Bull. Aust. Math. Soc. **100** (2019), 86–96
doi:10.1017/S0004972718001429

SHARP BOUNDS OF SOME COEFFICIENT FUNCTIONALS OVER THE CLASS OF FUNCTIONS CONVEX IN THE DIRECTION OF THE IMAGINARY AXIS

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(Received 22 October 2018; accepted 29 October 2018; first published online 29 January 2019)

Abstract

We apply the Schwarz lemma to find general formulas for the third coefficient of Carathéodory functions dependent on a parameter in the closed unit polydisk. Next we find sharp estimates of the Hankel determinant $H_{2,2}$ and Zalcman functional $J_{2,3}$ over the class $\mathcal{CV}$ of analytic functions f normalised such that $\mathrm{Re}\{(1-z^2)f'(z)\} > 0$ for $z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, that is, the subclass of the class of functions convex in the direction of the imaginary axis.

2010 *Mathematics subject classification*: primary 30C45.

Keywords and phrases: univalent functions, functions convex in the direction of the imaginary axis, Hankel determinant, Zalcman functional, coefficients, Carathéodory class.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{T} := \partial\mathbb{D}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}$ the subclass of functions f normalised by $f(0) := 0$ and $f'(0) := 1$, that is, of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1,\ z \in \mathbb{D}. \tag{1.1}$$

Given $n, q \in \mathbb{N}$,

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

denotes the Hankel determinant of a function $f \in \mathcal{A}$ of the form (1.1). The problem of finding the upper bound of the Hankel determinant over selected compact subclasses

The first-named author was supported by the Basic Science Research Program through the National Research Foundation of the Republic of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

of $\mathcal{A}$ has been intensively studied. Many authors have examined the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ (see, for example, [2, 4, 5, 10, 11, 17, 21]).

We investigate $H_{2,2}(f)$ and also the functional $J_{2,3}(f) := a_2a_3 - a_4$, a specific case of the generalised Zalcman functional $J_{n,m}(f) := a_na_m - a_{n+m-1}$ for $n, m \in \mathbb{N} \setminus \{1\}$, which was investigated by Ma [20] (see also [23] for other results). Many authors (see, for example, [1, 2, 4, 5, 11, 14]) have computed upper bounds for the functional $J_{2,3}$ over various subclasses of $\mathcal{A}$.

By $\mathcal{CV}$, we denote a subclass of $\mathcal{A}$ of functions f such that

$$\operatorname{Re}\{(1 - z^2)f'(z)\} > 0, \quad z \in \mathbb{D}. \tag{1.2}$$

The class $\mathcal{CV}$ plays an important role in geometric function theory. Each $f \in \mathcal{CV}$ maps $\mathbb{D}$ univalently onto a domain $f(\mathbb{D})$ convex in the direction of the imaginary axis, that is, for $w_1, w_2 \in f(\mathbb{D})$ such that $\operatorname{Re} w_1 = \operatorname{Re} w_2$ the line segment $[w_1, w_2]$ lies in $f(\mathbb{D})$ with the additional property that there exist two points $\omega_1, \omega_2 \in \partial f(\mathbb{D})$ for which $\{\omega_1 + it : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D})$ and $\{\omega_2 - it : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D})$ (see, for example, [7, page 199]). In fact, the class $\mathcal{CV}$ is a subclass of the class $\mathcal{CV}(i)$ of functions convex in the direction of the imaginary axis introduced by Robertson [24] in 1936. Robertson gave an analytic condition for the class $\mathcal{CV}(i)$ under some regularity of functions in $\mathcal{CV}(i)$ on the unit circle. The proof of Robertson's conjecture for the whole class $\mathcal{CV}(i)$ was finally completed by Hengartner and Schober [9] by dividing the class $\mathcal{CV}(i)$ into three subclasses with the class $\mathcal{CV}$ as one of them. A supplement to their proof was given by Royster and Ziegler [25]. For further information on convexity in the direction of the imaginary axis, see, for example, [7, pages 193–206]. The condition (1.2) has been generalised by replacing the polynomial $1 - z^2$ by quadratic polynomials [15, 16] and by any polynomials having their roots in $\mathbb{C} \setminus \mathbb{D}$ [12, 13].

In this paper we derive sharp estimates of $H_{2,2}$ and $J_{2,3}$ over the class $\mathcal{CV}$:

$$\max_{f \in \mathcal{CV}} |H_{2,2}(f)| = 1$$

and

$$\max_{f \in \mathcal{CV}} |J_{2,3}(f)| = \frac{1}{486}(233 + 7\sqrt{7}).$$

Since the class $\mathcal{CV}$ has a representation through the Carathéodory class $\mathcal{P}$, that is, the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_nz^n, \quad z \in \mathbb{D}, \tag{1.3}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{CV}$ can be expressed in terms of the coefficients of functions in $\mathcal{P}$. Therefore, to get the upper bounds of $H_{2,2}$ and $J_{2,3}$, our calculations are based on parametric formulas for the second and third coefficients in $\mathcal{P}$. However, the class $\mathcal{CV}$ is not rotation invariant, that is, if $f \in \mathcal{CV}$, then $f_\theta \notin \mathcal{CV}$ for each $\theta \in (0, 2\pi)$, where $f_\theta(z) := e^{-i\theta}f(e^{i\theta}z)$ for $z \in \mathbb{D}$. Results in the cited papers mostly concern rotation-invariant subclasses of $\mathcal{A}$ and use the formula for

https://doi.org/10.1017/S0004972718001429 Published online by Cambridge University Press

c_3 due to Libera and Zlotkiewicz [18, 19] with the restriction that $c_1 \geq 0$. However, this cannot work for the whole class $\mathcal{CV}$. So, to solve the problems of this paper, we first find a general formula for c_3. We present two different methods of proof. The second one is constructive and gives some extremal functions. It can be applied to other coefficients of functions in the Carathéodory class. We believe that this new result will be useful for other coefficient problems for classes which are not rotation invariant.

2. Parametric formulas for coefficients of Carathéodory functions

The following lemma is due to Carathéodory (see, for example, [8]).

LEMMA 2.1. *The power series for a function p given by (1.3) converges in $\mathbb{D}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n := \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ \overline{c}_1 & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{c}_n & \overline{c}_{n-1} & \overline{c}_{n-2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N},$$

are nonnegative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k L(e^{it_k} z), \quad z \in \mathbb{D},$$

where $m \in \mathbb{N}$,

$$L(z) := \frac{1+z}{1-z}, \quad z \in \mathbb{D},$$

$\rho_k > 0$, $\sum_{k=1}^{m} \rho_k = 1$, $t_k \in [0, 2\pi)$ and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \geq m$.

In particular, $D_1 \geq 0$ yields the well-known inequality (2.1) due to Carathéodory [3] (see, for example, [6, page 41]). In turn, $D_2 \geq 0$ leads to the inequality (2.2) (see, for example, [22, page 166]).

LEMMA 2.2. *If $p \in \mathcal{P}$ is of the form (1.3), then*

$$|c_1| \leq 2 \tag{2.1}$$

and

$$|2c_2 - c_1^2| \leq 4 - |c_1|^2. \tag{2.2}$$

Now, using $D_3 \geq 0$, we prove the inequality for the third coefficient of functions in $\mathcal{P}$. When $c_1 \geq 0$, this was done by Libera and Zlotkiewicz [18, 19]. The formula due to Libera and Zlotkiewicz is useful in applications when the class of analytic functions characterised in terms of the class $\mathcal{P}$ and the coefficient functional are rotation invariant. Then by suitable rotation it can be assumed that the coefficient c_1 is real. However, when the class or the coefficient functional are not rotation invariant we need to use the general formulas for the third as well as for further coefficients of $\mathcal{P}$.

Lemma 2.3. *If $p \in \mathcal{P}$ is of the form (1.3), then*

$$|(4-|c_1|^2)(2c_3-c_1c_2)-(c_1^2-2c_2)(\overline{c}_1c_2-2c_1)| \le (4-|c_1|^2)^2-|2c_2-c_1^2|^2. \tag{2.3}$$

Proof. By Lemma 2.1, for $p \in \mathcal{P}$ of the form (1.3),

$$\begin{aligned} D_3 &= \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \\ &= 2\begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} - c_1\begin{vmatrix} \overline{c}_1 & c_1 & c_2 \\ \overline{c}_2 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_1 & 2 \end{vmatrix} + c_2\begin{vmatrix} \overline{c}_1 & 2 & c_2 \\ \overline{c}_2 & \overline{c}_1 & c_1 \\ \overline{c}_3 & \overline{c}_2 & 2 \end{vmatrix} - c_3\begin{vmatrix} \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 \end{vmatrix} \\ &= 16+|c_1|^4-12|c_1|^2-8|c_2|^2+|c_2|^4-2|c_1|^2|c_2|^2-4|c_3|^2+|c_1|^2|c_3|^2+4\overline{c}_1^2c_2 \\ &\qquad +4c_1^2\overline{c_2}+4c_1c_2\overline{c}_3+4\overline{c}_1\overline{c}_2c_3-c_1^3\overline{c}_3-\overline{c_1}^3c_3-c_1\overline{c}_2^2c_3-\overline{c}_1c_2^2\overline{c}_3 \ge 0. \end{aligned} \tag{2.4}$$

Since a straightforward algebraic computation shows that

$$\begin{aligned} &[(4-|c_1|^2)^2-|2c_2-c_1^2|^2]^2-|(4-|c_1|^2)(2c_3-c_1c_2)-(c_1^2-2c_2)(\overline{c}_1c_2-2c_1)|^2 \\ &\quad = 4(4-|c_1|^2)D_3, \end{aligned}$$

the inequality (2.3) follows from (2.4). □

Let $\mathcal{B}_0$ be the class of all self-maps of $\mathbb{D}$ of the form

$$\omega(z)=\sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2.5}$$

that is, the class of so-called Schwarz functions. Given $\alpha \in \mathbb{D}$, let

$$\psi_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}, \quad z \in \overline{\mathbb{D}}.$$

It is well known that ψ_α is a conformal automorphism of $\mathbb{D}$, $\psi_\alpha(\mathbb{D})=\mathbb{D}$, $\psi_\alpha(\mathbb{T})=\mathbb{T}$ and $\psi_\alpha^{-1}=\psi_{-\alpha}$. Moreover, for $n \in \mathbb{N}$,

$$\psi_\alpha^{(n)}(\alpha)=\frac{n!\overline{\alpha}^{n-1}}{(1-|\alpha|^2)^n}. \tag{2.6}$$

It is easy to check that the inequalities (2.1)–(2.3) can be written in the forms (2.7)–(2.9), respectively, that is, in a form dependent on a parameter lying in the polydisk $\overline{\mathbb{D}}^k$ for $k=1,2,3$. As remarked earlier, formulas (2.7) and (2.8) are known. Now we will prove the formula (2.9) in a new way. This method based on the Schwarz lemma readily allows us to find formulas for each coefficient of Carathéodory functions.

Lemma 2.4. *If $p \in \mathcal{P}$ is of the form (1.3), then*

$$c_1 = 2\zeta_1, \tag{2.7}$$

$$c_2 = 2\zeta_1^2+2(1-|\zeta_1|^2)\zeta_2 \tag{2.8}$$

and

$$c_3 = 2\zeta_1^3 + 4(1-|\zeta_1|^2)\zeta_1\zeta_2 - 2(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3 \tag{2.9}$$

for some $\zeta_i \in \overline{\mathbb{D}}$ *and* $i \in \{1,2,3\}$.

For $\zeta_1 \in \mathbb{T}$, *there is a unique function* $p \in \mathcal{P}$ *with* c_1 *as in (2.7), namely,*

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z}, \quad z \in \mathbb{D}. \tag{2.10}$$

For $\zeta_1 \in \mathbb{D}$ *and* $\zeta_2 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_1 *and* c_2 *as in (2.7)–(2.8), where*

$$\omega(z) = z\psi_{-\zeta_1}(\zeta_2 z), \quad z \in \mathbb{D}, \tag{2.11}$$

that is,

$$p(z) = \frac{1+(\overline{\zeta_1}\zeta_2+\zeta_1)z+\zeta_2 z^2}{1+(\overline{\zeta_1}\zeta_2-\zeta_1)z-\zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{2.12}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ *and* $\zeta_3 \in \mathbb{T}$, *there is a unique function* $p = L \circ \omega \in \mathcal{P}$ *with* c_1, c_2 *and* c_3 *as in (2.7)–(2.9), where*

$$\omega(z) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z)), \quad z \in \mathbb{D}, \tag{2.13}$$

that is,

$$p(z) = \frac{1+(\overline{\zeta_2}\zeta_3+\overline{\zeta_1}\zeta_2+\zeta_1)z+(\overline{\zeta_1}\zeta_3+\zeta_1\overline{\zeta_2}\zeta_3+\zeta_2)z^2+\zeta_3 z^3}{1+(\overline{\zeta_2}\zeta_3+\overline{\zeta_1}\zeta_2-\zeta_1)z+(\overline{\zeta_1}\zeta_3-\zeta_1\overline{\zeta_2}\zeta_3-\zeta_2)z^2-\zeta_3 z^3}, \quad z \in \mathbb{D}. \tag{2.14}$$

PROOF. Let $p \in \mathcal{P}$ be of the form (1.3). Then there exists $\omega \in \mathcal{B}_0$ of the form (2.5) such that

$$(1-\omega(z))p(z) = 1+\omega(z), \quad z \in \mathbb{D}. \tag{2.15}$$

Putting the series (1.3) and (2.5) into (2.15), by equating coefficients,

$$c_1 = 2b_1, \quad c_2 = 2b_2 + 2b_1^2, \quad c_3 = 2b_3 + 4b_1b_2 + 2b_1^3. \tag{2.16}$$

Part 1. By the Schwarz lemma (see, for example, [7, Vol. I, pages 84–85]),

$$|b_1| = |\omega'(0)| \le 1, \tag{2.17}$$

that is,

$$b_1 = \zeta_1 \tag{2.18}$$

for some $\zeta_1 \in \overline{\mathbb{D}}$. By (2.16), we get the formula (2.7).

Moreover, equality in (2.17), that is, the case $\zeta_1 \in \mathbb{T}$ in (2.18), holds if and only if

$$\omega(z) = \zeta_1 z, \quad z \in \mathbb{D}$$

(see [7, Vol. I, page 85]). From (2.15), it follows that then p can only be as in (2.10).

Part 2. By Part 1, we can assume that $b_1 \in \mathbb{D}$. Define

$$\varphi_1(z) := \frac{\omega(z)}{z}, \quad z \in \mathbb{D} \setminus \{0\}, \quad \varphi_1(0) := b_1. \tag{2.19}$$

By the maximum principle for analytic functions, the function φ_1 is a self-map of $\mathbb{D}$, so a function

$$\omega_1(z) := \psi_{b_1}(\varphi_1(z)) = b_1^{(1)} z + b_2^{(1)} z^2 + \cdots, \quad z \in \mathbb{D}, \tag{2.20}$$

is a Schwarz function. By the Schwarz lemma,

$$|b_1^{(1)}| = |\omega_1'(0)| = \frac{|b_2|}{1 - |b_1|^2} \le 1, \tag{2.21}$$

that is,

$$b_1^{(1)} = \zeta_2 \tag{2.22}$$

for some $\zeta_2 \in \overline{\mathbb{D}}$. Taking into account (2.18) and (2.21),

$$b_2 = (1 - |\zeta_1|^2)\zeta_2. \tag{2.23}$$

By (2.18), the formula (2.8) follows from (2.16).

Moreover, equality in (2.21), that is, the case $\zeta_2 \in \mathbb{T}$ in (2.23), holds if and only if $\omega_1(z) = \zeta_2 z$, $z \in \mathbb{D}$. Consequently, by (2.19), (2.20) and (2.18),

$$\omega(z) = z\varphi_1(z) = z\psi_{-b_1}(\omega_1(z)) = z\psi_{-\zeta_1}(\zeta_2 z), \quad z \in \mathbb{D},$$

that is, ω is as in (2.11). From (2.15), it follows that then p can only be as in (2.12).

Part 3. By Parts 1 and 2, we can assume that $b_1, b_1^{(1)} \in \mathbb{D}$. Define

$$\varphi_2(z) := \frac{\omega_1(z)}{z}, \quad z \in \mathbb{D} \setminus \{0\}, \quad \varphi_2(0) := b_1^{(1)}. \tag{2.24}$$

Since the function φ_2 is a self-map of $\mathbb{D}$, a function

$$\omega_2(z) := \psi_{b_1^{(1)}}(\varphi_2(z)) = b_1^{(2)} z + b_2^{(2)} z^2 + \cdots, \quad z \in \mathbb{D}, \tag{2.25}$$

is a Schwarz function. By the Schwarz lemma,

$$|b_1^{(2)}| = |\omega_2'(0)| = \frac{|b_2^{(1)}|}{1 - |b_1^{(1)}|^2} \le 1, \tag{2.26}$$

that is,

$$b_1^{(2)} = \zeta_3 \tag{2.27}$$

for some $\zeta_3 \in \overline{\mathbb{D}}$. Taking into account (2.26) and (2.22),

$$b_2^{(1)} = (1 - |\zeta_2|^2)\zeta_3. \tag{2.28}$$

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On the other hand, from (2.20), by applying (2.6), (2.18) and (2.23),

$$b_2^{(1)} = \frac{1}{2}\omega_1''(0) = \frac{\overline{b_1}b_2^2}{(1-|b_1|^2)^2} + \frac{b_3}{1-|b_1|} = \overline{\zeta_1}\zeta_2^2 + \frac{b_3}{1-|\zeta_1|^2}.$$

This together with (2.28) yields

$$b_3 = -(1-|\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + (1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3.$$

By (2.18), the formula (2.9) follows from (2.16). Moreover, equality in (2.26), that is, the case $\zeta_3 \in \mathbb{T}$ in (2.27), holds if and only if $\omega_2(z) = \zeta_3 z$, $z \in \mathbb{D}$. Thus, by (2.24), (2.25) and (2.22),

$$\omega_1(z) = z\varphi_2(z) = z\psi_{-b_1^{(1)}}(\omega_2(z)) = z\psi_{-\zeta_2}(\zeta_3 z), \quad z \in \mathbb{D}.$$

Now (2.20) and (2.19) with (2.18) yield

$$\omega(z) = z\varphi_1(z) = z\psi_{-b_1}(\omega_1(z)) = z\psi_{-\zeta_1}(z\psi_{-\zeta_2}(\zeta_3 z), \quad z \in \mathbb{D},$$

that is, ω is as in (2.13). From (2.15), it follows that then p can only be as in (2.14). □

Remark 2.5. In a similar way we can get formulas for the coefficients c_n for $n \geq 4$.

3. Applications

Having the formulas (2.7)–(2.9), we now find the sharp estimate of the Hankel determinant $H_{2,2}$ over the class $\mathcal{CV}$.

Theorem 3.1. *We have*

$$\max_{f\in\mathcal{CV}} |H_{2,2}(f)| = 1 \tag{3.1}$$

with the extremal function

$$f(z) = \frac{z}{1-z^2}, \quad z \in \mathbb{D}. \tag{3.2}$$

Proof. Let $f \in \mathcal{CV}$ be of the form (1.1). By (1.2),

$$(1-z^2)f'(z) = p(z), \quad z \in \mathbb{D}, \tag{3.3}$$

for some function $p \in \mathcal{P}$ of the form (1.3). By putting the series (1.1) and (1.3) into (3.3) and equating coefficients,

$$a_2 = \tfrac{1}{2}c_1, \quad a_3 = \tfrac{1}{3}(c_2+1), \quad a_4 = \tfrac{1}{4}(c_1+c_3). \tag{3.4}$$

Hence, by using the equalities (2.7)–(2.9),

$$\begin{aligned} a_2a_4 - a_3^2 &= \tfrac{1}{72}(9c_1c_3 + 9c_1^2 - 8c_2^2 - 16c_2 - 8) \\ &= \tfrac{1}{18}[\zeta_1^4 + \zeta_1^2 - 2 + 2(\zeta_1^2 - 4)(1-|\zeta_1|^2)\zeta_2 \\ &\qquad - (|\zeta_1|^2 + 8)(1-|\zeta_1|^2)\zeta_2^2 + 9\zeta_1(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3]. \end{aligned}$$

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Setting $x := |\zeta_1| \in [0,1]$, $y := |\zeta_2| \in [0,1]$ and taking into account that $|\zeta_3| \le 1$,

$$\begin{aligned}|a_2a_4 - a_3^2| &\le \tfrac{1}{18}[x^4 + x^2 + 2 + 2(x^2+4)(1-x^2)y \\ &\quad + (x^2+8)(1-x^2)y^2 + 9x(1-x^2)(1-y^2)] \\ &= \tfrac{1}{18}[x^4 + x^2 + 2 + 9x(1-x^2) + 2(x^2+4)(1-x^2)y \\ &\quad + (1-x)(8-x)(1-x^2)y^2] =: \tfrac{1}{18}F(x,y), \quad x,y \in [0,1]. \end{aligned} \tag{3.5}$$

For $x = 1$,

$$|a_2a_4 - a_3^2| \le \tfrac{2}{9}. \tag{3.6}$$

Now let $x \in [0,1)$. Then

$$\frac{\partial F}{\partial y} = 2(x^2+4)(1-x^2) + 2(1-x)(8-x)(1-x^2)y = 0$$

only for

$$y = -\frac{x^2+4}{(1-x)(8-x)} = y_0.$$

Since $y_0 < 0$, for each $x \in [0,1)$ the function $[0,1] \ni y \mapsto F(\cdot, y)$ is strictly increasing. Therefore, by (3.5),

$$|a_2a_4 - a_3^2| \le \tfrac{1}{18}F(x,1) = \tfrac{1}{18}(-2x^4 - 12x^2 + 18) \le 1, \quad x \in [0,1).$$

This together with (3.6) shows that

$$|H_{2,2}(f)| \le 1. \tag{3.7}$$

For the function (3.2), which is in $\mathcal{CV}$, we have $a_2 = a_4 = 0$ and $a_3 = 1$. This gives the equality in (3.7) and proves (3.1). □

Now we find the sharp estimate of the Zalcman functional $J_{2,3}$ over the class $\mathcal{CV}$.

THEOREM 3.2. *We have*

$$\max_{f \in \mathcal{CV}} |J_{2,3}(f)| = \tfrac{1}{486}(233 + 7\sqrt{7}) \approx 0.51753 \tag{3.8}$$

with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{1-t^2}\,dt, \quad z \in \mathbb{D}, \tag{3.9}$$

where p is of the form (2.14) with

$$\zeta_1 = \frac{4-\sqrt{7}}{6}i, \quad \zeta_2 = \frac{-5+2\sqrt{7}}{3}, \quad \zeta_3 = i. \tag{3.10}$$

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PROOF. For $f \in \mathcal{CV}$, by (3.4),

$$\begin{aligned} a_4 - a_2 a_3 &= \tfrac{1}{12}(3c_3 + 3c_1 - 2c_1 c_2 - 2c_1) \\ &= \tfrac{1}{6}[-\zeta_1^3 + \zeta_1 + 2\zeta_1(1-|\zeta_1|^2)\zeta_2 - 3\overline{\zeta_1}(1-|\zeta_1|^2)\zeta_2^2 \\ &\qquad + 3(1-|\zeta_1|^2)(1-|\zeta_2|^2)\zeta_3]. \end{aligned} \tag{3.11}$$

Setting $x := |\zeta_1| \in [0,1]$, $y := |\zeta_2| \in [0,1]$ and taking into account that $|\zeta_3| \le 1$,

$$|a_4 - a_2 a_3| \le \tfrac{1}{6}[x^3 - 3x^2 + x + 3 + 2x(1-x^2)y - 3(1-x)^2(1+x)y^2] =: \tfrac{1}{6}F(x,y). \tag{3.12}$$

For $x = 0$,

$$F(0,y) = 3(1-y^2) \le 3, \quad y \in [0,1]. \tag{3.13}$$

For $x = 1$,

$$F(1,y) = 2, \quad y \in [0,1]. \tag{3.14}$$

Let $x \in (0,1)$. Note that $3(1-x)^2(1+x) > 0$ and $\partial F/\partial y = 0$ only for

$$y = \frac{x}{3(1-x)} =: y_0.$$

For $y_0 \ge 1$, that is, for $x \in [3/4, 1)$,

$$F(x,y) \le F(x,1) = 6x - 4x^3 \le F(\tfrac{3}{4},1) = \tfrac{45}{16} = 2.8125 \tag{3.15}$$

and, for $y_0 \in [0,1)$, that is, for $x \in (0, 3/4)$,

$$F(x,y) \le F(x,y_0) = \tfrac{1}{3}(4x^3 - 8x^2 + 3x + 9) =: \tfrac{1}{3}\varphi(x). \tag{3.16}$$

Since φ attains its maximum value

$$\varphi\Big(\frac{4-\sqrt{7}}{6}\Big) = \frac{1}{81}(233 + 7\sqrt{7}) \approx 3.10519$$

at $x_0 = (4-\sqrt{7})/6 \approx 0.2257$, taking into account (3.12)–(3.16) yields

$$|J_{2,3}(f)| \le \tfrac{1}{486}(233 + 7\sqrt{7}). \tag{3.17}$$

By Lemma 2.4, a function p of the form (2.14) with ζ_1, ζ_2 and ζ_3 as in (3.10) belongs to $\mathcal{P}$. Thus, the corresponding function f given by (3.9) belongs to $\mathcal{CV}$ and, by (3.11),

$$\begin{aligned} a_4 - a_2 a_3 &= \frac{i}{6}[x_0^3 + x_0 + 2x_0(1-x_0^2)y_0 + 3x_0(1-x_0^2)y_0^2 + 3(1-x_0^2)(1-y_0^2)] \\ &= \frac{i}{6}\Big[x_0^3 + x_0 + 2x_0(1-x_0^2)\frac{x_0}{3(1-x_0)} + 3x_0(1-x_0^2)\frac{x_0^2}{9(1-x_0)^2} \\ &\qquad + 3(1-x_0^2)\Big(1 - \frac{x_0^2}{9(1-x_0)^2}\Big)\Big] \\ &= \frac{1}{18}(4x_0^3 - 8x_0^2 + 3x_0 + 9) = \frac{1}{81}(233 + 7\sqrt{7}). \end{aligned}$$

This gives equality in (3.17) and proves (3.8). □

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