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Equivariant Embeddings into Smooth Toric Varieties

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Abstract. We characterize embeddability of algebraic varieties into smooth toric varieties and prevarieties. Our embedding results hold also in an equivariant context and thus generalize a well-known embedding theorem of Sumihiro on quasiprojective G-varieties. The main idea is to reduce the embedding problem to the affine case. This is done by constructing equivariant affine conoids, a tool which extends the concept of an equivariant affine cone over a projective G-variety to a more general framework.

Introduction

Classical algebraic geometry mainly deals with quasiprojective varieties. These varieties thus come embedded into an ambient space whose structure is rather well understood, a fact on which rely many basic ideas and explicit working tools of the classical theory. The modern concept of defining a variety by gluing affine pieces is much more flexible, but the price for this flexibility is the loss of structural insight offered by the ambient space. The intention of embedding theorems is to regain such insight.

Since about 1970, toric varieties have been thoroughly studied. A most remarkable feature is their explicit description by combinatorial data. The class of toric varieties contains the classical ambient spaces, namely the affine and projective spaces, but it is considerably larger. In fact, by a theorem of Włodarczyk [15], toric varieties may serve as ambient spaces for surprisingly many varieties: A normal variety X can always be embedded into a toric prevariety, and X admits an embedding into a separated toric variety if and only if every two points of X have a common affine neighbourhood.

In the present article, we study some problems arising from Włodarczyk's result. The first one concerns singularities: On the one hand, one would like to get rid of the assumption of X being normal, on the other hand it is important to know when one can choose a smooth ambient space. So it is natural to ask, compare [15, Problems 5.4 and 5.5]: Which varieties admit embeddings into smooth toric varieties? A second point is the problem of embedding equivariantly with respect to algebraic group actions. Such embeddings are for example interesting in the context of quotient constructions, as these are quite well understood in the toric case.

To address the above problems, we introduce a tool that generalizes the concept of an affine cone over a projective variety: An *affine conoid* over a not necessarily projective variety X is an affine variety $\overline{X}$ together with an action of an algebraic torus H

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and a dense open invariant subset $\widehat{X} \subset \overline{X}$ where H acts freely with a geometric quotient $q\colon \widehat{X} \to X = \widehat{X}/H$. These affine conoids are the key to reduce the embedding problem to the affine case. However, they might also be of interest independently from our applications to embeddings, see *e.g.* Remark 3.9.

Our first main result characterizes existence of affine conoids for arbitrary varieties and relates it to embeddability. Following Borelli [5], we call an irreducible variety X *divisorial*, if for every $x \in X$ there is an effective Cartier divisor D on X such that $X \setminus \mathrm{Supp}(D)$ is an affine neighbourhood of x. The class of divisorial varieties considerably extends the class of quasiprojective varieties; for example it includes all $\mathbb{Q}$-factorial varieties and their subvarieties. We prove (see Theorem 3.2):

Theorem 1 *For an irreducible variety X, the following statements are equivalent:*

i) *X is divisorial.*
ii) *There exists an affine conoid over X.*
iii) *X admits a closed embedding into a smooth toric prevariety of affine intersection.*

Here we say that a prevariety Y is of *affine intersection* if for any two open affine subvarieties of Y, their intersection is again affine. This means that the non-separatedness of Y is of quite mild nature. In fact, using the appropriate formulation of divisoriality, we obtain the above result even for reducible varieties X. For the crucial part, the implication "i) $\Rightarrow$ ii)", we extend known constructions by Cox [6] and Kajiwara [11] from the setting of toric varieties to arbitrary divisorial varieties. As to the question of embeddability into separated smooth toric varieties, we obtain (see Corollary 5.4):

Theorem 2 *An irreducible variety X admits a closed embedding into a smooth toric variety if and only if for any two $x, x' \in X$, there is an effective Cartier divisor D on X such that $X \setminus \mathrm{Supp}(D)$ is affine and contains x and x'.*

We now turn to the second problem, namely, equivariant embeddings. If a connected linear algebraic group acts on a normal quasiprojective variety, then Sumihiro's Equivariant Embedding Theorem [13, Theorem 1] guarantees existence of a locally closed equivariant embedding into a projective space. We extend this result to the divisorial case (see Theorem 3.4 and Corollary 5.7):

Theorem 3 *Let X be a normal divisorial variety with a regular action of a connected linear algebraic group G.*

i) *There exist a smooth toric prevariety Z of affine intersection with a linear G-action and a G-equivariant closed embedding $X \to Z$.*
ii) *If for any two $x, x' \in X$, there is an effective Cartier divisor D on X such that $X \setminus \mathrm{Supp}(D)$ is affine and contains x and x', then one can choose Z to be a separated smooth toric variety.*

Similar to the case of Z being a projective space, a linear action on a toric prevariety Z is an action induced by some linear representation "over" Z; for the precise formulation see Section 1. Again, the basic step in the proof is the construction of affine conoids, but now in an equivariant manner. A consequence of Theorem 3 is that every $\mathbb{Q}$-factorial toric variety can be embedded into a smooth one by means of a toric morphism (see Corollary 5.8).

The criterion of Theorem 2 and Theorem 3 ii) on pairs of points $x, x' \in X$ can also be formulated for k-tuples of points; we call the resulting property *k-divisoriality*. In view of the Kleiman-Chevalley-Criterion, increasing k means "approximating" quasiprojectivity. We show that Theorem 2 and Theorem 3 ii) have analogous statements for $k > 2$, that means k-divisorial varieties can be embedded into k-divisorial smooth toric varieties (see Theorems 5.3 and 5.6). In particular, we prove a conjecture of Włodarczyk [15, 5.3] in the $\mathbb{Q}$-factorial case.

The present paper is organized as follows: In Section 1, we introduce equivariant affine conoids and show that they give rise to equivariant embeddings into toric prevarieties. In Section 2 we present our construction of equivariant affine conoids over divisorial G-varieties. The first main results are proved in Section 3. Moreover, we relate embeddings via affine conoids to classical projective embeddings, and discuss a consequence concerning Geometric Invariant Theory in this section. Finally, Sections 4 and 5 are devoted to the problem of embedding into separated and, more specially, k-divisorial smooth toric varieties.

1 Affine Conoids and Embeddability

Affine cones are a useful tool to study projective varieties. The purpose of this section is to extend that tool to more general varieties. We introduce the notion of an equivariant affine conoid over a G-variety X, and we show in Proposition 1.7 that such an affine conoid gives rise to a G-equivariant embedding of X into a certain smooth toric prevariety.

Throughout the whole article, we work in the categories of varieties and prevarieties defined over a fixed algebraically closed field $\mathbb{K}$. For the general background, we refer for example to [10, Chapter I]. We say that a prevariety X is of *affine intersection* if the diagonal morphism $X \to X \times X$ is affine. A variety is a separated but possibly reducible prevariety.

Let us recall some notation on group actions. A G-variety is a variety X together with a regular action $G \times X \to X$ of an algebraic group G. We say that the action of a G-variety X is *free at* $x \in X$ if the orbit map $g \mapsto g \cdot x$ is a locally closed embedding of G into X. Moreover, we call an action *free* if it is free at every point.

We shall be concerned with the following type of quotients: A *geometric quotient* for a G-variety X is an affine regular map $p \colon X \to Y$ onto a variety $Y = X/G$ such that the p-fibres are precisely the G-orbits and $\mathcal{O}_Y = p_*(\mathcal{O}_X)^G$ holds. Sometimes we allow in this setting also non-separated quotient spaces Y; then we speak of geometric *prequotients*.

Definition 1.1 An *affine conoid* over a variety X is an affine variety $\overline{X}$ together with a regular action of an algebraic torus H and a dense open H-invariant subset $\widehat{X} \subset \overline{X}$ where H acts freely with geometric quotient $q \colon \widehat{X} \to X = \widehat{X}/H$.

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Clearly this concept includes the classical notion of an affine cone over a projective variety. In order to present non-projective complete varieties admitting an affine conoid, we consider toric varieties. Recall at this point that a *toric variety* is a normal, and hence irreducible variety together with a regular action of an algebraic torus having a dense free orbit. There exist many non-projective complete smooth toric varieties (see *e.g.* [7, p. 74]), and a construction of Cox [6] provides affine conoids in that cases:

Example 1.2 Let X be a complete smooth toric variety arising from a fan Δ in a lattice N. Denote by $\Delta^{(1)}$ the set of onedimensional cones of Δ. Consider the lattice homomorphism

$$Q\colon \mathbb{Z}^{\Delta^{(1)}} \to N, \quad e_\varrho \mapsto v_\varrho,$$

where e_ϱ is the canonical base vector corresponding to $\varrho \in \Delta^{(1)}$ and v_ϱ denotes the primitive lattice vector of $\varrho \in \Delta^{(1)}$. For every cone $\sigma \in \Delta$ let $\sigma^{(1)}$ be the set of its extremal rays and define a cone

$$\widehat{\sigma} := \operatorname{cone}(e_\varrho; \varrho \in \sigma^{(1)}) \subset \mathbb{R}^{\Delta^{(1)}}.$$

These cones form a fan in $\mathbb{Z}^{\Delta^{(1)}}$, and the associated toric variety $\widehat{X}$ is an open subvariety of $\overline{X} := \mathbb{K}^{\Delta^{(1)}}$. The toric morphism $q\colon \widehat{X} \to X$ defined by Q is a geometric quotient for the free action of the algebraic torus $H := \ker(q)$ on $\widehat{X}$.

As we are also interested in the equivariant setting, we have to fix an appropriate equivariant notion of an affine conoid. Let G be an algebraic group, and let X be a G-variety. Suppose that $\overline{X}$ is an affine conoid over X, and let H, $\widehat{X}$ and $q\colon \widehat{X} \to X$ be the associated data as in 1.1. Assume moreover that G acts also regularly on $\overline{X}$.

Definition 1.3 We say that $\overline{X}$ is a *G-equivariant* affine conoid over X if the actions of G and H on $\overline{X}$ commute, G leaves $\widehat{X} \subset \overline{X}$ invariant, and the map $q\colon \widehat{X} \to X$ is G-equivariant.

In the subsequent constructions, we shall use a characterization of free torus actions in terms of certain regular functions. Assume that an algebraic torus H acts regularly on a variety X. Recall that a function $f \in \mathcal{O}(X)$ is called *homogeneous* with respect to a character $\chi \in \operatorname{Char}(H)$ if $f(t\cdot x) = \chi(t)f(x)$ holds for every $t \in H$ and every $x \in X$.

Remark 1.4 Let H be an algebraic torus, and let X be an affine H-variety. The action of H is free at $x \in X$ if and only if x has an H-invariant open neighbourhood $U \subset X$ admitting for every $\chi \in \operatorname{Char}(H)$ a χ-homogeneous $f \in \mathcal{O}(U)$ with $f(x) \neq 0$.

We begin the construction of equivariant embeddings with two auxiliary results concerning the following situation: Let G be a linear algebraic group and let Y denote an affine G-variety. Suppose that H is an algebraic torus contained as a closed subgroup in the center of G, and that $V \subset Y$ is a G-invariant open subset such that H acts freely on V. Under these assumptions we have:

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Lemma 1.5 *There exist a linear G-action on some $\mathbb{K}^n$, a G-equivariant closed embedding $\Phi\colon Y \to \mathbb{K}^n$ and an open subset $U \subset \mathbb{K}^n$ with the following properties:*

i) *U is invariant under the actions of G and $\mathbb{T}^n := (\mathbb{K}^*)^n$.*
ii) *$H \subset G$ acts diagonally on $\mathbb{K}^n$ and freely on U.*
iii) *$V = \Phi^{-1}(U)$ holds.*

Moreover, if G is an algebraic torus, then one can achieve that G acts diagonally on $\mathbb{K}^n$.

Proof Choose generators $f_1, \dots, f_r$ of $\mathcal{O}(Y)$ such that for some $s < r$, the functions $f_1, \dots, f_s$ generate the ideal of $Y \setminus V$. Let $M_i \subset \mathcal{O}(Y)$ be the (finite-dimensional) vector subspace generated by $G\cdot f_i$, and let N_i denote the dual G-module of M_i. Then we obtain G-equivariant regular maps

$$\Phi_i\colon X \to N_i, \quad x \mapsto \big(h \mapsto h(x)\big).$$

Let $N := N_1 \oplus \cdots \oplus N_r$, and let $\Phi\colon Y \to N$ be the map with components Φ_i. Note that Φ is a G-equivariant closed embedding. Choosing for every N_i a basis consisting of H-homogeneous vectors, we may assume that $N = \mathbb{K}^n$ holds and that H acts diagonally, *i.e.*, as a subgroup of the big torus $\mathbb{T}^n \subset \mathbb{K}^n$.

The set $U' \subset \mathbb{K}^n$ consisting of all free H-orbits is invariant under the actions of G and $\mathbb{T}^n$, because these actions commute with the action of H. Moreover, Remark 1.4 implies that U' is open in $\mathbb{K}^n$. Since H acts freely on V, we have $\Phi(V) \subset U'$. Set

$$U := U' \setminus (N_{s+1} \oplus \cdots \oplus N_r).$$

Then also U is open and invariant under the actions of G and the big torus $\mathbb{T}^n$. By construction, we have $V = \Phi^{-1}(U)$. So U has the desired properties. The supplement for the case of G being a torus is obvious. ■

For the next statment, recall that a *toric prevariety* is a normal prevariety together with a regular action of an algebraic torus having a dense free orbit. An introduction to toric prevarieties is given in [2].

Lemma 1.6 *Notation as in 1.5. The action of H on U admits a geometric prequotient $p\colon U \to Z := U/H$. Moreover, Z is a smooth toric prevariety of affine intersection and G acts regularly on Z making $p\colon U \to Z$ equivariant.*

Proof Cover U by H-invariant affine open sets $U_i \subset U$, and set $Z_i := \operatorname{Spec}\big(\mathcal{O}(U_i)^H\big)$. Since H acts freely, the natural maps $p_i\colon U_i \to Z_i$ are geometric quotients. Using Remark 1.4, one easily verifies that the maps p_i are even locally trivial. In particular, each Z_i is a smooth affine variety.

The varieties Z_i glue together along the open subsets $Z_{ij} := p_i(U_i \cap U_j)$ to a smooth prevariety Z. Since each Z_{ij} is the quotient space of the affine variety $U_i \cap U_j$, it is again affine. Consequently the prevariety Z is of affine intersection. Moreover, the maps $p_i\colon U_i \to Z_i$ glue together to a geometric prequotient $p\colon U \to Z$.

Since the actions of G and $\mathbb{T}^n$ on U commute with the action of H, universality of geometric prequotients yields regular actions of G and $\mathbb{T}^n$ on Z making $p\colon U \to Z$ equivariant. In particular, Z becomes a toric prevariety. ■

As the G-action on the toric prevariety Z in the above lemma is induced by a linear representation of G on $\mathbb{K}^n$, we call it *linear*. We are now ready for the main result of this section:

Proposition 1.7 *Let G be a linear algebraic group, and suppose that the G-variety X has a G-equivariant affine conoid. Then X admits a closed G-equivariant embedding into a smooth toric prevariety of affine intersection on which G acts linearly.*

Proof Let $\overline{X}$ be a G-equivariant affine conoid over X and let $q\colon \widehat{X} \to X = \widehat{X}/H$ denote the associated geometric quotient. Lemma 1.5 yields a $G \times H$-equivariant embedding $\Phi\colon \overline{X} \to \mathbb{K}^n$ and a $G \times H$-invariant open set $U \subset \mathbb{K}^n$ with $\Phi^{-1}(U) = \widehat{X}$ such that H acts freely on U. As we showed in Lemma 1.6, the geometric prequotient $U \to U/H$ exists and $Z := U/H$ is a smooth toric prevariety of affine intersection.

By the universal property of geometric prequotients, the restriction $\Phi\colon \widehat{X} \to U$ induces a regular map $X \to Z$ on the level of quotients. By construction, this map is equivariant with respect to the induced linear G-action on Z. Moreover, by H-closedness of the geometric prequotient $U \to Z$, the map $X \to Z$ is a closed embedding. ■

2 Ample Groups of Line Bundles

In this section we perform our construction of equivariant affine conoids. The basic tool is a suitable generalization of ample line bundles: Instead of a single line bundle, we shall use certain groups of line bundles. First we make precise what we mean by a group of line bundles.

Let X be a variety and consider a cover $\mathfrak{U} = (U_i)_{i \in I}$ of X by open subsets. This cover gives rise to an additive group $\Lambda(\mathfrak{U})$ of line bundles on X: For each cocycle $\xi \in Z^1(\mathfrak{U}, \mathcal{O}_X^*)$, let L_ξ denote the line bundle obtained by gluing the products $U_i \times \mathbb{K}$ along the maps

$$(x, z) \mapsto \bigl(x, \xi_{ij}(x) z\bigr).$$

The sum $L_\xi + L_\eta$ of two such line bundles is by definition the line bundle $L_{\xi\eta} = L_{\eta\xi}$. So the set $\Lambda(\mathfrak{U})$ consisting of all the bundles L_ξ is in fact an abelian group, isomorphic to $Z^1(\mathfrak{U}, \mathcal{O}_X^*)$. When we speak of a group of line bundles on X, we think of a subgroup of some group $\Lambda(\mathfrak{U})$ as above.

Now, let Λ be a finitely generated free group of line bundles on X. In the sequel, we associate to this group of line bundles a variety $\widehat{X}$ over X. For each line bundle $L \in \Lambda$, let $\mathcal{A}_L$ denote its sheaf of sections. We identify $\mathcal{A}_0$ with the structure sheaf $\mathcal{O}_X$.

The sections of a line bundle $L_\xi \in \Lambda$ over an open set $U \subset X$ are described by families $f_i \in \mathcal{O}_X(U \cap U_i)$ that are compatible with the gluing cocycle ξ. Thus, for any two sections $f \in \mathcal{A}_L(U)$ and $f' \in \mathcal{A}_{L'}(U)$, we can take the product $(f_i f_i')$ of their

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defining families (f_i) and (f_i') to obtain a section $ff' \in \mathcal{A}_{L+L'}(U)$. Extending this operation yields a multiplication on

$$\mathcal{A} := \bigoplus_{L \in \Lambda} \mathcal{A}_L.$$

We call $\mathcal{A}$ the graded $\mathcal{O}_X$-algebra associated to Λ. This algebra is reduced, and moreover, it is locally of finite type over $\mathcal{A}_0 = \mathcal{O}_X$; that means over sufficently small affine open sets $U \subset X$, the $\mathcal{O}(U)$-algebra $\mathcal{A}(U)$ is finitely generated. Consequently we obtain a variety

$$\widehat{X} := \operatorname{Spec}(\mathcal{A})$$

by glueing the affine varieties $\operatorname{Spec}\big(\mathcal{A}(U)\big)$, where U ranges over small open affine neighbourhoods $U \subset X$. In this process, the inclusion map $\mathcal{O}_X = \mathcal{A}_0 \to \mathcal{A}$ gives rise to an affine regular map

$$q\colon \widehat{X} \to X,$$

and we have $\mathcal{A} = q_*(\mathcal{O}_{\widehat{X}})$. We refer to $\widehat{X}$ as to the variety over X associated to the group Λ. The Λ-grading of the $\mathcal{O}_X$-algebra $\mathcal{A}$ defines a regular action of the algebraic torus

$$H := \operatorname{Spec}(\mathbb{K}[\Lambda])$$

on $\widehat{X}$ such that for each affine open set $U \subset X$, the sections $\mathcal{A}_L(U)$ are precisely the functions of $q^{-1}(U)$ that are homogeneous with respect to the character $\chi^L \in \operatorname{Char}(H)$. Using Remark 1.4, we observe:

Remark 2.1

i) H acts freely on $\widehat{X}$, and the map $q\colon \widehat{X} \to X$ is a geometric quotient for the action of H on $\widehat{X}$.
ii) For a section $f \in \mathcal{A}_L(X)$, let $Z(f) \subset X$ denote its set of zeroes. Then the set of zeroes of f, viewed as a regular function on $\widehat{X}$, is just

$$N(\widehat{X}; f) = q^{-1}\big(Z(f)\big) \subset \widehat{X}.$$

To proceed in our construction of affine conoids, we need a condition on the group Λ of line bundles which guarantees that the associated variety $\widehat{X}$ over X is quasiaffine.

Definition 2.2 We call a finitely generated free group Λ of line bundles on X *ample* if its associated graded $\mathcal{O}_X$-algebra $\mathcal{A}$ admits homogeneous sections $f_1, \dots, f_r \in \mathcal{A}(X)$ such that the open sets $X \setminus Z(f_i)$ form an affine cover of X.

Note that this generalizes the usual concept of ampleness in the sense that an ample line bundle generates an ample group. Moreover, the above notion of an ample group yields precisely what we are looking for:

Proposition 2.3 *Let G be a linear algebraic group, and let X be a G-variety. If Λ is an ample group of line bundles on X and every $L \in \Lambda$ is G-linearizable, then X admits a G-equivariant affine conoid.*

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For the proof we need two statements ensuring existence of suitable equivariant affine closures. We use the following common notation: For a variety Y and a function $h \in \mathcal{O}(Y)$, let $Y_h := \{y \in Y; h(y) \neq 0\}$.

Lemma 2.4 *Let Y be a variety endowed with an action of a linear algebraic group G. Suppose that*

i) *there are $f_1, \dots, f_r \in \mathcal{O}(Y)$ such that the sets $Y_i := Y_{f_i}$ are affine, cover Y and satisfy $\mathcal{O}(Y_i) = \mathcal{O}(Y)_{f_i}$,*
ii) *the representation of G on $\mathcal{O}(Y)$ given by $(g \cdot f)(y) = f(g^{-1} \cdot y)$ is rational.*

Then there is an affine G-variety $\overline{Y}$ containing Y as a dense open invariant subvariety such that the f_i extend regularly to $\overline{Y}$, and $\overline{Y}_{f_i} = Y_{f_i}$ holds.

Proof The main point is that $\mathcal{O}(Y)$ needs not be of finite type over $\mathbb{K}$. However, we find $h_1, \dots, h_s \in \mathcal{O}(Y)$ such that there are generators for each $\mathbb{K}$-algebra $\mathcal{O}(Y_i)$ among the functions h_j / f_i^l. Consider the subalgebra $A \subset \mathcal{O}(Y)$ generated by $h_1, \dots, h_s$ and $f_1, \dots, f_r$. By rationality of the G-representation on $\mathcal{O}(Y)$, we can enlarge A such that it is G-invariant but remains finitely generated.

Consider the affine G-variety $\overline{Y} := \operatorname{Spec}(A)$. Then the inclusion $A \subset \mathcal{O}(Y)$ defines a G-equivariant regular map $\varphi \colon Y \to \overline{Y}$. Moreover, each function $f_i \in \mathcal{O}(Y)$ is the pullback of a function on $\overline{Y}$, denoted again by f_i. By construction, restricting φ gives isomorphisms $Y_{f_i} \to \overline{Y}_{f_i}$ of affine varieties. Since $Y_{f_i} = \varphi^{-1}(\overline{Y}_{f_i})$ holds, we see that φ is the desired open embedding. ∎

Lemma 2.5 *Let Λ be an ample group of line bundles on a variety X, and suppose that the sections $f_1, \dots, f_r$ of the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ are as in 2.2. Then, setting $X_i := X \setminus Z(f_i)$, we have $\mathcal{A}(X_i) = \mathcal{A}(X)_{f_i}$.*

Proof Let $L_i \in \Lambda$ be the degree of f_i. Then there is an inverse $f_i^{-1} \in \mathcal{A}_{-L_i}(X_i)$ of $f_i|_{X_i}$. So we obtain an injection $\mathcal{A}(X)_{f_i} \subset \mathcal{A}(X_i)$. This map is also surjective: Let $L \in \Lambda$ and $f \in \mathcal{A}_L(X_i)$. Arguing locally, we see that for some suitably large integer m, the section $f f_i^m \in \mathcal{A}_{L+mL_i}(X_i)$ admits an extension to a section of $\mathcal{A}_{L+mL_i}(X)$, compare *e.g.* [5, Proposition 2.2]. But this means $f \in \mathcal{A}(X)_{f_i}$. ∎

Proof of Proposition 2.3 Fix a basis $L_1, \dots, L_k$ of the free abelian group Λ, and choose for every L_j a G-linearization. Via tensoring these G-linearizations, we obtain a G-linearization for each $L \in \Lambda$. This makes the associated graded $\mathcal{O}_X$-algebra $\mathcal{A}$ into a G-sheaf: for a section $f \in \mathcal{A}_L(U)$ let

$$(g \cdot f)(x) := g \cdot \big(f(g^{-1} \cdot x) \big).$$

Then $g \cdot f \in \mathcal{A}_L(g \cdot U)$. Moreover, on $\mathcal{A}_0 = \mathcal{O}_X$ we have the canonical G-sheaf structure arising from the G-action on X, and the multiplication of the graded $\mathcal{O}_X$-algebra $\mathcal{A}$ associated to Λ is compatible with the G-action. Since G respects the homogeneous components $\mathcal{A}_L$, we infer from [12, Section 2.5] that the representation of G on $\mathcal{A}(X)$ is rational.

Let $\widehat{X} = \mathrm{Spec}(\mathcal{A})$ be the variety over X associated to Λ, and denote by $q\colon \widehat{X} \to X$ the geometric quotient for the action of $H := \mathrm{Spec}(\mathbb{K}[\Lambda])$. The fact that we made $\mathcal{A}$ into a G-sheaf, allows us to define a G-action on $\widehat{X}$: Fix $g \in G$ and let $U \subset X$ be an affine open set. The algebra homomorphism

$$\mathcal{A}(g\cdot U) \to \mathcal{A}(U), \quad f \mapsto g^{-1}\cdot f$$

defines a morphism of affine varieties

$$T_{g,U}\colon q^{-1}(U) = \mathrm{Spec}\big(\mathcal{A}(U)\big) \to \mathrm{Spec}\big(\mathcal{A}(g\cdot U)\big) = q^{-1}(g\cdot U).$$

The maps $T_{g,U}$ glue together to a map $T_g\colon \widehat{X} \to \widehat{X}$. Moreover, $g\cdot x := T_g(x)$ defines a group action on $\widehat{X}$ and the representation of G on $\mathcal{O}(\widehat{X}) = \mathcal{A}(X)$ induced by this action is the one we started with. In particular, since $\mathcal{A}_0$ is canonically G-linearized, the map $q\colon \widehat{X} \to X$ is G-equivariant.

In order to check that the actions of G and H on $\widehat{X}$ commute, let $x \in \widehat{X}$, $g \in G$ and $t \in H$. Choose a q-saturated affine open neighbourhood $\widehat{U}$ of x. By G-equivariance of q, both points $g\cdot t\cdot x$ and $t\cdot g\cdot x$ lie in $g\cdot\widehat{U}$. Suppose $f \in \mathcal{O}(g\cdot\widehat{U})$ is homogeneous with respect to a character $\chi^L \in \mathrm{Char}(H)$. Then also $g^{-1}\cdot f \in \mathcal{O}(\widehat{U})$ is χ^L-homogeneous, and we obtain

$$f(g\cdot t\cdot x) = (g^{-1}\cdot f)(t\cdot x) = \chi^L(t)(g^{-1}\cdot f)(x) = \chi^L(t)f(g\cdot x) = f(t\cdot g\cdot x).$$

Since the H-homogeneous functions separate the points of $g\cdot\widehat{U}$, it follows that $g\cdot t\cdot x$ equals $t\cdot g\cdot x$. So the actions of G and H on $\widehat{X}$ commute. In particular, they define an action of the product $G \times H$ on $\widehat{X}$.

We shall apply Lemma 2.4 to obtain a $G \times H$-equivariant affine closure $\overline{X}$ of $\widehat{X}$. First note that the representation of $G \times H$ on $\mathcal{A}(X)$ is rational, because this holds for the representations of the factors G and H. So we only have to check Condition 2.4 i).

As to this, let $f_1, \dots, f_r \in \mathcal{A}(X)$ as in Definition 2.2. By Remark 2.1 ii), it suffices to know that for $X_i := X \setminus Z(f_i)$, one has $\mathcal{A}(X_i) = \mathcal{A}(X)_{f_i}$. But this is guaranteed by Lemma 2.5. So Condition 2.4 i) is verified, and Lemma 2.4 provides a $G \times H$-equivariant affine closure $\overline{X}$ of $\widehat{X}$, which is the desired affine conoid. ■

As mentioned earlier, our approach generalizes known constructions for toric varieties. The most recent one is due to T. Kajiwara [11]; he proved that a toric variety with enough invariant effective Cartier divisors is a geometric quotient of a quasi-affine toric variety. A systematic treatment of quotient presentations of toric varieties is given in [3].

The following observation is useful to construct ample groups of line bundles on smooth and, more generally $\mathbb{Q}$-factorial varieties, *i.e.*, normal varieties such that for every Weil divisor some multiple is Cartier.

Remark 2.6 Suppose on a variety X exist effective Cartier divisors $D_1, \dots, D_r$ such that the sets $X \setminus \mathrm{Supp}(D)_i$ form an affine cover of X. Then any choice of local equations of the D_i defined on a common cover of X associates to each D_i a line bundle L_i. Replacing the D_i with suitable multiples, one achieves that the L_i generate a free group Λ. This group Λ is ample and the canonical sections $f_i := 1 \in \mathcal{O}_X(D_i)$ satisfy to the conditions 2.2.

3 Characterizing Existence of Affine Conoids

Summing up the considerations of the preceding two sections, we present here our first main theorems. Moreover, we give some discussion and outline a consequence to quotient constructions in this section.

Following Borelli [5], we call a prevariety X *divisorial* if for every $x \in X$ there exists a line bundle L on X admitting a global section f such that removing its zero set $Z(f)$ yields an affine open neighbourhood $X \setminus Z(f)$ of x.

Remark 3.1

i) A variety is divisorial if and only if it admits an ample group of line bundles.
ii) An irreducible variety X is divisorial if and only if for every $x \in X$ there exists an effective Cartier divisor D on X such that $X \setminus \mathrm{Supp}(D)$ is an open affine neighbourhood of x.

The first main result relates divisoriality, existence of affine conoids and embeddability into smooth toric prevarieties to each other:

Theorem 3.2 *For a variety X, the following statements are equivalent:*

i) *X is divisorial.*
ii) *There exists an affine conoid over X.*
iii) *X admits a closed embedding into a smooth toric prevariety of affine intersection.*

Proof The implication "i) ⇒ ii)" is Proposition 2.3, and the implication "ii) ⇒ iii)" is Proposition 1.7. To obtain the remaining direction "iii) ⇒ i)", note that a smooth prevariety of affine intersection is divisorial and that subvarieties of divisorial prevarieties are again divisorial. ■

Remark 3.3

i) Theorem 3.2 holds as well for prevarieties X. Our proof works without changes. Note that a divisorial prevariety X is necessarily of affine intersection.
ii) The Hironaka twist is a smooth variety of dimension three that cannot be embedded into a separated toric variety, compare [15].
iii) There exist normal surfaces that admit neither embeddings into toric prevarieties of affine intersection nor into $\mathbb{Q}$-factorial ones, see [9].

Theorem 3.4 *Let X be a normal divisorial variety with a regular action of a connected linear algebraic group G. Then there exist a smooth toric prevariety Z of affine intersection with a linear G-action and a G-equivariant closed embedding $X \to Z$.*

Proof Choose an ample group Λ of line bundles on X and fix a basis $L_1, \dots, L_r$ of Λ. Replacing every L_i by a suitable multiple, we achieve that $L_1, \dots, L_r$ are G-linearizable, see *e.g.* [12, Proposition 2.4]. Since Λ remains ample, the assertion follows from Propositions 2.3 and 1.7. ■

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Remark 3.5

i) A condition on X like normality is necessary in Theorem 3.4: Identifying 0 and ∞ in the projective line yields a $\mathbb{K}^*$-variety that cannot be equivariantly embedded into any normal prevariety.
ii) If in the setting of Theorem 3.4, the group G is a torus and acts effectively, then, by the supplement of Lemma 1.5, one can arrange the embedding in such a way that G acts as a subtorus of the big torus of Z.
iii) For $G = \mathbb{C}^*$, the main result of [8] provides existence of equivariant embeddings into toric prevarieties even for non-divisorial normal X.

In the remainder of this section we discuss further aspects of affine conoids. First we note that the results hold also with a more general definition: In Definition 1.1 we could replace the algebraic torus H by an arbitrary diagonalizable group. Moreover, in characteristic zero one could even omit in Definition 1.1 the requirement of H acting freely on the set $\widehat{X}$. This is due to the following observation:

Remark 3.6 Let H be a diagonalizable group acting regularly and effectively on an affine variety $\overline{Y}$. Suppose $\widehat{Y} \subset \overline{Y}$ is an open subset with geometric quotient $\widehat{Y} \to X := \widehat{Y}/H$. Then the group $\Gamma \subset H$ generated by H_y, $y \in \widehat{Y}$, is finite. If $\operatorname{char}(\mathbb{K}) = 0$, then $\overline{X} := \overline{Y}/\Gamma$ is an affine conoid over X.

If a complete variety X admits an ample divisor D in the classical sense, then the linear system of a suitable multiple of D gives rise to an embedding of X into a projective space. In the following example we discuss the embedding of X provided by the (ample) group Λ of line bundles induced by D:

Example 3.7 Let X be a variety with $\mathcal{O}(X) = \mathbb{K}$, *e.g.* a complete one. Suppose that there is a line bundle L on X generating an ample group $\Lambda = \mathbb{Z}L$. We show that the method of Propositions 2.3 and 1.7 embeds X into a smooth quasiprojective toric variety:

Let $\widehat{X}$ denote the variety over X associated to Λ. Note that $\operatorname{Spec}(\mathbb{K}[\Lambda])$ equals $\mathbb{K}^*$. Choose any $\mathbb{K}^*$-equivariant affine closure $\overline{X}$ of $\widehat{X}$. Since $\mathcal{O}(X) = \mathbb{K}$ holds, every $\mathbb{K}^*$-invariant regular function on $\overline{X}$ is constant. In particular, the $\mathbb{K}^*$-variety $\overline{X}$ has an attractive fixed point.

It follows that the map Φ constructed in Lemma 1.5 embeds $\overline{X}$ into some $\mathbb{K}^n$ with linear $\mathbb{K}^*$-action having zero as attractive fixed point. Hence the induced map $X \to Z$ used in the proof of Proposition 1.7 embeds X into the set of regular points of a weighted projective space. In particular, X is quasiprojective.

In view of this observation, it is interisting to know when there exist "small" affine conoids over a given variety X. For this one needs small ample groups. Here the Picard group $\operatorname{Pic}(X)$ gives some bound:

Proposition 3.8 *Let X be a divisorial variety. If $\operatorname{Pic}(X)$ is generated by d elements, then X admits an affine conoid $\overline{X}$ with $\dim(\overline{X}) \leq \dim(X) + d$.*

Proof Choose an ample group Λ of line bundles on X. Since $\operatorname{Pic}(X)$ is generated by d elements, there is a subgroup $\Lambda' \subset \Lambda$ of rank at most d such that each $L \in \Lambda$ is isomorphic to an element of Λ'. The variety $\widehat{X}$ over X associated to Λ' satisfies

$$\dim(\widehat{X}) = \dim(X) + \dim\bigl(\operatorname{Spec}(\mathbb{K}[\Lambda'])\bigr) = \dim(X) + \operatorname{rk}(\Lambda').$$

Now, the group Λ' is obviously ample. Consequently $\widehat{X}$ is quasiaffine, and any equivariant affine closure $\overline{X}$ of $\widehat{X}$ is an affine conoid as wanted. ■

We conclude this section with a "philosophical" consequence of existence of equivariant affine conoids. Assume that a reductive group G acts regularly on a normal divisorial variety X. It is the central task of Geometric Invariant Theory to look for G-invariant open subsets $U \subset X$ admitting reasonable quotients. Affine conoids reduce this problem to the quasiaffine case:

Remark 3.9 Let $\overline{X}$ be a G-equivariant affine conoid over X and let $q\colon \widehat{X} \to X = \widehat{X}/H$ be the associated geometric quotient. A G-invariant open subset $U \subset X$ admits a categorical (good, geometric) quotient for the action of G if and only if $q^{-1}(U) \subset \widehat{X}$ admits categorical (good, geometric) quotient for the action of $G \times H$.

4 A Finiteness Result

So far we characterized divisoriality of a given variety X by existence of an embedding into a smooth toric prevariety Z of affine intersection. In this section we provide an important ingredient for the investigation of embeddings into a separated ambient space Z.

The following property, also considered in [15] and [14], is crucial: We say that a prevariety X has the *A_k-property*, if any k points $x_1, \dots, x_k \in X$ admit a common open affine neighbourhood in X.

Remark 4.1

i) For $k \geq 2$, an A_k-prevariety is necessarily separated.
ii) A toric prevariety is separated if and only if it has the A_2-property.

We are interested in open *A_k-subsets* of a given prevariety X, *i.e.*, open subsets $X' \subset X$ that have as a prevariety themselves the A_k-property. The main result of this section generalizes [14, Theorem 3.5] to the nonseparated case:

Proposition 4.2 *A prevariety has only finitely many maximal open A_k-subsets.*

This proposition can be obtained by combining [14, Theorem 3.5] with [4, Theorem I]. However, for the sake of self-containedness, we present below a simple direct proof, based on a slight modification of the arguments used in [14, Section 3].

We apply Proposition 4.2 to actions of connected algebraic groups G. Assume that G acts by means of a regular map $G \times X \to X$ on a prevariety X. As immediate consequences of the above result, we obtain:

Corollary 4.3 *The maximal open A_k-subsets of X are G-invariant.*

Proof Compare [2, Proof of Prop. 1.3]. Let $X_1, \dots, X_r$ be the maximal open A_k-subsets of X. We show that X_1 is G-invariant. Each $g \in G$ permutes the complements $A_i := X \setminus X_i$. Consequently G is covered by the closed subsets

$$G(i) := \{g \in G;\ g \cdot A_1 \subset A_i\}.$$

Now, G is connected, hence $G = G(i)$ for some i. In particular, for the neutral element $e_G \in G$ we have $e_G \cdot A_1 \subset A_i$. This means $A_i = A_1$. In other words, G leaves X_1 invariant. ■

Corollary 4.4 *X has the A_k-property if and only if for any collection $B_1, \dots, B_k \subset X$ of closed G-orbits there exist $x_i \in B_i$ such that $x_1, \dots, x_k$ admit a common open affine neighbourhood in X.* ■

We turn to the proof of Proposition 4.2. Fix an integer k. Suppose that X is a topological space such that the product topology on X^k is noetherian, *e.g.*, X is a prevariety. Let $\mathfrak{U}$ be any family of open subsets of X. Set

$$A := X^k \setminus \bigcup_{U \in \mathfrak{U}} U^k.$$

Then A is a closed subspace of X^k. Denote by $A_1, \dots, A_r$ the irreducible components of A. Let $p_i \colon X^k \to X$ be the projection onto the i-th factor. For a subset $Y \subset X$, let

$$X(Y) := X \setminus \bigcup_{p_i(A_j) \cap Y = \varnothing} \overline{p_i(A_j)}.$$

By a $\mathfrak{U}_k$-*subset* we mean a subset $Y \subset X$ such that for any $x_1, \dots, x_k \in Y$ there exists an $U \in \mathfrak{U}$ that contains the points $x_1, \dots, x_k$. The basic properties of the above construction are subsumed as follows:

Lemma 4.5

i) *X has only finitely many subsets of the form $X(Y)$.*
ii) *If Y is open in X then we have $Y \subset X(Y)$.*
iii) *If $Y \subset X$ is an open $\mathfrak{U}_k$-subset then so is $X(Y)$.*

Proof Only for iii) there is something to show. Suppose that Y is an open $\mathfrak{U}_k$-subset but $X(Y)$ does not have the $\mathfrak{U}_k$-property. Then there exist points $x_1, \dots, x_k \in X(Y)$ that are not contained in a common $U \in \mathfrak{U}$. So $(x_1, \dots, x_k)$ lies in A and hence in some irreducible component A_j of A. In particular, $x_i \in p_i(A_j)$ holds for all i.

By definition of $X(Y)$, the fact $x_i \in p_i(A_j)$ implies $p_i(A_j) \cap Y \neq \varnothing$ for all i. Thus each $p_i^{-1}(Y)$ intersects A_j. Since A_j is irreducible and Y is open, we obtain that Y^k intersects A_j. Since Y^k is covered by the sets U^k, $U \in \mathfrak{U}$, this is a contradiction to the definition of A. ■

Proof of Proposition 4.2 Let $\mathfrak{U}$ denote the family of all open affine subvarieties of X. According to Lemma 4.5 it suffices to show that the open A_k-subsets of X are just its open $\mathfrak{U}_k$-subsets. Clearly every open A_k-subset $Y \subset X$ is $\mathfrak{U}_k$. The converse is seen as follows:

Let $Y \subset X$ be an open $\mathfrak{U}_k$-subset. For given $x_1, \dots, x_k \in Y$ we have to find an affine open $V \subset Y$ that contains $x_1, \dots, x_k$. By assumption, there is an open affine $U \subset X$ containing $x_1, \dots, x_k$. Choose a function $f \in \mathcal{O}(U)$ that vanishes along $U \setminus Y$ but at no point x_i. Then $V := U_f$ is the desired affine neighbourhood in Y. ■

5 Separated Ambient Spaces

Here we discuss embeddings into separated smooth toric varieties. We shall also consider ambient spaces with additional properties. In order to formulate our results, we introduce the following terminology:

Definition 5.1 Let k be a positive integer. We say that a prevariety X is *k-divisorial*, if for any k points $x_1, \dots, x_k \in X$ there is a line bundle L on X admitting a global section f such that $X \setminus Z(f)$ is affine and contains $x_1, \dots, x_k$.

Of course, k-divisoriality is strongly related to the A_k-property discussed in the preceeding section. Moreover, we note:

Remark 5.2

- i) A quasiprojective variety is k-divisorial for all $k \in \mathbb{N}$.
- ii) An irreducible variety X is k-divisorial if and only if for every $x_1, \dots, x_k \in X$ there is an effective Cartier divisor D on X such that $X \setminus \operatorname{Supp}(D)$ is affine and contains $x_1, \dots, x_k$.
- iii) A $\mathbb{Q}$-factorial variety is k-divisorial if and only if it has the A_k-property.
- iv) Every $\mathbb{Q}$-factorial toric variety is 2-divisorial.
- v) The smooth toric variety discussed in [1, Example 3.1] is not 3-divisorial.

The first result of this section characterizes embedabbility into k-divisorial smooth toric varieties. In particular, it implies [15, Conjecture 5.3] for $\mathbb{Q}$-factorial varieties:

Theorem 5.3 *Let X be a variety, and let $k \geq 2$ be an integer. Then the following statments are equivalent:*

- i) *X is k-divisorial.*
- ii) *X admits a closed embedding into a k-divisorial smooth toric variety.*

As a direct consequence, we obtain the following characterization of embeddability into smooth toric varieties and thereby answer [15, Problem 5.4] and, partially, [15, Problem 5.5]:

Corollary 5.4 *A variety X admits a closed embedding into a smooth toric variety if and only if X is 2-divisorial.* ■

As an immediate consequence of this statement, we obtain the following special version of Nagata's Completion Theorem:

Corollary 5.5 *Every 2-divisorial variety admits a 2-divisorial completion.*

Proof Given a 2-divisorial variety X, embed it into a smooth toric variety Z. Choose a smooth toric completion $\overline{Z}$ of Z. Then the closure of X in $\overline{Z}$ is the desired completion. ∎

Let us turn to G-varieties X. Though it might be surprising at the first glance, k-divisoriality turns out to be also in the equivariant setting the right criterion. We prove:

Theorem 5.6 *Let G be a connected linear algebraic group, and let X be a normal G-variety. If X is k-divisorial for some $k \geq 2$, then X admits a G-equivariant closed embedding into a smooth k-divisorial toric variety with linear G-action.*

Corollary 5.7 *Let G be a connected linear algebraic group and let X be a normal 2-divisorial G-variety. Then X admits a closed G-equivariant embedding of X into a smooth toric variety with linear G-action.* ∎

As in Remark 3.5 ii), the supplement of Lemma 1.5 and the proof given below yield for an effective action of a torus G on X that one can arrange in Theorem 5.6 the action of G on the ambient toric variety to be a subtorus action. This implies in particular:

Corollary 5.8 *Every $\mathbb{Q}$-factorial toric variety can be embedded by means of a toric morphism into a smooth toric variety.* ∎

Proof of Theorems 5.3 and 5.6 By pulling back the desired data from the ambient space, we see that k-divisoriality is necessary to embed a given variety X into a smooth k-divisorial toric variety. We shall show the converse in the setting of Theorem 5.6. However, normality of X is merely needed to obtain G-linearizations of line bundles. Thus our proof also settles Theorem 5.3.

So suppose the connected linear algebraic group G acts regularly on the normal k-divisorial variety X. Consider the k-fold product X^k. This is covered by sets of the form U^k, where $U \subset X$ is an affine open subset obtained by removing the zero set of a section of some line bundle on X. Since finitely many of these U^k cover X^k, we obtain line bundles $L_1, \dots, L_r$ on X and sections $f_i \colon X \to L_i$ such that each $X_i := X \setminus Z(f_i)$ is affine and any k points $x_1, \dots, x_k \in X$ lie in some common X_i.

Surely, we may assume that $L_1, \dots, L_r$ generate a group Λ of line bundles. Moreover, replacing the L_i and the f_i with suitable multiples, we achieve that Λ is free and every L_i is G-linearizable. Then Λ is ample, and the sections $f_1, \dots, f_r$ satisfy to the conditions of Definition 2.2. Let $\widehat{X}$ denote the variety over X associated to Λ. Recall, that the canonical map $q \colon \widehat{X} \to X$ is a geometric quotient for the action of $H := \operatorname{Spec}(\mathbb{K}[\Lambda])$ on $\widehat{X}$. Moreover, by Remark 2.1, H acts freely on $\widehat{X}$.

As in the proof of Proposition 2.3, we endow $\widehat{X}$ with a G-action, commuting with the action of H, such that $q\colon \widehat{X} \to X$ becomes G-equivariant. In order to obtain an appropriate $G\times H$-equivariant affine closure $\overline{X}$ of $\widehat{X}$, we view the sections $f_i\colon X \to L_i$ as regular functions on $\widehat{X}$. According to Remark 2.1 ii), we have $\widehat{X}_{f_i} = q^{-1}(X_i)$, and Lemma 2.5 yields $\mathcal{O}(\widehat{X}_{f_i}) = \mathcal{O}(\widehat{X})_{f_i}$. Thus Lemma 2.4 provides a $G \times H$-equivariant affine closure $\overline{X}$ of $\widehat{X}$ such that the functions f_i extend regularly to $\overline{X}$ and $\overline{X}_{f_i} = q^{-1}(X_i)$ holds.

Choose a $G \times H$-equivariant embedding $\Phi\colon \overline{X} \to \mathbb{K}^n$ and a $G \times H$-invariant open set $U \subset \mathbb{K}^n$ as in Lemma 1.5. Then $\Phi^{-1}(U) = \widehat{X}$ holds and the geometric prequotient $U \to Z := U/H$ exists. Moreover, we proved in Lemma 1.6 that Z is a smooth toric prevariety with linear G-action. The map $X \to Z$ of quotients induced by Φ is a G-equivariant closed embedding.

In the sequel, we regard $\overline{X}$ and X as subvarieties of $\mathbb{K}^n$ and Z respectively. We claim that for any k-points $x_1, \dots, x_k \in X$ there is an affine open neighbourhood $V \subset Z$ with $x_1, \dots, x_k \in V$. To construct such a V, we take one of the $X_i \subset X$ with $x_1, \dots, x_k \in X_i$. By our choice of $\overline{X}$, we have $q^{-1}(X_i) = \overline{X}_{f_i}$.

Now, $f_i \in \mathcal{O}(\overline{X})$ is the restriction of some H-homogeneous function $h_i \in \mathcal{O}(\mathbb{K}^n)$. Consider the H-invariant affine open set $U_i := \mathbb{K}^n_{h_i}$. Then $q^{-1}(X_i)$ is a closed H-invariant subset of U_i. In particular, U_i contains all the fibres $q^{-1}(x_j)$. We have to shrink U_i a little bit: Let $A := U_i \setminus U$. Then A is a closed H-invariant subset of U_i. Since $q^{-1}(X_i) \subset U$ holds, we obtain $A \cap q^{-1}(X_i) = \emptyset$.

Looking at the quotient $\operatorname{Spec}\bigl(\mathcal{O}(U_i)^H\bigr)$, we find a function $f \in \mathcal{O}(U_i)^H$ that vanishes on A but has no zeroes along the H-orbits $q^{-1}(x_j)$. Thus, removing the zero set of this function f from U_i, we achieve that $U_i \subset U$ holds, U_i is still H-invariant, affine and contains all the fibres $q^{-1}(x_j)$. Now set $V := U_i/H \subset Z$. Then V is an affine open set in Z and $x_1, \dots, x_k \in V$. So our claim is verified.

Let $\mathbb{S}$ denote the big torus of the toric prevariety Z. Removing from Z step by step the (finitely many) closed $\mathbb{S}$-orbits that do not hit X, we arrive at an open $\mathbb{S}$-invariant subset $Z' \subset Z$ such that X is contained in Z' and each closed $\mathbb{S}$-orbit of Z' has nonempty intersection with X. Corollary 4.4 and the above claim imply that Z' has the A_k-property and hence is k-divisorial.

To conclude the proof, we have to make Z' invariant under the action of G. We argue in a similar way as above: Let Z'' be a maximal open A_k-subset of Z such that $Z' \subset Z''$ holds. Since G and $\mathbb{S}$ are connected, we can apply Corollary 4.3, and obtain that Z'' is invariant under the actions of both, G and $\mathbb{S}$. So $X \subset Z''$ is the desired G-equivariant closed embedding. ■

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