Proceedings of the Edinburgh Mathematical Society (1983) **26**, 7–13 ©

THE PARTIALLY ORDERED FAMILY OF ALL $\mathscr{R}$-CLASSES OF $S(X)$

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(Received 12th February 1981)

1. Introduction

Let $S(X)$ denote the semigroup of all continuous selfmaps of the topological space X. Let $\mathscr{L}(S(X))$ and $\mathscr{R}(S(X))$ denote the partially ordered families of all $\mathscr{L}$-classes and $\mathscr{R}$-classes, respectively, of $S(X)$ where the partial orders are the usual ones [**3**, p. 29]. In [**6**], we made the following

Conjecture. *The following statements are equivalent about any two compact* 0-*dimensional metric spaces X and Y:*

(1) $\mathscr{L}(S(X))$ *and* $\mathscr{L}(S(Y))$ *are order isomorphic.*

(2) $\mathscr{R}(S(X))$ *and* $\mathscr{R}(S(Y))$ *are order isomorphic.*

(3) *The semigroups* $S(X)$ *and* $S(Y)$ *are isomorphic.*

(4) *The spaces X and Y are homeomorphic.*

The equivalence of (1), (3) and (4) had previously been shown in [**5**] and, of course, it is immediate that (3) implies (2) so that in order to prove the conjecture, it is sufficient to show that statement (2) implies any one of the others. We show, in fact, that (2) implies (4) for a class of spaces which properly includes the class of compact 0-dimensional metric spaces as well as all dendrites. These spaces are defined and some of their properties deduced in Section 2. The main theorem of the paper is formally stated and proved in Section 3.

2. $\mathscr{R}$-admissible spaces

We assume that all topological spaces are Hausdorff. By a retract of a space X, we mean the range of an idempotent continuous selfmap of X. We will denote the range of a function f by Ran f. We first recall a definition from [**8**].

Definition (2.1). A space X is *retractably generated* if the family of all retracts of X forms a subbasis for the closed subsets of X.

In [**8**] it was shown that products of retractably generated spaces are retractably generated and since it is immediate that both the closed unit interval and the real line are retractably generated, this implies that all Euclidean N-cells and N-spaces are retractably generated and so is Hilbert space and the Hilbert cube. In addition, all 0-dimensional spaces are retractably generated as was also noted in [**8**]. By 0-dimensional, we mean a space which has a basis of sets which are both closed and open. There is yet another well-known class of spaces which are all retractably generated. Recall that a Peano continuum is a compact, connected locally connected metric space and that a dendrite is a Peano continuum which contains no simple closed curves.

Proposition (2.1). *All dendrites are retractably generated.*

Proof. Let X be a dendrite, F a nonempty closed subset of X and p a point in $X-F$. For any $x\in X$, let $N(x,\varepsilon)$ denote the open neighborhood about x with radius ε. Choose $\varepsilon>0$ so small that $N(p,2\varepsilon)\cap F=\varnothing$. Since X is locally connected, there exists for each $x\in X$, a connected open subset G_x such that $x\in G_x$ and, moreover, we may choose G_x so that $\operatorname{cl} G_x\subset N(x,\varepsilon)$. Then $\{G_x:x\in F\}$ is an open cover of F and since F is compact, some finite subcollection $\{G_{x_i}\}_{i=1}^N$ is sufficient to cover F. Thus we have

$$F\subset\cup\{G_{x_i}\}_{i=1}^N\subset\cup\{\operatorname{cl} G_{x_i}\}_{i=1}^N$$

and

$$p\notin\cup\{\operatorname{cl} G_{x_i}\}_{i=1}^N$$

Now each $\operatorname{cl} G_{x_i}$ is a subcontinuum of X and hence a dendrite by [**10,** p. 80]. Since a dendrite is an absolute retract [**1,** p. 138], it follows that each $\operatorname{cl} G_{x_i}$ is a retract of X and the proof is completed.

For a space which is not retractably generated consider the following

Example (2.2). Let $\mathbb{R}^2$ denote the Euclidean plane and let

$$A_n=\{(x,y)\in\mathbb{R}^2:y=x/n \quad\text{and}\quad 0\leqq x\leqq 1\}.$$

Then define $X=\cup\{A_n\}_{n=1}^\infty$. We claim that X is not retractably generated. Let

$$F=\{(x,y)\in X:\tfrac{1}{2}\leqq x\leqq 1\}$$

and let $p=(0,0)$. Then F is closed, $p\notin F$ but any closed subset H satisfying $p\notin H$ and $F\subset H$ will have an infinite number of components and consequently, cannot be a finite union of retracts of X.

Thus far, our retractably generated spaces have been either connected or totally disconnected but there are many retractably generated spaces which satisfy neither of these two conditions. Evidence of this is given by

Proposition (2.3). *The free union of any collection of retractably generated spaces is a retractably generated space.*

Proof. Let X be the free union of the retractably generated spaces $\{X_\alpha:\alpha\in\Lambda\}$. Let F be a closed subset of X and let $p\in X-F$. We may assume $p\in X_1$. Since X_1 is retractably generated, there exists a finite collection $\{H_i\}_{i=1}^N$ of retracts of X_1 such that $F \cap X_1 \subset \cup\{H_i\}_{i=1}^N$ and $p \notin \cup\{H_i\}_{i=1}^N$. Now each H_i $(1 \leqq i \leqq N)$ is also a retract of X and $H_{N+1} = \cup\{X_\alpha : \alpha \neq 1\}$ is a retract of X. Consequently, $\{H_i\}_{i=1}^{N+1}$ is a finite collection of retracts of X whose union contains F but not the point p.

We introduce another term.

Definition (2.4). A space X is *range retractable* if the range of each continuous selfmap of X is a retract of X.

Among the 0-dimensional metric spaces, the range retractable spaces can be easily characterized.

Proposition (2.5). *A 0-dimensional metric space is range retractable if and only if it is either compact or discrete.*

Proof. Let X be any 0-dimensional metric space. We assume it is not discrete and prove, by contradiction, that it is compact. If it is not compact, then it is not countably compact since the two notions coincide for metric spaces. Thus, for any open basis, there will exist a countable sequence $\{V_n\}_{n=1}^\infty$ of basic open sets which covers X and has the property that no finite subcollection covers X. Since X is 0-dimensional, we may take each V_n to be closed as well as open. Define a sequence $\{W_n\}_{n=1}^\infty$ by $W_1 = V_1$ and

$$W_n = V_n - \cup\{V_i\}_{i=1}^{n-1}$$

for $n > 1$. Each of the sets W_n is both open and closed and $W_n \cap W_m = \emptyset$ when $n \neq m$. Moreover, $\{W_n\}_{n=1}^\infty$ is a cover of X and no finite subcollection will serve as a cover. There is no loss of generality if we assume that $W_n \neq \emptyset$ for all n. Since X is not discrete, there exists a sequence of distinct points $\{y_n\}_{n=1}^\infty$ converging to a point p where $p \neq y_n$ for all n. Define a selfmap f of X by $f(x) = y_n$ for $x \in W_n$. The map f is continuous but $\operatorname{Ran} f = \{y_n\}_{n=1}^\infty$ is not closed since it lacks the limit point p. Consequently, $\operatorname{Ran} f$ cannot possibly be a retract and we have the contradiction.

Now suppose that X is either compact or discrete. In either case, X is a 0-dimensional metric space with the property that $\operatorname{Ran} f$ is closed for each $f \in S(X)$. It now follows from [**4**, p. 281] that $\operatorname{Ran} f$ is a retract for each $f \in S(X)$.

As it turns out, most range retractable spaces are compact. The latter result together with our next one, gives evidence of this. Recall that a space is realcompact if it can be embedded into a product of real lines as a closed subspace. Evidently all realcompact spaces are completely regular. The converse is not true but the usual examples to the contrary require some effort to produce. The best known example of a completely regular space which is not realcompact is perhaps the space of ordinals less than the first uncountable ordinal. References on realcompact spaces include [**2**] and [**9**].

Proposition (2.6). *Every realcompact, range retractable space which contains an arc is compact.*

Proof. Suppose X is not compact. Then it is not pseudocompact [2, p. 79] and there exists an unbounded continuous function f from X into the reals $\mathbb{R}$. Let A be an arc in X with endpoints a and b and let h be any homeomorphism from $\mathbb{R}$ onto $A-\{a,b\}$. Then $h \circ f \in S(X)$ and $\operatorname{Ran} h \circ f$ is not closed since at least one of the points a or b is a limit point of $\operatorname{Ran} h \circ f$. Consequently, $\operatorname{Ran} h \circ f$ is not a retract of X and a contradiction has been reached.

Thus far we have shown that most range retractable spaces must necessarily be compact and that the converse holds whenever the spaces involved are 0-dimensional. In general, however, the converse is far from valid.

Example (2.7). Let X be any compact space which contains a copy of the Euclidean plane $\mathbb{R}^2$. Then X contains a copy D of the closed unit disk whose boundary B is a simple closed curve. Choose any two distinct points $a, b \in D$ and let f be any continuous function from X into the closed unit interval I such that $f(a)=0$ and $f(b)=1$. Since D is connected $\operatorname{Ran} f = I$. Now, let g be any continuous function from I *onto* B. Then $g \circ f \in S(X)$ and $\operatorname{Ran} g \circ f = B$. However, no continuous selfmap of X can retract X onto B since no disk can be retracted onto its boundary. Thus, X is not range retractable.

Connectivity is also closely tied to the notion of a range retractable space.

Proposition (2.8). *A range retractable space is either connected or totally disconnected.*

Proof. Suppose X is range retractable and not totally disconnected. Then X has a component A containing two distinct points a and b. We assume X is not connected. Then it is the union of two nonempty disjoint open subsets G and H. Define $f(x)=a$ for $x \in G$ and $f(x)=b$ for $x \in H$. The function f belongs to $S(X)$ and $\operatorname{Ran} f = \{a,b\}$. However, $\{a,b\}$ cannot possibly be the range of an idempotent map in $S(X)$. Any such map g would fix both a and b so that $\{a,b\} \subset g[A]$. Since A is connected, $g[A]$ is also and we couldn't possibly have $\operatorname{Ran} g = \{a,b\}$. Thus, a contradiction has been reached.

Our next definition introduces the spaces with which the main theorem is concerned.

Definition (2.9). A topological space is *$\mathscr{R}$-admissible* if it is both retractably generated and range retractable.

Proposition (2.10). *The following statements are equivalent about a* 0*-dimensional metric space X:*

(2.10.1) *X is $\mathscr{R}$-admissible,*

(2.10.2) *X is range retractable,*

(2.10.3) *X is either compact or discrete.*

Proof. It is immediate that (2.10.1) implies (2.10.2) and it follows from Proposition (2.5) that (2.10.2) implies (2.10.3). The comments following Definition (2.1) together with Proposition (2.5) allow us to conclude that (2.10.3) implies (2.10.1) and the proof is complete.

Our next result gives still another source for $\mathscr{R}$-admissible spaces.

Proposition (2.12). *All dendrites are $\mathscr{R}$-admissible.*

Proof. Let X be any dendrite. Then X is retractably generated in view of Proposition (2.2). Let $f \in S(X)$. Then $\operatorname{Ran} f$ is a subcontinuum of X and is therefore, itself, a dendrite [**10**, p. 89]. Again, we recall that dendrites are absolute retracts [**1**, p. 138] so that $\operatorname{Ran} f$ is a retract of X. This proves the result.

We now see that all dendrites and all compact 0-dimensional metric spaces are $\mathscr{R}$-admissible. We give an example of an $\mathscr{R}$-admissible space which belongs to neither of these two classes.

Example (2.13). Our space X is a subspace of the Euclidean plane. Let

$$A = \{(x, y) \in \mathbb{R}^2 : 0 < x \leqq 1/n \quad \text{and} \quad y = |x \sin 1/x|\}$$

$$B = \{(x, y) \in \mathbb{R}^2 : 0 < x \leqq 1/n \quad \text{and} \quad y = -|x \sin 1/x|\}$$

and then let $X = A \cup B \cup \{(0, 0)\}$. Topologically, what we have here is an infinite sequence of circles, each one tangent to the next, converging down to a point. The space X is certainly not 0-dimensional and it is far from being a dendrite. Nevertheless, it is $\mathscr{R}$-admissible. We will omit the details.

3. The main theorem

Let us recall that $\mathscr{R}(S(X))$ denotes the partially ordered family of all $\mathscr{R}$-classes of the semigroup $S(X)$ where one defines $R_1 \leqq R_2$ when the principal right ideal generated by any of the elements of R_1 is contained in the principal right ideal generated by any of the elements in R_2. And now we are in a position to state and prove our

Main theorem. *The following statements about any two $\mathscr{R}$-admissible spaces X and Y are equivalent:*

(1) *The partially ordered sets $\mathscr{R}(S(X))$ and $\mathscr{R}(S(Y))$ are order isometric,*

(2) *The semigroups $S(X)$ and $S(Y)$ are isomorphic,*

(3) *The spaces X and Y are homeomorphic.*

It is evident that (3) implies (2) and that (2) implies (1). The proof will therefore be complete when we have shown that (1) implies (3) and to do this, it will be convenient to have some lemmas to assist us.

Lemma (3.1). *Let X be any topological space whatsoever and let R_f be the $\mathscr{R}$-class in $S(X)$ which contains f. Then R_f is a minimal element of $\mathscr{R}(S(X))$ if and only if f is a constant function.*

The proof is an easy exercise and will not be given.

Definition (3.2). For any $R_f \in \mathscr{R}(S(X))$ we define $SPT(R_f)=\{R_g \in \mathscr{R}(S(X)): R_g$ is a minimal element and $R_g \leqq R_f\}$ and we refer to $SPT(R_f)$ as the *support* of R_f.

Lemma (3.3). *Let X be any topological space whatsoever and let $f, g \in S(X)$. Then* $\operatorname{Ran} f = \operatorname{Ran} g$ *if and only if* $SPT(R_f)=SPT(R_g)$.

This proof is also straightforward and will be omitted. Our next lemma concerns regular $\mathscr{R}$-classes. By a regular $\mathscr{R}$-class, we mean, of course, one which consists entirely of regular elements.

Lemma (3.4). *Let X be any topological space whatsoever and let $f, g \in S(X)$. Suppose that R_f is regular, $R_f \leqq R_g$ and $SPT(R_f)=SPT(R_g)$. Then $R_f = R_g$.*

Proof. Since $R_f \leqq R_g$, we have $f = g \circ k$ for some $k \in S(X)$. Since X is regular, Theorem (3.1) of [7] assures us that $\operatorname{Ran} f$ is a retract of X and that f maps some subspace A of X homeomorphically onto $\operatorname{Ran} f$. Then k is injective on A, g is injective on $k[A]$ and

$$g[k[A]] = f[A] = \operatorname{Ran} f.$$

That is, g maps $k[A]$ bijectively onto $\operatorname{Ran} f$. Now let H be any closed subset of $k[A]$. One verifies that

$$g[H] = f[A \cap k^{-1}[H]].$$

Since f is a homeomorphism on A and $A \cap k^{-1}[H]$ is a closed subset of A, it follows that $g[H]$ is a closed subset of $\operatorname{Ran} f$. Consequently, g maps $k[A]$ homeomorphically onto $\operatorname{Ran} f$. Since $SPT(R_f)=SPT(R_g)$, it follows from Lemma 3 that $\operatorname{Ran} f = \operatorname{Ran} g$ and we have now shown that g maps $k[A]$ onto $\operatorname{Ran} g$ which is a retract of X. In view of Theorem (3.1) of [7], this means that g is also regular. It now follows from Theorem (3.2) of [7] that f and g are $\mathscr{R}$-equivalent or, in other words, that $R_f = R_g$.

Lemma (3.5). *Let X be a range retractable space and let $f \in S(X)$. Suppose that for every $g \in S(X)$, $R_f \leqq R_g$ and $SPT(R_f)=SPT(R_g)$ together imply $R_f = R_g$. Then R_f is a regular $\mathscr{R}$-class.*

Proof. Since X is a range retractable space, there exists an idempotent $v \in S(X)$ such that $\operatorname{Ran} f = \operatorname{Ran} v$. Then $f = v \circ f$ which means $R_f \leqq R_v$ and it follows from Lemma (3.3) that $SPT(R_f)=SPT(R_v)$. The hypothesis now applies and we have $R_f = R_v$ which means that R_f is a regular $\mathscr{R}$-class.

From Lemmas (3.4) and (3.5), we immediately get the following

Lemma (3.6). *Let X be a range retractable space and let $R_f \in \mathscr{R}(S(X))$. Then R_f is a* *regular $\mathscr{R}$-class if and only if for every $R_g \in \mathscr{R}(S(X))$, $R_f \leqq R_g$ and $SPT(R_f) = SPT(R_g)$ together imply $R_f = R_g$.*

Now we are in a position to complete the proof of the main theorem and, as we have observed previously, we need only show that (1) implies (3). With this in mind, let X and Y be $\mathscr{R}$-admissible spaces and let φ be an order isomorphism from $\mathscr{R}(S(X))$ onto $\mathscr{R}(S(Y))$. Let $\mathscr{R}_R(S(X))$ and $\mathscr{R}_R(S(Y))$ denote the partially ordered families of *regular* $\mathscr{R}$-classes of $S(X)$ and $S(Y)$ respectively. Lemmas (3.1), (3.3) and (3.6) together characterise the regular $\mathscr{R}$-classes of $S(X)$ and $S(Y)$ within the partially ordered sets $\mathscr{R}(S(X))$ and $\mathscr{R}(S(Y))$ respectively by means of those partial orders. Consequently, the order isomorphism φ must map $\mathscr{R}_R(S(X))$ isomorphically onto $\mathscr{R}_R(S(Y))$. Since X and Y are both retractably generated, Theorem (3.4) of [**8**] applies and we conclude that X and Y are homeomorphic. It should be noted that in [**8**], the symbol $\mathscr{R}(S(X))$ was used to denote the collection of regular $\mathscr{R}$-classes of $S(X)$. This completes the proof of the main theorem.

As we observed in the introduction, the equivalence of statements (1), (3) and (4) of the *conjecture* was established in [**5**]. Since compact 0-dimensional metric spaces are $\mathscr{R}$-spaces, it follows from the main theorem of this paper that the latter statements are also equivalent to (2). Hence, the *conjecture*, first made in [**6**] is now a theorem.

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