finite set, and let $\rho : C \to D$ be a natural transformation each of whose components is an epimorphism. Then the aim is to find a condition for this transformation ρ to be a limit cone in $\mathcal{C}$.

THEOREM 1.1. *If* $\mathcal{D}$ *is directed then the canonical map* $\rho : C \to \lim D$ *is a monomorphism (respectively an isomorphism) if and only if the canonical map* $\operatorname{colim} U[D, 2] \to U[C, 2]$ *in* $\mathcal{E}ns$ *is a surjection (respectively a bijection).*

Proof. Consider

$$C \cong \mathcal{B}([C, 2], 2) \xrightarrow{\alpha} \lim \mathcal{B}([D, 2], 2) \cong \lim D .$$

Here ρ is a monomorphism (respectively an isomorphism) if and only if α is a monomorphism (respectively an isomorphism). But α is just the image of the canonical map

$$\beta : \operatorname{colim}[D, 2] \to [C, 2]$$

in $\mathcal{B}$ under the category equivalence $\mathcal{B}^{op} \to \mathcal{C}$. Thus ρ is a monomorphism (respectively an isomorphism) if and only if β is an epimorphism (respectively an isomorphism). But the domain of β is a filtered colimit. Thus, on considering the aforementioned properties of U, the result follows. //

COROLLARY 1.2. *The canonical map* $\rho : C \to \lim D$ *is a monomorphism if and only if each continuous map* $C \to 2$ *factors through some* e *in the cone.* //

A directed cone $\rho : C \to D$ is called *saturated* if:

(i) given any commuting diagram

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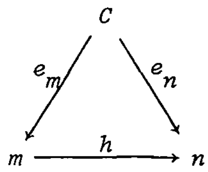
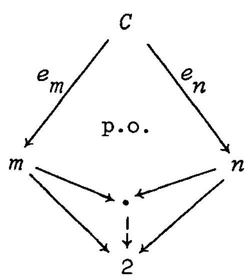


with e_m, e_n in the cone, then $h = Df$;

(ii) given any pair in the cone, their pushout in C is in the cone.

THEOREM 1.3. *The canonical map* $\rho : C \to \lim D$ *is an isomorphism if* $\rho : C \to D$ *is saturated and each map* $g : C \to 2$ *factors through some* e *in the cone.*

Proof. Firstly $\operatorname{colim} U[D, 2] \to U[C, 2]$ is a surjection (see Corollary 1.2). It is an injection since, given any two factorisations of a given $g : C \to 2$,



we can form the pushout of (e_m, e_n) in C and relate both factorisations to a third via maps in the diagram (since we are assuming $\rho : C \to D$ is saturated). //

EXAMPLE 1.4. Let $\Pi = (T, \mu, \eta)$ be a monad on *Ens* . Then we can lift this to a monad on K ; namely $\overline{T}X = \int^{Y} TY \cdot [Y, X]$. If we associate each compact totally disconnected $\overline{\Pi}$-algebra C with its set of finite quotients then we obtain a (directed) saturated cone under C . //

2. Special conditions for profiniteness

Throughout this section we will consider the situation in the preceding example in which Π is a *finitary* monad on *Ens* (this makes $\overline{\Pi}$

finitary on K). The set of *all* non-nullary finitary operations of Π will be denoted by Ω for short.

Let A be a *compact* $\overline{\Pi}$-algebra. We will call a subset $M \subseteq A \times A$ a Δ-*module* if $x \in M$ implies $\mu(\Delta, \ldots, x, \ldots, \Delta) \subseteq M$ for all $\mu \in \Omega$. Given any set $X \subseteq A \times A$ we will denote by X^* the *union* of all the Δ-modules contained in X.

LEMMA 2.1. *X^* is a Δ-module.* //

For any $Y \subseteq A \times A$ we define

$$\langle Y \rangle = \bigcup_{\mu\in\Omega} \bigcup_{i=1}^{a(\mu)} \mu(\Delta, \ldots, Y, \ldots, \Delta) .$$

LEMMA 2.2. *$\langle Y \rangle$ is a Δ-module.* //

We will call A Δ-*finite* if there exists a finite number of operations $\{\mu_1, \ldots, \mu_k\} \subset \Omega$ such that

$$\langle Y \rangle = \bigcup_{i=1}^{k} \bigcup_{j=1}^{a(\mu_i)} \mu_i(\Delta, \ldots, Y, \ldots, \Delta)$$

for all $Y \subseteq A \times A$.

THEOREM 2.3. *Let X be an open equivalence relation on a Δ-finite A. Then X^* is an open algebra congruence on A.*

Proof. Choose $x \in X^*$. Then, by continuity of μ and compactness of A, there exists an open set V_μ about x such that $\mu(\Delta, \ldots, V_\mu, \ldots, \Delta) \subseteq X$ for each $\mu \in \Omega$. Thus there exists an open set V about x such that $\mu_i(\Delta, \ldots, V, \ldots, \Delta) \subseteq X$ for all $i = 1, \ldots, k$. Thus, by Δ-finiteness of A, there exists an open V about x such that $\langle V \rangle \subseteq X$. Therefore $\langle V \rangle \subseteq X^*$ and so X^* is open. It is straightforward to check that X^* is an A-congruence (see Numakura [6], Lemma 4). //

COROLLARY 2.4. *A Δ-finite totally disconnected algebra A is profinite.* //

Now suppose that Ω is generated by only a *finite* set Ω_b of basic operations. Call A Δ-*associative* if, for each $\mu \in \Omega_b$, there exists an integer $m = m(\mu) > 0$ such that

$$\mu(\Delta, \ldots, \mu(\Delta, \ldots, \mu(\Delta, \ldots, Y, \ldots, \Delta), \ldots)) \subseteq \mu(\Delta, \ldots, Y, \ldots, \Delta)$$

for all $Y \subseteq A \times A$, where the only restriction on the left-hand side is that μ should occur precisely m times. Call A *Δ-distributive* if

$$\mu(\Delta, \ldots, \rho(\Delta, \ldots, Y, \ldots, \Delta), \ldots, \Delta) = \rho(\Delta, \ldots, \mu(\Delta, \ldots, Y, \ldots, \Delta), \ldots, \Delta)$$

for all $\mu, \rho \in \Omega_b$ and $Y \subseteq A \times A$.

PROPOSITION 2.5. *If A is Δ-associative and Δ-distributive then it is Δ-finite.*

Proof. The diagonal Δ is a subalgebra of $A \times A$ so $\mu(\Delta, \ldots, \Delta) \subseteq \Delta$ for all $\mu \in \Omega$. Thus any derived expression whose entries are all Δ except for one entry which is Y, can be contained in the expression $\mu_1(\Delta, \ldots, \mu_2(\Delta, \ldots, \mu_n(\Delta, \ldots, Y, \ldots, \Delta), \ldots))$ in which the $\mu_1, \ldots, \mu_n$ are basic operations from Ω_b. By Δ-distributivity followed by Δ-associativity, any such derived expression can be contained in an expression in which each basic μ occurs less than $m(\mu)$ times, and there is only a finite number of such expressions; so the result follows. //

Examples of Δ-associative and Δ-distributive algebras include groups, rings, semigroups, distributive lattices, and lattice ordered groups.

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