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ON A GROUP PRESENTATION DUE TO FOX

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In 1956 R. H. Fox had occasion, while investigating fundamental groups of topological surfaces, to believe that the group $\langle a, b \mid ab^2=b^3a, ba^2=a^3b\rangle$ was trivial. Using the Todd-Coxeter coset enumeration algorithm a proof was obtained, see [3], and this algorithmic proof was used to produce an algebraic proof, see [2]. In [1] Benson and Mendelsohn, using a similar method to that of [2] showed that $\langle a, b \mid ab^n=b^{n+1}a, ba^n=a^{n+1}b\rangle$ is trivial. In this note we give a direct proof for the more general problem of describing the structure of the group $\langle a, b \mid ab^n= b^\ell a, ba^n=a^\ell b\rangle$.

We use $|\cdot|$ to denote the order of a group, the order of a subgroup and the modulus of an integer, the context making it clear which is intended.

THEOREM. *Let $G=\langle a, b \mid ab^n=b^\ell a, ba^n=a^\ell b\rangle$. Then if*
(i) $(n, \ell)\neq 1$, *G is infinite;*
(ii) $(n, \ell)=1$, *G is metacyclic of order* $|\ell-n|^3$.

Proof. We can assume without loss of generality that $n\leq \ell$.

(i) If $(n, \ell)=d\neq 1$ then adding the relations $a^d=b^d=1$ to G shows that $\mathbb{Z}_d * \mathbb{Z}_d$, the free product of two copies of the cyclic group of order d, is a homomorphic image of G. Therefore G is infinite.

(ii) Assume $(n, \ell)=1$. The relation $ab^na^{-1}=b^\ell$ gives, for any i,

$$a^ib^{n^i}a^{-i} = b^{\ell^i}. \tag{1}$$

Putting $i=n$ in (1) and conjugating by b^{-1} we obtain $ba^nb^{n^n}a^{-n}b^{-1}=b^{\ell^n}$ and so

$$a^\ell b^{n^n}a^{-\ell} = b^{\ell^n}. \tag{2}$$

However (1) with $i=\ell$ is $a^\ell b^{n^\ell}a^{-\ell}=b^{\ell^\ell}$ and thus $b^{\ell^n(\ell^{\ell-n}-n^{\ell-n})}=1$. Raising (2) to the power $\ell^{\ell-n}-n^{\ell-n}$ we obtain $b^{(\ell^{\ell-n}-n^{\ell-n})}=1$, since ℓ and n are coprime.

Now $(n, \ell^{\ell-n}-n^{\ell-n})=1$ so there exist integers α,β such that $\alpha n+\beta(\ell^{\ell-n}-n^{\ell-n})=1$. Then $G\cong\langle a, b \mid aba^{-1}=b^{\alpha\ell}, bab^{-1}=a^{\alpha\ell}, a^{(\ell^{\ell-n}-n^{\ell-n})}=b^{(\ell^{\ell-n}-n^{\ell-n})}=1\rangle$. It is easy to see that the order of a and b is

$$\begin{aligned}((\alpha\ell-1)^2, (\ell^{\ell-n}-n^{\ell-n}))&\\ &= (\alpha^2\ell^2-2\alpha\ell+1, (\ell^{\ell-n}-n^{\ell-n}))\\ &= (n^2\alpha^2\ell^2-2n^2\alpha\ell+n^2, (\ell^{\ell-n}-n^{\ell-n})) \quad \text{since} \quad (n^2, (\ell^{\ell-n}-n^{\ell-n})) = 1.\\ &= ((\ell-n)^2, (\ell^{\ell-n}-n^{\ell-n})) = (\ell-n)^2.\end{aligned}$$

Now $aba^{-1}b^{-1}=b^{\alpha\ell-1}$ so $a^{1-\alpha\ell}=b^{\alpha\ell-1}$. Raising this to the power n gives $a^{n-\alpha n\ell}=b^{\alpha n\ell-n}$ showing that $a^{n-\ell}=b^{\ell-n}$.

Therefore $\langle a\rangle$ is normal in G, $|G/\langle a\rangle|=\ell-n$ and $|\langle a\rangle|=(\ell-n)^2$ giving the result.

COROLLARY. *The group* $\langle a, b \mid ab^n=b^{n+1}a, ba^n=a^{n+1}b\rangle$ *is trivial.*

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