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## On the Geometrical Interpretation of $i^i$.

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If we put $\theta=\pi/2$ in the well known equation

$$e^{i\theta}=\cos\theta+i\sin\theta$$

we get $e^{i\pi/2}=i$; and raising both sides to the power $i$,

$$i^i=(e^{i\pi/2})^i=e^{i^2\pi/2}=e^{-\pi/2}.$$

Before proceeding further it will be useful to consider the former of these results. Putting for $e^{i\pi/2}$ the equivalent series, we have

$$i=1-\frac{1}{2!}\left(\frac{\pi}{2}\right)^2+\frac{1}{4!}\left(\frac{\pi}{2}\right)^4-\dots$$
$$+i\left\{\frac{\pi}{2}-\frac{1}{3!}\left(\frac{\pi}{2}\right)^3+\frac{1}{5!}\left(\frac{\pi}{2}\right)^5-\dots\right\}$$

It is obvious that each of the infinite series here involved is convergent, and a very little numerical calculation is sufficient to show that their limits are 0 and 1 respectively. In fact, taking $\pi=3{\cdot}1416$, or $\pi/2=1{\cdot}5708$, the two series become

$$1-1{\cdot}2337+{\cdot}2537-{\cdot}0208+{\cdot}0009={\cdot}0001\,,$$

and $$1{\cdot}5708-{\cdot}6459+{\cdot}0797-{\cdot}0047+{\cdot}0001=1{\cdot}0000\,.$$

Returning now to the equation $i^i=e^{-\pi/2}$, and substituting for $e$ and $\pi$ their numerical values, we get

$$i^i={\cdot}20788\,;$$

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and the question I propose to consider is how this result is to be understood or interpreted. It will be convenient first to consider the more general expression $a^i$, where $a$ is a complex number. We first observe that

$$(a^i)^i = a^{i^2} = a^{-1};$$

so that the effect of the operation $(\ )^i$, if performed twice upon $a$, is to give us the reciprocal of $a$; or the operation is one which goes half way towards the reciprocal. Next, writing $re^{i\theta}$ for $a$, we have

$$(re^{i\theta})^i = r^i e^{-\theta} = e^{-\theta} e^{i\log r} = e^{-\theta}(\cos\log r + i\sin\log r).$$

This shows us that the operation $(\ )^i$, when performed on a complex with modulus $r$ and amplitude $\theta$, gives us a new complex, of which the modulus is $e^{-\theta}$, and the amplitude $\log r$. Performing the same operation on this new complex, we have

$$(r^i e^{-\theta})^i = r^{-1} e^{-i\theta};$$

the result being the reciprocal of our original complex. Proceeding in the same way, we get

$$(r^{-1} e^{-i\theta})^i = r^{-i} e^{-i^2\theta} = r^{-i} e^{\theta};$$

$$(r^{-i} e^{\theta})^i = r^{-i^2} e^{i\theta} = re^{i\theta}$$

We have thus produced the original complex, and we see that the

moduli of the four complexes are ... ... $r$, $e^{-\theta}$, $r^{-1}$, $e^{\theta}$;

and the amplitudes ... ... ... ... $\theta$, $\log r$, $-\theta$, $-\log r$.

If $OP_1$ in Figure 13 represents $re^{i\theta}$, so that $OP_1 = r$, $P_1OA = \theta$, then the other three complexes will be represented by $OP_2$, $OP_3$, $OP_4$; where $AOP_1 = AOP_3$, $AOP_2 = AOP_4$, and $OP_1.OP_3 = OP_2.OP_4 = 1$. The figure is drawn for the case where $r = \frac{3}{4}$, $e^{-\theta} = \frac{1}{2}$; so that

the moduli are ... ... ... $\frac{3}{4}$, $\frac{1}{2}$, $\frac{4}{3}$, $2$;

and the amplitudes ... ... $\left\{ \begin{array}{l} \cdot 693,\ -\cdot 288,\ -\cdot 693,\ \cdot 288, \\ \text{or } 39°\cdot 7,\ -16°\cdot 6,\ -39°\cdot 7,\ 16°\cdot 6. \end{array} \right.$

If the complex $(re^{i\theta})^i$ is a real number, the point $P_2$ lies in OA or in AO produced; and the condition for this is that the amplitude shall be 0 or $\pi$. If now we suppose $r$ to approach 1, the angles $AOP_2$ and $AOP_4$ gradually diminish, and ultimately vanish when $r=1$. In this case

the moduli become ... ... $1, \; e^{-\theta}, \; 1, \; e^{\theta}$;

and the amplitudes ... ... $\theta, \; 0, \; -\theta, \; 0$;

so that the points, $P_2$, $P_4$, lie in OA, and the equation $(re^{i\theta})^i = r^i e^{-\theta}$ becomes $(e^{i\theta})^i = e^{-\theta}$. If we now suppose $\theta$ to approach $\pi/2$, or the line $OP_1$ to approach the perpendicular OB,

the moduli become ... ... $1, \; e^{-\pi/2}, \; 1, \; e^{\pi/2}$;

and the amplitudes ... ... $\pi/2, \; 0, \; -\pi/2, \; 0$.

In this case $e^{i\theta}$, which $=\cos\theta + i\sin\theta$, becomes $=i$, and we have $i^i = e^{-\pi/2} = \cdot 20788$. We thus see that this apparently anomalous result admits of a simple geometrical interpretation.

Another special case deserving of notice is when $AOP_1 = AOP_4$. This will happen when $\theta = -\log r$ or $r = e^{-\theta}$; and then

the moduli are ... ... ... $e^{-\theta}, \; e^{-\theta}, \; e^{\theta}, \; e^{\theta}$;

and the amplitudes ... ... $\theta, \; -\theta, \; -\theta, \; \theta$.

(See Figure 14.)

In the foregoing investigation I have not taken into account the possibility that $a^i$ may have a multiplicity of values, and I will now consider that point. If $l$ is any integer, we have

$$e^{2il\pi} = \cos 2l\pi + i\sin 2l\pi = 1.$$

Hence
$$re^{i\theta} = re^{i(\theta+2l\pi)}$$

and
$$(re^{i\theta})^i = \{re^{i(\theta+2l\pi)}\}^i = r^i e^{-\theta-2l\pi} = e^{-\theta-2l\pi} e^{i\log r}.$$

We thus see that, instead of the first member having a single value, as we have hitherto assumed, it has an infinite number of values, all

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of which have the same amplitude, $\log r$; while the moduli are $e^{-\theta}, e^{-\theta\pm 2\pi}, e^{-\theta\pm 4\pi}$, etc., and form a geometric series of which the ratio is $e^{\pm 2\pi}$.

Repeating the operation $(\ )^i$, since $e^{2im\pi}=1$, where $m$ is any integer,

$$(r^i e^{-\theta-2l\pi})^i = (r^i e^{2im\pi}.e^{-\theta-2l\pi})^i$$

$$= r^{-1}e^{-2m\pi}.e^{-i\theta}.e^{-2il\pi}$$

$$= r^{-1}e^{-2m\pi}.e^{-i\theta}$$

so that, instead of the reciprocal $r^{-1}e^{-i\theta}$, we have an infinite number of complexes, all of which have the same amplitude, $-\theta$; while the moduli are $r^{-1}, r^{-1}e^{\pm 2\pi}, r^{-1}e^{\pm 4\pi}$, etc., and form a geometric series, of which the ratio is $e^{\pm 2\pi}$. Since $l$ and $m$ each denote any integer, and they do not occur in the same formula, we may say that by successive repetitions of $(\ )^i$ we get a series of complexes, of which

| | | | | | | |
|---|---|---|---|---|---|---|
| the moduli are ... | ... $r$, | $e^{-\theta+2l\pi}$, | $r^{-1}e^{2l\pi}$, | $e^{\theta+2l\pi}$, | $re^{2l\pi}$, | ... |
| and the amplitudes | ... $\theta$, | $\log r$, | $-\theta$, | $-\log r$, | $\theta$, | ... |

It thus appears that we are not entitled to reason, as we did above, that $(a^i)^i = a^{ii} = a^{i^2} = a^{-1}$. This is analogous to what occurs with fractional indices; for instance, $(a^{\frac{1}{2}})^2 = a$; while $(a^2)^{\frac{1}{2}}$ is not $a$, but $\pm a$.

We have seen that $(\ )^i$ is a periodic operation with a period 4, subject to the remark that the original complex is only one of a series that are produced by performing the operation four times. Subject to a similar remark, we may say that $(\ )^z$ is a periodic operation, the period of which is $n$, if $z$ is a primary $n^{th}$ root of unity. Suppose that $z = x + iy = \cos 2\pi/n + i\sin 2\pi/n$; then

$$(re^{i\theta})^z = (re^{i\theta}.e^{2if\pi})^z, \quad \text{where } f \text{ is any integer}$$

$$= r^z e^{iz\theta}.e^{2ifz\pi}$$

$$= r^z e^{iz\theta}.e^{2ifz\pi}.e^{2ig\pi}, \quad \text{where } g \text{ is any integer.}$$

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Performing the operation $(\ )^z$ again,

$$(re^{i\theta})^{z^2} = r^{z^2} e^{iz^2\theta} . e^{2ifz^2\pi} . e^{2igz\pi} . e^{2ih\pi}, \quad \text{where } h \text{ is any integer.}$$

Proceeding in this way, we get

$$(re^{i\theta})^{z^n} = r^{z^n} e^{iz^n\theta} . e^{2ifz^n\pi} . e^{2igz^{n-1}\pi} \dots e^{2isz\pi}$$

$$= re^{i\theta} . e^{2i\pi(gz^{n-1} + hz^{n-2} + \dots + sz)}.$$

The index of $e$ in the last factor becomes

$$2i\pi\{g\cos 2(n-1)\pi/n + h\cos 2(n-2)\pi/n + \dots + s\cos 2\pi/n\}$$
$$-2\pi\{g\sin 2(n-1)\pi/n + h\sin 2(n-2)\pi/n + \dots + s\sin 2\pi/n\}$$

where $g\, h \dots s$ are any integers.

The preceding investigation was suggested to me by a perusal of Hayward's *Vector Algebra and Trigonometry.* In chap. 5 Mr Hayward gives the result (p. 115), $(4{\cdot}810475\dots)^i = i$; and this at once leads to $i^i = (4{\cdot}810475\dots)^{-1} = {\cdot}20788$. He then discusses the interpretation of $A^B$, where A and B are complex numbers; and shows that it has an infinite number of values, forming a series with a constant ratio; and he explains how these may be geometrically represented as derived from a "fundamental vector". He also considers several "particular cases"; but not specially the case where $B = i$, which is the one I have mostly had in view.

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