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COLLECTIONS OF SEQUENCES HAVING THE RAMSEY PROPERTY ONLY FOR FEW COLOURS

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A family $\mathcal{C}$ of sequences has the *r-Ramsey property* if for every positive integer k, there exists a least positive integer $g^{(r)}(k)$ such that for every r-colouring of $\{1,2,\dots,g^{(r)}(k)\}$ there is a monochromatic k-term member of $\mathcal{C}$. For fixed integers $m > 1$ and $0 \leqslant a < m$, define a *k-term $a \pmod{m}$-sequence* to be an increasing sequence of positive integers $\{x_1,\dots,x_k\}$ such that $x_i - x_{i-1} \equiv a \pmod{m}$ for $i = 2,\dots,k$. Define an m-a.p. to be an arithmetic progression where the difference between successive terms is m. Let $\mathcal{C}^*_{a(m)}$ be the collection of sequences that are either $a \pmod{m}$-sequences or m-a.p.'s. Landman and Long showed that for all $m \geqslant 2$ and $1 \leqslant a < m$, $\mathcal{C}^*_{a(m)}$ has the 2-Ramsey property, and that the 2-Ramsey function $g^{(2)}_{a(m)}(k,n)$, corresponding to k-term $a \pmod{m}$-sequences or n-term m-a.p.'s, has order of magnitude mkn. We show that $\mathcal{C}^*_{a(m)}$ does not have the 4-Ramsey property and that, unless $m/a = 2$, it does not have the 3-Ramsey property. In the case where $m/a = 2$, we give an exact formula for $g^{(3)}_{a(m)}(k,n)$. We show that if $a \neq 0$, there exist 4-colourings or 6-colourings (depending on m and a) of the positive integers which avoid 2-term monochromatic members of $\mathcal{C}^*_{a(m)}$, but that there never exist such 3-colourings. We also give an exact formula for $g^{(r)}_{0(m)}(k,n)$.

1. INTRODUCTION

Many results in Ramsey theory take on the following general form: there exists a positive integer $f(r)$ such that for every partition of $[1,f(r)] = \{1,\dots,f(r)\}$ into r classes, some class will contain a set with property P (where P is some specified property). This can also be described by saying that for every r-colouring of $[1,f(r)]$, there is a monochromatic set with property P. Two of the most famous theorems of this type are van der Waerden's theorem [9] and Schur's theorem [8]. Schur's theorem says that for every positive integer r, there exists a positive integer $f(r)$ such that whenever $[1,f(r)]$ is r-coloured, there is a monochromatic set $\{x,y,z\}$ such that $x+y = z$. Van der Waerden's theorem states that for all positive integers k and r, there exists a positive integer $w(k,r)$ such that whenever $[1,w(k,r)]$ is r-coloured, there is a monochromatic k-term arithmetic progression. Estimation of the van der Waerden

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numbers $w(k,r)$ remains one of the most intriguing (and presumably one of the most difficult) problems in Ramsey theory, even for $r = 2$ (see [**2**] for an in depth discussion). In recent years, several problems similar to van der Waerden's theorem have been looked at, where the family of k-term arithmetic progressions is replaced by some other family of k-term sequences. Examples can be found in [**1, 3, 4, 5, 6, 7**].

Let us say that a family C of sequences has the *r-Ramsey property* if for every positive integer k there exists a positive integer $g^{(r)}(k)$ such that whenever $[1, g^{(r)}(k)]$ is r-coloured there is a monochromatic k-term member of C. Thus van der Waerden's theorem tells us that the family of arithmetic progressions has the r-Ramsey property for every positive integer r. Schur's theorem, although it does not involve the parameter k, also is a result which holds for any number of colours r. Likewise, most other well-known results in Ramsey theory which say that in a large enough set there exists some specified monochromatic structure, are true regardless of the number of colours used. In this paper we examine the r-Ramsey property for certain collections of sequences that were shown to be 2-Ramsey in [7], and show that these collections do not have the 4-Ramsey property, and in most cases do not have the 3-Ramsey property. We find this particularly intriguing because the Ramsey function corresponding to k-term sequences of this type, where two colours are used, grows only like a quadratic in k. This is in contrast to the van der Waerden function for 2 colours, $w(k,2)$, for which the best known upper bound is enormous [2]; yet the collection of arithmetic progressions does have the r-Ramsey property for all r.

Let m and a be fixed integers such that $m \geqslant 2$ and $0 \leqslant a < m$. Define a *k-term $a\,(\text{mod}\ m)$-sequence* to be an increasing sequence of positive integers $\{x_1, \dots, x_k\}$ such that $x_i - x_{i-1} \equiv a\,(\text{mod}\ m)$ for $2 \leqslant i \leqslant k$. Let $C_{a(m)}$ denote the family of all $a\,(\text{mod}\ m)$-sequences. Define an *m-a.p.* to be an arithmetic progression such that the difference between consecutive terms is m. Denote by $C^*_{a(m)}$ the family of all sequences that are either $a\,(\text{mod}\ m)$-sequences or m-a.p.'s. In [7] the authors showed that for all m and all $a \neq 0$, the family $C_{a(m)}$ does not have the 2-Ramsey property, but that $C^*_{a(m)}$ does. If we let $g_{a(m)}(k)$ denote the least positive integer N such that every 2-colouring of $[1, N]$ yields a monochromatic member of $C^*_{a(m)}$, it was shown that $g_{a(m)}(k) = mk^2(1 + o(1))$. In fact, the following more general result was found:

Theorem 1. (Landman and Long) *Let $m, k, n \geqslant 2$ and $0 \leqslant a < m$. Let $g_{a(m)}(k,n)$ denote the least positive integer N such that every 2-colouring of $[1, N]$ contains either a monochromatic k-term $a\,(\text{mod}\ m)$-sequence or a monochromatic n-term m-a.p. Then for all $1 \leqslant a < m$, $g_{a(m)}(k,n) \sim mkn$ (as $k \to \infty$ and $n \to \infty$). Also, $g_{0(m)}(k,n) = 2m(k-1) + 1$.*

Let us extend the notation of Theorem 1 so that $g^{(r)}_{a(m)}(k,n)$ denotes the least positive integer N (if it exists) such that every r-colouring of $[1,N]$ contains a monochromatic k-term $a(\text{mod } m)$-sequence or a monochromatic n-term m-a.p. If no such N exists, we write $g^{(r)}_{a(m)}(k,n)=\infty$. In Section 2 we show that if $a>0$ and $m/a\neq 2$, then $g^{(3)}_{a(m)}(k,2)=\infty$ for k sufficiently large; thus, the family $C^*_{a(m)}$ does not have the 3-Ramsey property. We also prove that if $m/a=2$, then this family does have the 3-Ramsey property, and we give an exact formula for $g^{(3)}_{a(m)}(k,n)$ (having order of magnitude $3mnk$).

In Section 3 we consider $r>3$. We show that for all $1\leqslant a<m$, the family $C^*_{a(m)}$ does not have the r-Ramsey property if $r>3$. Further, we show that there exist 4-colourings or 6-colourings (depending on m and a) of the positive integers that in fact avoid any 2-term monochromatic members of $C^*_{a(m)}$, but that no such 3-colourings exist. In addition, the result of Theorem 1 which deals with the case of $a=0$ is easily extended to r colours.

2. Three Colors

The following lemma will be useful in obtaining results throughout this paper.

Lemma 1. *Let c be a positive integer, $r\geqslant 2$, $m\geqslant 2$ and $0\leqslant a<m$. Then*

$$g^{(r)}_{ca(cm)}(k,n)=c\Big(g^{(r)}_{a(m)}(k,n)-1\Big)+1.$$

Proof: Let $N=g^{(r)}_{a(m)}(k,n)$. Now if y is a positive integer, then $\{x_i:1\leqslant i\leqslant k\}$ is an $a(\text{mod } m)$-sequence if and only if $\{cx_i+y:1\leqslant i\leqslant k\}$ is a $ca(\text{mod } cm)$-sequence; and $\{x_i:1\leqslant i\leqslant k\}$ is an m-a.p. if and only if $\{cx_i+y:1\leqslant i\leqslant k\}$ is a cm-a.p. Hence, by the definition of N, any r colouring of $\{1,c+1,2c+1,\dots,(N-1)c+1\}$ must contain a k-term monochromatic $ca(\text{mod } cm)$-sequence or an n-term monochromatic ca-a.p. Hence $g^{(r)}_{ca(cm)}(k,n)\leqslant c(N-1)+1$.

On the other hand, we know there is an r-colouring χ of $[1,N-1]$ that contains no monochromatic k-term $a(\text{mod } m)$-sequence and no monochromatic n-term m-a.p. Define χ' on $[1,c(N-1)]$ by

$$\chi'([c(x-1)+1,cx])=\chi(x) \text{ for } x=1,\dots,N-1.$$

To complete the proof we shall show that χ' avoids monochromatic k-term $ca(\text{mod } cm)$-sequences and monochromatic n-term cm-a.p.'s. Assume, by way of contradiction, that $\{s_i\}$ is a monochromatic (with respect to χ') sequence of one of these types. Let $t_i=\lceil s_i/c\rceil$ for each i. Then, by the reasoning of the previous paragraph, $\{t_i\}$ is either a k-term $a(\text{mod } m)$-sequence or an n-term m-a.p. that is monochromatic with respect to χ, a contradiction. $\square$

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We now show that for most choices of m and a, $C^*_{a(m)}$ does not have the 3-Ramsey property (hence it has the r-Ramsey property only for $r = 2$).

Theorem 2. *Let $1 \leqslant a < m$ where $m/a \neq 2$ and let $s = \lceil (2m/3) \rceil$. Then $g^{(3)}_{a(m)}(s+1, 2) = \infty$, so that $C^*_{a(m)}$ does not have the 3-Ramsey property.*

Proof: Let $d = \gcd(m, a)$. By Lemma 1 we know that $g^{(3)}_{a(m)}(k,n) = d\left(g^{(3)}_{(a/d)(m/d)}(k,n) - 1\right) + 1$. Hence we may assume that $d = 1$. To prove the theorem we provide a 3-colouring of the positive integers that contains no monochromatic 2-term m-a.p. and no monochromatic $(s+1)$-term $a \pmod m$-sequence.

For each positive integer x, define $\overline{x}$ to be the element of $[1, 2m]$ such that $x \equiv \overline{x} \pmod{2m}$. Let $t = \lceil (4m/3) \rceil$. Define χ to be the $2m$-periodic colouring of the positive integers with $\chi(x) = \chi(\overline{x})$ for all x, where $\chi([1,s]) = 1$, $\chi([s+1,t]) = 2$, and $\chi([t+1, 2m]) = 3$. In other words, $\chi = IJKIJKIJK\ldots$, where $I = 11\ldots1$ has length s, $J = 22\ldots2$ has length $t - s$, and $K = 33\ldots3$ has length $2m - t$.

To see that there is no monochromatic 2-term m-a.p., note that since $m \geqslant 3$, we have $s < m < t$. Hence if $\overline{x} \in [1, s]$, then $s + 1 \leqslant \overline{x+m} \leqslant 2m$, so that $\chi(x+m) \neq \chi(x)$. Likewise if $\overline{x} \in [s+1, t]$, then $\overline{x+m} \notin [s+1, t]$, and if $\overline{x} \in [t+1, 2m]$, then $\overline{x+m} \notin [t+1, 2m]$. Thus in all cases $\chi(x+m) \neq \chi(x)$, so that there is no monochromatic 2-term m-a.p.

To complete the proof, assume X is an $(s+1)$-term $a \pmod m$-sequence. Then X is of the form $\{x, x+a+j_1m, x+2a+j_2m, \ldots, x+sa+j_sm\}$ where $j_1 \leqslant \cdots \leqslant j_s$ are nonnegative integers. Since $d = 1$ and $s < m$, from elementary group theory we know that the set

$$\overline{X} = \{\overline{x}, \overline{x+a+j_1m}, \overline{x+2a+j_2m}, \ldots, \overline{x+sa+j_sm}\}$$

consists of $s+1$ distinct elements modulo m. Then these $s+1$ elements are also distinct modulo $2m$ and, since each of the sets $[1,s]$, $[s+1,t]$, and $[t+1,2m]$ contains no more than s elements, X cannot be monochromatic under χ. Hence there is no monochromatic $(s+1)$-term $a \pmod m$-sequence. ∎

When $1 \leqslant a < m$, then the only circumstance under which $C^*_{a(m)}$ has the 3-Ramsey property is when $a = m/2$. In the next theorem we establish this fact by giving a precise formula for the associated Ramsey function $g^{(3)}_{(m/2)(m)}(k,n)$.

Theorem 3. *Let $m > 1$ be even, and let $k, n \geqslant 2$. Then*

$$g^{(3)}_{\frac{m}{2}(m)}(k,n) = \frac{m}{2}(3k-5)(2n-1) + 1. \tag{1}$$

Proof: We first prove the theorem for $m = 2$. To show that (when $m = 2$) the right-hand side of (1) is a lower bound for $g^{(3)}_{1(2)}(k,n)$, we shall give a 3-colouring of

$[1,(3k-5)(2n-1)]$ that contains no monochromatic k-term $1\,(\mathrm{mod}\ 2)$-sequence and no monochromatic n-term 2-a.p.

For $i=0,\dots,3k-6$, define $I_i=[i(2n-1)+1,(i+1)(2n-1)]$. Hence

$$\bigcup_{i=0}^{3k-6} I_i=[1,(3k-5)(2n-1)].$$

If $k=2$, colour $I_0=[1,2n-1]$ with the colouring 1212...12 3 (each of the colours 1 and 2 occurs $n-1$ times). This colouring clearly contains no 2-term monochromatic $1\,(\mathrm{mod}\ 2)$-sequence and no n-term monochromatic 2-a.p.

If $k\geqslant 3$, colour the I_i as follows:

$$\chi(I_i)=\begin{cases}\underbrace{1212\dots12}_{2n-2}3 & \text{if } i\equiv 0\,(\mathrm{mod}\ 3)\\ \underbrace{3131\dots31}_{2n-2}2 & \text{if } i\equiv 1\,(\mathrm{mod}\ 3)\\ \underbrace{2323\dots23}_{2n-2}1 & \text{if } i\equiv 2\,(\mathrm{mod}\ 3)\end{cases}$$

Since $|I_i|=2n-1$, we see that there is no monochromatic n-term 2-a.p.

Note that for each $j=0,\dots,k-3$, the interval $I_{3j}\cup I_{3j+1}\cup I_{3j+2}$ contains no 2-term $1\,(\mathrm{mod}\ 2)$-sequence having colour 1. Hence any $1\,(\mathrm{mod}\ 2)$-sequence with colour 1 that is contained in $\bigcup_{i=0}^{3k-7} I_i$ has length at most $k-2$. Thus any $1\,(\mathrm{mod}\ 2)$-sequence with colour 1 that is contained in $\bigcup_{i=0}^{3k-6} I_i$ has length at most $k-1$.

Note also that if $k\geqslant 4$, then for each $j=0,\dots,k-4$, the interval $I_{3j+2}\cup I_{3j+3}\cup I_{3j+4}$ contains no 2-term $1\,(\mathrm{mod}\ 2)$-sequence having colour 2. Hence any $1\,(\mathrm{mod}\ 2)$-sequence with colour 2 that is contained in $\bigcup_{i=2}^{3k-8} I_i$ has length at most $k-3$. Since $I_0\cup I_1$ and (for $k\geqslant 3$) $I_{3k-7}\cup I_{3k-6}$ can each contain at most one term of a $1\,(\mathrm{mod}\ 2)$-sequence having colour 2, we see that any $1\,(\mathrm{mod}\ 2)$-sequence having colour 2 that is contained in $\bigcup_{i=0}^{3k-6} I_i$ has length at most $k-1$.

Finally, we see that for each $j=0,\dots,k-3$, the interval $I_{3j+1}\cup I_{3j+2}\cup I_{3j+3}$ contains no 2-term $1\,(\mathrm{mod}\ 2)$-sequence having colour 3. Therefore, any $1\,(\mathrm{mod}\ 2)$-sequence with colour 3 that is contained in $\bigcup_{i=1}^{3k-6} I_i$ has length at most $k-2$. Thus any $1\,(\mathrm{mod}\ 2)$-sequence having colour 3 that is contained in $\bigcup_{i=0}^{3k-6} I_i$ has length at most $k-1$.

We have shown that χ avoids both monochromatic k-term $1\,(\mathrm{mod}\ 2)$-sequences and monochromatic n-term 2-a.p.'s, and therefore $g^{(3)}_{1(2)}(k,n)\geqslant(3k-5)(2n-1)+1$.

Next we prove that

$$g^{(3)}_{1(2)}(k,n) \leqslant (3k-5)(2n-1)+1. \tag{2}$$

Let $I(k,n) = [1,(3k-5)(2n-1)+1]$. We need to show that for every 3-colouring of $I(k,n)$, there is either a monochromatic k-term 1 (mod 2)-sequence or a monochromatic n-term 2-a.p.

Given a 3-colouring ϕ of $I(k,n)$, let $a_i(\phi)$ denote the length of the longest monochromatic 1 (mod 2)-sequence that has colour i $(i = 1,2,3)$. Let $S(k,n,\phi) = a_1(\phi) + a_2(\phi) + a_3(\phi)$. To prove (2) our strategy is to establish the following fact:

FACT 1. If ϕ is a 3-colouring of $I(k,n)$ for which there is no monochromatic n-term 2-a.p., then $S(k,n,\phi) \geqslant 3(k-1)+1$.

Then (2) will follow easily from Fact 1 by the pigeon-hole principle, since there is either a monochromatic n-term 2-a.p. or a monochromatic k-term 1 (mod 2)-sequence.

We prove Fact 1 by induction on k. Let $k = 2$ and let ϕ be any 3-colouring of $[1,2n]$ with no monochromatic n-term 2-a.p. So there are odd numbers o_1, o_2 and even numbers e_1, e_2 in $[1,2n]$ such that $\phi(o_1) \neq \phi(o_2)$ and $\phi(e_1) \neq \phi(e_2)$. Since there are only three colours, $\phi(e_i) = \phi(o_j)$ for some i and j, $1 \leqslant i,j \leqslant 2$. Therefore $\{e_i, o_j\}$ is a monochromatic 2-term 1 (mod 2)-sequence, say of colour 1. We also see (since there is no monochromatic n-term 2-a.p.) that there is some even number e and some odd number o such that $\phi(e) \neq 1$ and $\phi(o) \neq 1$. Hence either $\phi(o) = \phi(e)$ and we have another monochromatic 2-term 1 (mod 2)-sequence, or else $\{e\}$ and $\{o\}$ are each 1-term 1 (mod 2)-sequences, not of the same colour, and neither having colour 1. In either case, $S(2,n,\phi) \geqslant 4$.

Now assume $k \geqslant 2$ and that Fact 1 holds for k. Let ϕ be any 3-colouring of $I(k+1,n) = [1,(3k-2)(2n-1)+1]$ such that there is no monochromatic n-term 2-a.p. To complete the proof we show that $S(k+1,n,\phi) \geqslant 3k+1$.

Let $A_1 = [(3k-5)(2n-1)+1,(3k-4)(2n-1)+1]$, $A_2 = [(3k-4)(2n-1)+1,(3k-3)(2n-1)+1]$, and $A_3 = [(3k-3)(2n-1)+1,(3k-2)(2n-1)+1]$. Thus, $I(k+1,n) = I(k,n) \cup A_1 \cup A_2 \cup A_3$. For each j $(j = 1,2,3)$, since A_j has length $2n$, by the same argument used for $[1,2n]$ in the case of $k = 2$, A_j contains a monochromatic sequence $Y_j = \{e_j, o_j\}$, where e_j is even and o_j is odd. Let $\overline{\phi}$ represent the colouring ϕ restricted to the interval $I(k,n)$. Then, by using the contributions of the Y_j, we have that $S(k+1,n,\phi) \geqslant S\big(k,n,\overline{\phi}\big) + 3$. Therefore, by the inductive hypothesis, $S(k+1,n,\phi) \geqslant 3k+1$.

We have established the theorem for $m = 2$. Now let m be any positive even integer. Then by this result for $m = 2$, and Lemma 1, we have

$$g^{(3)}_{\frac{m}{2}(m)}(k,n) = \frac{m}{2}\Big(g^{(3)}_{1(2)}(k,n) - 1\Big) + 1 = \frac{m}{2}(3k-5)(2n-1)+1,$$

and the proof is complete. ∎

3. More than three colours

From the results of Section 2, we know that if $1 \leqslant a < m$ and $m/a \neq 2$, then $C^*_{a(m)}$ does not have the r-Ramsey property whenever $r \geqslant 3$. The remaining cases are those where either (i) $a = 0$; or (ii) $r \geqslant 4$ and $m/a = 2$.

The case of $a = 0$ is a unique case, and is also easier (one explanation for this is that when $a > 0$, an $a \,(\mathrm{mod}\ m)$-sequence can be thought of as an arithmetic progression modulo m, while a $0 \,(\mathrm{mod}\ m)$-sequence cannot be considered as part of the collection of arithmetic progression's modulo m). Note by the following theorem that the value of $g^{(r)}_{0(m)}(k,n)$ is independent of n.

Theorem 4. *For all* $r, m, k, n \geqslant 2$, $g^{(r)}_{0(m)}(k,n) = rm(k-1)+1$.

Proof: In $[1, rm(k-1)+1]$ there are $r(k-1)+1$ integers that are congruent to $1 \,(\mathrm{mod}\ m)$. Under any r-colouring of $[1, rm(k-1)+1]$ at least k of these integers must be monochromatic, giving a k-term monochromatic $0 \,(\mathrm{mod}\ m)$-sequence. Hence $g^{(r)}_{0(m)}(k,n) \leqslant rm(k-1)+1$.

To show the reverse inequality, let χ be the following r-colouring of $[1, rm(k-1)]$:

$$\underbrace{11\ldots1}_{m}\underbrace{22\ldots2}_{m}\cdots\underbrace{rr\ldots r}_{m}\underbrace{11\ldots1}_{m}\underbrace{22\ldots2}_{m}\cdots\underbrace{rr\ldots r}_{m}\cdots\cdots\cdots\underbrace{11\ldots1}_{m}\underbrace{22\ldots2}_{m}\cdots\underbrace{rr\ldots r}_{m}$$

where there are $r(k-1)$ blocks of size m. Then χ avoids monochromatic k-term $0 \,(\mathrm{mod}\ m)$-sequences. Also, since $n \geqslant 2$, χ avoids monochromatic n-term m-a.p.'s, and the proof is complete. ∎

For the only remaining case, $m/a = 2$ and $r \geqslant 4$, it turns out that $C^*_{a(m)}$ does not have the r-Ramsey property. In fact, the next theorem gives the stronger result that, whenever $m/(\gcd(a,m))$ is even, it is possible to 4-colour the positive integers so as to avoid monochromatic 2-term members of $C^*_{a(m)}$.

Theorem 5. *Let* $1 \leqslant a < m$ *and let* m/d *be even where* $d = \gcd(m,a)$. *Then* $g^{(4)}_{a(m)}(2,2) = \infty$.

Proof: By Lemma 1 it is sufficient to prove the result when $d = 1$. Hence we assume that m is even and a is odd. We give a colouring of the positive integers that contains no monochromatic 2-term $a \,(\mathrm{mod}\ m)$-sequences and no monochromatic 2-term m-a.p.'s.

Let Q be the string $1212\ldots12$ of length m, and let R be the string $3434\ldots34$ of length m. Colour the positive integers with the colouring $\chi = QRQRQR\ldots$. Clearly χ contains no monochromatic 2-term m-a.p. Also, since m is even, there is no pair of positive integers x and y such that $y-x$ is odd and $\chi(x)=\chi(y)$. Thus there is no monochromatic 2-term $a\,(\mathrm{mod}\ m)$-sequence. ∎

By Theorem 3 we know that we cannot replace the value of $r=4$ with $r=3$ in Theorem 5 and still have a true statement. Looking at Theorem 2 one might wonder whether there are any values of m and a for for which $g^{(3)}_{a(m)}(2,2)=\infty$. The next theorem shows that this is not the case.

THEOREM 6. *Let $1\leqslant a<m$. Then $g^{(3)}_{a(m)}(2,2)\leqslant 3m$.*

PROOF: Assume the statement is false. Hence for some a and m there exists a 3-colouring $\chi:[1,3m]\to\{1,2,3\}$ having no monochromatic 2-term $a\,(\mathrm{mod}\ m)$-sequence and no monochromatic 2-term m-a.p. Without loss of generality, let $\chi(m+1)=1$ and $\chi(2m+1)=2$.

By assumption, $\chi(1)\neq\chi(m+a+1)$, $\chi(m+1)\neq\chi(m+a+1)$, and $\chi(1)\neq\chi(m+1)$. Hence we have two cases: (a) $\chi(1)=2$ and $\chi(m+a+1)=3$, and (b) $\chi(1)=3$ and $\chi(m+a+1)=2$. We prove only case (a), as the proof of case (b) is essentially the same. Now if $\chi(1)=2$ and $\chi(m+a+1)=3$, then $\chi(2m+a+1)=1$, but then $\{m+1,2m+a+1\}$ is a monochromatic $a\,(\mathrm{mod}\ m)$-sequence, a contradiction. ∎

Considering Theorems 2, 3, and 5, one would suspect that, for $a>0$ and m/d odd, $g^{(4)}_{a(m)}(2,2)=\infty$. However, computer output suggests that, for m/d odd, the least r for which $g^{(r)}_{a(m)}(2,2)=\infty$, is always either 5 or 6. We are able to show that for all $m\geqslant 2$ and $a\neq 0$ it is possible to 6-colour the positive integers so as to avoid monochromatic 2-term $a\,(\mathrm{mod}\ m)$-sequences and monochromatic 2-term m-a.p.'s.

THEOREM 7. *Let $1\leqslant a<m$ and let m/d be odd where $d=\gcd(m,a)$. Then $g^{(6)}_{a(m)}(2,2)=\infty$.*

PROOF: By Lemma 1, we assume $d=1$. We provide a 6-colouring of the positive integers that avoids monochromatic 2-term members of $C^{*}_{a(m)}$. The proof splits naturally into two cases.

CASE 1. $a<m/2$. Let $m=qa+t$, where q is an integer and $0\leqslant t<a$. We have two subcases.

(i) q is even. Colour $[1,2m]$ with the colouring

$$ABAB\ldots AB\quad E'\quad CDCD\ldots CD\quad F',$$

where $A=11\ldots1$, $B=22\ldots2$, $C=33\ldots3$, and $D=44\ldots4$ each have length a, where $E'=55\ldots5$ and $F'=66\ldots6$ each have length t, and where each of A,B,C,D occurs $q/2$ times. Now extend this to a colouring of the positive integers by repeating it (that is, we now have a $2m$-periodic colouring of the positive integers). Call this colouring of the positive integers χ.

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There is no monochromatic 2-term m-a.p. with respect to χ, for: if $i \in A$, then $i+m \in C$; if $i \in B$, then $i+m \in D$; if $i \in C$, then $i+m \in A$; et cetera.

Now let $\{x,y\}$ be an $a(\bmod m)$-sequence. Let $x = cm+j$, where $1 \leqslant j \leqslant m$. If $1 \leqslant j \leqslant m-a$, then we see that $\chi(x) \neq \chi(x+a)$ and $\chi(x) \neq \chi(x+m+a)$. Therefore $\chi(x) \neq \chi(y)$. If, instead, $m-a < j \leqslant m$, then x belongs to some copy of B, E', D, or F', while y belongs to some copy of A or C. Thus $\{x,y\}$ is not monochromatic.

(ii) q is odd. In this case colour $[1,2m]$ with the colouring

$$ABAB\ldots AB \quad EB' \quad CDCD\ldots CD \quad FD'$$

where A, B, C, and D are defined as in subcase (i) and each occurs $(q-1)/2$ times, where $E = 55\ldots5$ and $F = 66\ldots6$ each have length a, and where $B' = 22\ldots2$ and $D' = 44\ldots4$ each have length t. As in subcase (i) extend this to a $2m$-periodic colouring of the positive integers. Since $ABAB\ldots ABEB'$ has length m, it is clear that there is no monochomatic 2-term m-a.p.

Let $\{x,y\}$ be an $a(\bmod m)$-sequence. If $x = cm+j$ where $1 \leqslant j \leqslant m-a$, then $\{x,y\}$ cannot be monochromatic for the same reason given in subcase (i). If $m-a < j \leqslant m$, then x belongs to some copy of E, B', F, or D', while y belongs to some copy of A or C. This proves the theorem for the case in which $a < m/2$.

CASE 2. $a > m/2$. (Equality is impossible since m/d is odd.) Let $b = m-a$. Let $m = q'b+t'$, where q' is an integer and $0 \leqslant t < b$. Again we have two subcases, depending on whether q' is even or odd. For convenience, we do the subcase in which q' is even, as the other is done in the same way. We use the same colouring that was used in Case 1, subcase (i), except replace a by b, q by q' and t by t'. Denote this new colouring by χ'. Then we know that χ' will avoid monochromatic 2-term m-a.p.'s and monochromatic 2-term $b(\bmod m)$-sequences. The proof will be completed by showing that there are also no monochromatic 2-term $a(\bmod m)$-sequences. Assume, by way of contradiction, that $\{x,y\}$ is a monochromatic $a(\bmod m)$-sequence under χ'. Let h be even such that $x+hm > y$. Then $\{y, x+hm\}$ is a $b(\bmod m)$-sequence and, since χ' is $2m$-periodic, $\chi'(x) = \chi'(x+hm)$, so that $\{y, x+hm\}$ is a monochromatic $b(\bmod m)$-sequence, a contradiction. ∎

We summarise the results on the Ramsey properties of $C^*_{a(m)}$ in the following table. For the different choices of m, a, and r, we give the asymptotic value, as $k \to \infty$, of the associated Ramsey function $g^{(r)}_{a(m)}(k,k)$.

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values of a, m	$r = 2$	$r = 3$	$r \geqslant 4$
$m/a = 2$	mk^2	$3mk^2$	∞
$m/a \neq 2$, $a \neq 0$	mk^2	∞	∞
$a = 0$	$2mk$	$3mk$	rmk

Table 1. Asymptotic value of $g^{(r)}_{a(m)}(k,k)$

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