RELATIONS ON TOPOLOGICAL SPACES: URYSOHN'S LEMMA

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1. Introduction and Preliminaries

Let X be a topological space equipped with a binary relation R; that is, R is a subset of the Cartesian square $X \times X$. Following Wallace [5], we write

$$xR = \{y : (x, y) \in R\}, \qquad Rx = \{y : (y, x) \in R\},$$
$$RA = \cup \{Rx : x \in A\} \text{ and } AR = \cup \{xR : x \in R\}.$$

Deviating from [7], we shall follow Wallace [4] to call the relation R *continuous* if $RA^* \subset (RA)^*$ for each $A \subset X$, where $*$ designates the topological closure. Borrowing the language from the Ordered System, though our relation R need not be any kind of order relation, we say that a subset S of X is *R-decreasing* (*R-increasing*) if $RS \subset S (SR \subset S)$, and that S is *R-monotone* if S is either R-decreasing or R-increasing. Two R-monotone subsets are of the *same type* if they are either both R-decresaing or both R-increasing.

Definition 1. A topological space X equipped with a relation R is said to be *R-normal* if, and only if, A_1 and A_2 are two disjoint closed subsets of X such that either A_1 is R-decreasing or A_2 is R-increasing then there exist two disjoint R-monotone open subsets U_1 and U_2 of X such that $A_1 \subset U_1$, $A_2 \subset U_2$ and that U_1 is R-decreasing and U_2 is R-increasing.

Our definition of R-normality agrees with that of strong normality of Ward's [6].

It should be noted that the *trivial* relation, $\Delta = \{(x, x) : x \in X\}$, is continuous and that a normal space is a particular R-normal space in which R is trivial. Recall that a family of subsets of X is *point-finite* (*locally finite*) if every point of X belongs to (has a neighborhood that meets) at most a finite number of sets in the family; in particular, all finite, all star finite [2], and all locally finite families are point-finite.

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The purpose of this note is to prove the following generalization of the famous Urysohn's Lemma.

THEOREM. *Let X be a topological space equipped with a continuous relation R such that X is R-normal, F an R-increasing (R-decreasing) closed subset of X, and $\{U_\alpha : \alpha \in A\}$ a locally finite family of R-increasing (R-decreasing) open sets in X such that $\cup \{U_\alpha : \alpha \in A\} \supset F$. Then there exists a family $\{f_\alpha : \alpha \in A\}$ of continuous functions on X with values in $[0, 1]$ such that*

(i) *if* $(x, y) \in R$, then $\sum_{\alpha \in A} f_\alpha(x) \leqq \sum_{\alpha \in A} f_\alpha(y)$;

(ii) $\sum_{\alpha \in A} f_\alpha(x) = 1$ $(\sum_{\alpha \in A} f_\alpha(x) = 0)$ *for all* $x \in F$; *and*

(iii) *for each* $\alpha \in A$, $f_\alpha(x) = 0$ $(f_\alpha(x) = 1)$ *for all* $x \in X \setminus U_\alpha$.

We are inspired by [3], [4], [5], and [6].

2. Proof of the main theorem

Let us record a number of readily established facts and some useful lemmas. *In all that follows, let X be always a topological space equipped with a relation R.*

2.1 *A subset A of X is R-decreasing (R-increasing) if, and only if, $X \setminus A$ is R-increasing (R-decreasing)* [6].

2.2 *If A is an R-decreasing (R-increasing) subset of X and if R is continuous, then A^* and A^0 are R-decreasing (R-increasing), where ^{0} denotes interior.*

2.3 *If $\{A_\alpha\}$ is a family of R-decreasing (R-increasing) subsets of X, then*

$$\bigcup_\alpha A_\alpha \text{ and } \bigcap_\alpha A_\alpha$$

are R-decreasing (R-increasing) [6].

2.4 *The empty subset $\square$ and the whole space X are both R-decreasing as well as R-increasing.*

LEMMA 1. (Urysohn-Nachbin-Ward). *The space X is R-normal if, and only if, A and B are two disjoint closed sets in X such that either A is R-decreasing or B is R-increasing, then there exists a continuous function h on X with values in $[0, 1]$ such that $h(x) \leqq h(y)$ whenever $(x, y) \in R$ and that $h(x) = 0$ for all $x \in A$ and $h(x) = 1$ for all $x \in B$.*

PROOF. To prove the sufficiency, let $U = \{x \in X | 0 \leqq h(x) < \frac{1}{2}\}$ and $V = \{x \in X | \frac{1}{2} < h(x) \leqq 1\}$, then $U \cap V = \square$, $A \subset U$ and $B \subset V$. By the continuity of h, one sees that U and V are open. Finally, since h satisfies the property that $(x, y) \in R$ implies $h(x) \leqq h(y)$, U is R-decreasing and V is R-increasing. Thus, X is R-normal.

The necessity part of the proof may be found in Ward [6, page 363].

LEMMA 2. *Let R be a continuous relation on X, then X is R-normal if, and only if, for any R-decreasing (R-increasing) closed set A contained in any open set $U \subset X$, there exists an R-decreasing (R-increasing) open set $U_0 \subset X$ such that $A \subset U_0 \subset U_0^* \subset U$.*

PROOF. If X is R-normal, A an R-decreasing (R-increasing) closed set contained in an open set U, then A and $X \setminus U$ are two disjoint closed sets in X such that A is R-decreasing (R-increasing). The R-normality of X implies that there exists an R-increasing (R-decreasing) open set V containing $X \setminus V$ and missing an R-decreasing (R-increasing) neighborhood of A. Taking $U_0 = X \setminus V^*$, one shows by (2.1) and (2.2) that U_0 fulfills the requirements.

Conversely, if A_1 and A_2 are two disjoint closed subsets of X such that A_1 is R-decreasing; the case where A_2 is R-increasing may be argued similarly. Then, since A_1 is contained in the open set $X \setminus A_2$, there exists an R-decreasing open set U_0 such that $A_1 \subset U_0 \subset U_0^* \subset X \setminus A_2$. Taking $U_1 = U_0$ and $U_2 = X \setminus U_0^*$, then U_1 and U_2 are two disjoint open sets such that $A_1 \subset U_1$, $A_2 \subset U_2$ and that U_1 is R-decreasing and U_2 is R-increasing, by (2.1) and (2.2). Thus, X is R-normal.

We say that a *family of subsets of X is R-increasing (R-decreasing, R-monotone)* if each of its member is an R-increasing (R-decreasing, R-monotone) subset of X; thus, an R-increasing open cover of X is a cover of X by R-increasing open sets.

LEMMA 3. *Let X be an R-normal space such that R is continuous, $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ a point-finite, R-increasing (R-decreasing) open cover of X, then there exists an R-increasing (R-decreasing) open cover $\mathscr{V} = \{V_\alpha : \alpha \in A\}$* [2] *of X such that $V_\alpha^* \subset U_\alpha$, for all $\alpha \in A$.*

PROOF. We shall prove only the case where $\mathscr{U}$ is R-increasing, the case where $\mathscr{U}$ is R-decreasing may be proved similarly.

By the well-ordering principle, let the indexing set A be well-ordered by a certain well-ordering denoted by $\leqq$. Let $\mathscr{V}^{(o)} = \mathscr{U}$ where o denotes a symbol not in A; define $o < \alpha$ for all $\alpha \in A$. For each β in A, we shall construct inductively an R-increasing open cover $\mathscr{V}^{(\beta)} = \{V_{\alpha,\beta} : \alpha \in A\}$ satisfying

[2] It should be remarked that we allow some of the V_α's to be empty [cf. (2.4)]. In fact, some of the V_α's may have to be empty even when all U_α are nonempty. The referee has contributed the following example: let $X = \{x_1, x_2\}$ with open sets: X, $\{x_2\}$ and the empty set $\square$. Let X by equipped with the trivial relation $R = \Delta$, then X is R-normal. Consider $\mathscr{U} = \{U_1, U_2\}$ where $U_1 = \{x_2\}$ and $U_2 = X$; in this case, the only possible choice of $\mathscr{V} = \{V_1, V_2\}$ is $V_1 = \square$ and $V_2 = X$.

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(i) $V^*_{\alpha,\beta} \subset U_\alpha$ for all $\alpha \leqq \beta$, and (ii) $V_{\alpha,\beta} = U_\alpha$ for all $\alpha > \beta$. Suppose for some $\gamma \in A$ that all $\mathscr{V}^{(\beta)}$ with $\beta < \gamma$ have been constructed, then we shall define $\mathscr{V}^{(\gamma)}$ as follows: define

$$V_{\alpha,\gamma} = \begin{cases} V_{\alpha,\alpha} & \text{for all } \alpha < \gamma, \\ U_\alpha & \text{for all } \alpha > \gamma. \end{cases}$$

To define $V_{\gamma,\gamma}$, we first observe that

$$U_\gamma \cup [\cup \{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}] = X$$

and hence $X \setminus U_\gamma$ and $X \setminus \cup \{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}$ are two disjoint (R-monotone) closed sets. By R-normality of X, there exist two disjoint open sets S and T such that $X \setminus U_\gamma \subset S$ and $T \supset X \setminus \cup\{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}$. Since $X \setminus U_\gamma$ is R-decreasing, by Lemma 2, there exists an R-decreasing open set Q such that $X \setminus U_\gamma \subset Q \subset Q^* \subset S$. Now let $V_{\gamma,\gamma} = X \setminus Q^*$, then by (2.1) and (2.2) $V_{\gamma,\gamma}$ is R-increasing. We have $V^*_{\gamma,\gamma} \subset X \setminus Q \subset U_\gamma$. Furthermore, since Q^* and $X \setminus \cup \{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}$ are disjoint, we have

$$\begin{aligned} \cup \{V_{\alpha,\alpha} : \alpha \in A\} &= V_{\gamma,\gamma} \cup [\cup \{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}] \\ &= (X \setminus Q^*) \cup [\cup \{V_{\alpha,\gamma} : \alpha \in A \ \& \ \alpha \neq \gamma\}] = X. \end{aligned}$$

Thus, by the transfinite induction, $\mathscr{V}^{(\gamma)}$ is well-defined for every $\gamma \in A$. It is to be noted that the γ^{th} cover $\mathscr{V}^{(\gamma)}$ contains an initial segment of every cover $\mathscr{V}^{(\beta)}$ for $\beta < \gamma$. Let $\mathscr{V} = \{V_\alpha : \alpha \in A\}$ be defined by $V_\alpha = V_{\alpha,\alpha}$ for all $\alpha \in A$. Then, it needs only to be proved that $\mathscr{V}$ covers X. To this end, let x be an arbitrary element in X, then by the point-finiteness of $\mathscr{U}$ there exist at least one and at most finitely many $U_{\alpha_1}, \cdots, U_{\alpha_n}$ containing x. Let $\alpha_0 = \max \{\alpha_1, \cdots, \alpha_n\}$, under the well-ordering we have introduced at the very beginning. Consequently, $x \notin U_\gamma$ for all $\gamma > \alpha_0$; but since $\mathscr{V}^{(\alpha_0)}$ covers X, there exists an $\beta \leqq \alpha_0$ such that $x \in V_{\beta,\alpha_0} = V_{\beta,\beta} = V_\beta \in \mathscr{V}$.

Proof of Theorem. Since F is an R-increasing (R-decreasing) closed subset of the R-normal space X, the set F equipped with the relative topology and the relation $R_F = R \cap (F \times F)$ forms an R_F-normal space, and the family $\{U_\alpha \cap F; \alpha \in A\}$ is a point-finite R_F-increasing (R_F-decreasing) open (in F) cover of F. By Lemma 3, there exists an R_F-increasing (R_F-decreasing) open (in F) cover $\{V_\alpha : \alpha \in A\}$ of F such that $V^*_\alpha \subset U_\alpha \cap F \subset U_\alpha$ for all $\alpha \in A$. Since each V_α is an R_F-increasing (R_F-decreasing) subset of the R-increasing (R-decreasing) subset F of X, each V_α is R-increasing (R-decreasing) in X and hence, by (2.2), each V^*_α is R-increasing (R-decreasing) in X. Now, for each $\alpha \in A$, applying Lemma 1 to two disjoint R-monotone closed sets V^*_α and $X \setminus U_\alpha$ [3] in X, we have a continuous function

[3] If $V^*_\alpha = \square$, regardless of $X \setminus U_\alpha = \square$ or otherwise, one may choose, for instance, the constant function $h_\alpha(x) = 0$ (resp. $h_\alpha(x) = 1$) for all $x \in X$; if $V^*_\alpha \neq \square$ and $X \setminus U_\alpha = \square$, then one chooses, for instance, the constant function $h_\alpha(x) = 1$ (resp. $h_\alpha(x) = 0$) for all $x \in X$.

$h_\alpha : X \to [0, 1]$ such that

$$h_\alpha(x) \leqq h_\alpha(y) \text{ whenever } (x, y) \in R,$$
$$h_\alpha(x) = 1 \; (h_\alpha(x) = 0) \text{ for all } x \in V^*_\alpha, \text{ and}$$
$$h_\alpha(x) = 0 \; (h_\alpha(x) = 1) \text{ for all } x \in X \diagdown U_\alpha.$$

For each $\alpha \in A$, define a function $f_\alpha : X \to [0, 1]$ by

$$f_\alpha(x) = h_\alpha(x)/\max \{\sum_{\alpha \in A} h_\alpha(x), 1\}$$

for all x in X. To show that each f_α is continuous, it suffices to show that the function $\sum_{\alpha \in A} h_\alpha : X \to [0, \infty)$ is continuous: for each x in X, by the local finiteness of $\{U_\alpha : \alpha \in A\}$, there exists an open neighborhood N_x of x such that $N_x \cap U_\alpha \neq \square$ for at least one and at most finitely many α's, say $\alpha_1, \alpha_2, \cdots \alpha_m$; therefore, for any open set G in $[0, \infty)$,

$$(\sum_{\alpha \in A} h_\alpha)^{-1}(G) \cap N_x = (h_{\alpha_1} + \cdots + h_{\alpha_m})^{-1}(G) \cap N_x$$

which is open, because $h_{\alpha_1} + \cdots + h_{\alpha_m} : X \to [0, \infty)$, as a sum of finitely many continuous functions, is continuous; and hence,

$$(\sum_{\alpha \in A} h_\alpha)^{-1}(G) = \bigcup_{x \in X} [(\sum_{\alpha \in A} h_\alpha)^{-1}(G) \cap N_x],$$

as a union of open sets, is open. Finally, by a routine verification, one verifies that the family $\{f_\alpha : \alpha \in A\}$ of continuous functions satisfies the conditions (i), (ii) and (iii) stated in the theorem.

Since every *normal* space is R-normal, where R is the (continuous) trivial relation Δ, and every finite family of open sets is locally finite, therefore we have the following Dieudonné's partition of unity.

COROLLARY (Dieudonné [1]). *Let X be a normal space, F a closed subset of X, and $U_1, U_2, \cdots, U_n$ open sets such that $\bigcup_{k=1}^n U_k \supset F$. Then there exist continuous functions $h_1, \cdots, h_n$ on X with values in $[0, 1]$ such that*

(i) $\sum_{k=1}^{n} h_k(x) = 1$ *for all* $x \in F$;

(ii) $h_k(x) = 0$ *for all* $x \in X \diagdown U_k$ *and for* $k = 1, 2, \cdots, n$.

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