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ON NILPOTENT EXTENSIONS OF ALGEBRAIC NUMBER FIELDS I

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Introduction

The lower central series of the absolute Galois group of a field is obtained by iterating the process of forming the maximal central extension of the maximal nilpotent extension of a given class, starting with the maximal abelian extension. The purpose of this paper is to give a cohomological description of this central series in case of an algebraic number field. This description is based on a result of Tate which states that the Schur multiplier of the absolute Galois group of a number field is trivial. We are in a profinite situation throughout which requires some functorial background especially for treating the dual of the Schur multiplier of a profinite group. In a future paper we plan to apply our results to construct a nilpotent reciprocity map.

§ 1. Central extensions and Schur multipliers

Let k be an algebraic number field of finite degree over the rationals $\mathbf{Q}$, and let k^{ab} (resp. k^{nil}) be the maximal abelian (resp. nilpotent) extension of k in the algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. For each positive integer c denote by $k^{(c)}/k$ the maximal nilpotent extension of class (at most) c. Hence $k^{(1)} = k^{\mathrm{ab}}$ and $k^{\mathrm{nil}} = \bigcup_{c=1}^{\infty} k^{(c)}$. For convenience we set $k^{(0)} = k$. Put $G^c = \mathrm{Gal}(k^{(c)}/k)$ and $N^c = \mathrm{Gal}(k^{(c)}/k^{(c-1)})$; N^c is a closed normal subgroup of G^c which is contained in the center $Z(G^c)$. Therefore we have a central extension of Galois groups

$$1 \longrightarrow N^{c+1} \longrightarrow G^{c+1} \longrightarrow G^c \longrightarrow 1\,.$$

We furnish the rational torus group $T = \mathbf{Q}/\mathbf{Z}$ with the discrete topology and consider it as a Galois module with trivial action.

PROPOSITION 1. *For each $c \geqq 1$, the compact group N^{c+1} is canonically*

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isomorphic to the Pontrjagin dual of the Schur multiplier $H^2(G^c, T)$ *of* G^c.

Proof. Put $\mathfrak{G} = \mathrm{Gal}(\bar{\mathbf{Q}}/k)$ and $\mathfrak{N}^c = \mathrm{Gal}(\bar{\mathbf{Q}}/k^{(c)})$. Then we have $G^c = \mathfrak{G}/\mathfrak{N}^c$, and the exact Hochschild-Serre sequence

$$\mathrm{Hom}(\mathfrak{G}, T) \xrightarrow{\mathrm{res}} \mathrm{Hom}(\mathfrak{N}^c, T)^{\mathfrak{G}} \xrightarrow{\tau^c} H^2(G^c, T) \longrightarrow H^2(\mathfrak{G}, T)\,,$$

where τ^c is the transgression. The last term $H^2(\mathfrak{G}, T)$ vanishes by a well known result of Tate; see e.g. [Se], § 6. Therefore $H^2(G^c, T)$ is isomorphic to the cokernel of the restriction map which is naturally identified with $\mathrm{Hom}(\mathfrak{N}^c/[\mathfrak{N}^c, \mathfrak{G}], T)$. By definition we have $[\mathfrak{N}^c, \mathfrak{G}] = \mathfrak{N}^{c+1}$ and $\mathfrak{N}^c/\mathfrak{N}^{c+1} = N^{c+1}$. This shows that $H^2(G^c, T)$ is isomorphic to $\mathrm{Hom}(N^{c+1}, T)$. Taking the dual groups we immediately obtain the proposition.

§ 2. The dual of the Schur multiplier

In this paper, $H^2(G, T)$ and its dual for a Galois group G of an infinite algebraic extension plays an important role. When G is finite, $H^2(G, T)$ is a finite abelian group and isomorphic to its dual although not canonically. When G is an infinite profinite group, however, $H^2(G, T)$ is different from its dual. So it seems worthwhile to give a brief survey on the dual of $H^2(G, T)$ for a profinite group G.

Let G be a profinite group and suppose that we have a presentation $G = F/R$ with a free profinite group F and its closed normal subgroup R which is generated by the relations in G. Associated to the exact sequence

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\rho} G \longrightarrow 1\,,$$

the transgression gives an exact sequence

$$\mathrm{Hom}(F, T) \xrightarrow{\mathrm{res}} \mathrm{Hom}(R, T)^F \xrightarrow{\tau} H^2(G, T)\,.$$

THEOREM 1. *The transgression induces an isomorphism*

$$\tau\colon \mathrm{Hom}(R \cap [F, F]/[R, F], T) \xrightarrow{\sim} H^2(G, T)\,.$$

Proof. It is easily seen that the cokernel of the restriction map can be identified with

$$\mathrm{Hom}(R \cap [F, F]/[R, F], T)\,.$$

To show that τ is surjective, we can slightly modify the method of

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[Ka], pp. 47–48. Let (φ) be an element in $H^2(G, T)$ with a 2-cocycle φ in $Z^2(G, T)$. Since the image of φ is a compact subset of the discrete topological group T, there exists a positive integer m such that the image of φ is contained in the subgroup $C_m = (1/m)\mathbf{Z}/\mathbf{Z}$ of T. Then φ belongs to $Z^2(G, C_m)$ and determines a central extension of profinite groups

$$1 \longrightarrow C_m \longrightarrow H \xrightarrow{\pi} G \longrightarrow 1 .$$

Since ρ: $F \to G$ gives a presentation of G with a free profinite group F, there exists a homomorphism η of F to H such that $\pi \circ \eta = \rho$. Then $\eta(R)$ is contained in (the image of) C_m. Take a continuous cross-section σ: $G \to F$ of ρ (cf. [Ko], Satz 1.16, p. 8, or [Sh], Theorem 3, p. 10), and put $\xi = \eta \circ \sigma$. Then we have a 2-cocycle $\psi \in Z^2(G, C_m)$ defined by

$$\psi(x, y) = \xi(xy)^{-1}\xi(x)\xi(y) , \qquad x, y \in G,$$

because ξ is a cross-section of π: $H \to G$. By the choice of H, ψ is cohomologous to φ. Therefore it belongs to the class (φ) in $H^2(G, T)$. Now let χ be the element of $\mathrm{Hom}(R, T)$ obtained by restricting η to R (combined with the inclusion $C_m \hookrightarrow T$). Then we have

$$\begin{aligned} \psi(x, y) &= \eta(\sigma(xy))^{-1}\eta(\sigma(x))\eta(\sigma(y)) \\ &= \eta(\sigma(xy)^{-1}\sigma(x)\sigma(y)) \\ &= \chi(\sigma(xy)^{-1}\sigma(x)\sigma(y)) \end{aligned}$$

for $x, y \in G$. Since σ is a cross-section of ρ, this shows that $\tau(\chi) = (\psi) = (\varphi)$, which proves that τ is surjective.

If we take another presentation $G = F'/R'$ with a free profinite group F', then we also have an isomorphism

$$\tau'\colon \mathrm{Hom}(R' \cap [F', F']/[R', F'], T) \xrightarrow{\sim} H^2(G, T)$$

by the theorem. It is easy to see, however, that there exists a canonical homomorphism of F to F' which induces a homomorphism

$$\theta\colon R \cap [F, F]/[R, F] \longrightarrow R' \cap [F', F']/[R, F'] .$$

The dual $\hat{\theta}$ satisfies the condition $\tau' = \tau \circ \hat{\theta}$ and is an isomorphism. Therefore θ is an isomorphism of compact abelian groups. This observation allows us to define

$$\mathfrak{M}(G) := R \cap [F, F]/[R, F] .$$

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Then the statement of the theorem is dualized as follows:

THEOREM 1′. *For a profinite group G, the Schur multiplier $H^2(G, T)$ is canonically dual to $\mathfrak{M}(G)$.*

§ 3. The structure of $\mathfrak{M}(G)$ as a profinite group

Let G be a profinite group and N be its closed normal subgroup. Then a homomorphism

$$\gamma = \gamma_N^G\colon \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/N)$$

is canonically determined by the definition. The cokernel of γ is also determined; the following sequence is exact (cf. [B-E], Theorem 1.1, p. 101):

$$\mathfrak{M}(G) \xrightarrow{\gamma} \mathfrak{M}(G/N) \longrightarrow N \cap [G, G]/[N, G] \longrightarrow 1\,. \tag{1}$$

On the other hand, we have the inflation map

$$\lambda = \lambda_N^G\colon H^2(G/N, T) \longrightarrow H^2(G, T);$$

its kernel can be determined by the Hochschild-Serre exact sequence

$$1 \longrightarrow \mathrm{Hom}(N \cap [G, G]/[N, G], T) \longrightarrow H^2(G/N, T) \xrightarrow{\lambda} H^2(G, T). \tag{1′}$$

Using Theorem 1′ we see

PROPOSITION 2. *The exact sequence* (1′) *is dual to* (1).

Altogether this shows

PROPOSITION 3. *Let the notation and the assumptions be as above. Denote the dual map of γ_N^G by $\hat{\gamma}_N^G$, and let τ_G and $\tau_{G/N}$ be the isomorphisms given in Theorem* 1 *for G and for G/N, respectively. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(\mathfrak{M}(G), T) & \xrightarrow{\tau_G} & H^2(G, T) \\ {\scriptstyle \hat{\gamma}_N^G}\uparrow & & \uparrow{\scriptstyle \lambda_N^G} \\ \mathrm{Hom}(\mathfrak{M}(G/N), T) & \xrightarrow[\tau_{G/N}]{} & H^2(G/N, T). \end{array} \tag{2}$$

Now let $\mathfrak{U}$ be the family of all open normal subgroups of G. For $U, V \in \mathfrak{U}$, $U \supset V$, we have a homomorphism

$$\gamma_{U,V}\colon \mathfrak{M}(G/V) \longrightarrow \mathfrak{M}(G/U)$$

together with

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$$\gamma_U\colon \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/U),$$

and

$$\gamma_V\colon \mathfrak{M}(G) \longrightarrow \mathfrak{M}(G/V).$$

From the definition we see

(3) $$\gamma_U = \gamma_{U,V} \circ \gamma_V$$

and

$$\gamma_{U,W} = \gamma_{U,V} \circ \gamma_{V,W}$$

for $U, V, W \in \mathfrak{U}$, $U \supset V \supset W$. Therefore we have a projective system of finite abelian groups

(4) $$\{\mathfrak{M}(G/U), \gamma_{U,V} \mid U, V \in \mathfrak{U}, U \supset V\}.$$

We have also an inductive system of Schur multipliers

(5) $$\{H^2(G/U, T), \lambda_{V,U} \mid U, V \in \mathfrak{U}, U \supset V\}$$

where $\lambda_{V,U}$ is the inflation map

$$\lambda_{V,U}\colon H^2(G/U, T) \longrightarrow H^2(G/V, T),$$

and also a system of homomorphisms

$$\lambda_U\colon H^2(G/U, T) \longrightarrow H^2(G, T), \qquad U \in \mathfrak{U}.$$

Since the action of G on T is trivial, we have

(6) $$H^2(G, T) = \varinjlim_{U} H^2(G/U, T),$$

(cf. [Sh], Corollary 1, p. 26); and then

PROPOSITION 4. $$\mathfrak{M}(G) = \varprojlim_{U \in \mathfrak{U}} \mathfrak{M}(G/U).$$

Proof. Put $H = \varprojlim \mathfrak{M}(G/U)$. Then by (3) and the universal property of H, we have a continuous homomorphism $\varphi\colon \mathfrak{M}(G) \to H$. Because of (2), the two systems (4) and (5) are dual to each other. By (6), therefore, $H^2(G, T)$ is the dual group of H. Then by Theorem 1′ and (2), we conclude that φ is an isomorphism.

§ 4. The structure of $\mathrm{Gal}(k^{(c+1)}/k^{(c)})$

Let us go back to Galois groups of nilpotent extensions of an algebraic number field k. An open normal subgroup U of G^c corresponds to a finite normal subextension K/k of $k^{(c)}/k$ in such a way that $U = \mathrm{Gal}(k^{(c)}/K)$. Therefore we obtain from Theorem 1′ and Proposition 4

THEOREM 2. (i) *For each $c \geqq 1$ there is a canonical isomorphism*

$$\iota^c\colon \mathfrak{M}(G^c) \xrightarrow{\sim} N^{c+1} = \mathrm{Gal}(k^{(c+1)}/k^{(c)}).$$

(ii) *$\mathfrak{M}(G^c)$ is determined by finite normal subextensions K/k of $k^{(c)}/k$ as*

$$\mathfrak{M}(G^c) = \varprojlim_K \mathfrak{M}(\mathrm{Gal}(K/k)).$$

For a finite normal subextension K/k of $k^{(c)}/k$ with $U = \mathrm{Gal}(k^{(c)}/K)$ denote by

$$\gamma_K = \gamma_U\colon \mathfrak{M}(G^c) \longrightarrow \mathfrak{M}(\mathrm{Gal}(K/k))$$

the natural homomorphism determined by (ii) of the theorem. We denote the maximal central extension of K/k in $\bar{\mathbf{Q}}$ by $MC(K/k)$. This is a subfield of $k^{(c+1)}$ because K/k is a subextension of $k^{(c)}/k$.

In Section 6 we shall prove

THEOREM 3. *For each finite normal subextension K/k of $k^{(c)}/k$, there exists a canonical isomorphism*

$$\iota_K\colon \mathrm{Im}(\gamma_K) \xrightarrow{\sim} \mathrm{Gal}(MC(K/k)\cdot k^{(c)}/k^{(c)})$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{M}(G^c) & \xrightarrow{\iota^c} & N^{c+1} = \mathrm{Gal}(k^{(c+1)}/k^{(c)}) \\ {\scriptstyle\gamma_K}\downarrow & & \downarrow{\scriptstyle\mathrm{res}} \\ \mathrm{Im}(\gamma_K) & \xrightarrow[\iota_K]{} & \mathrm{Gal}(MC(K/k)\cdot k^{(c)}/k^{(c)}). \end{array}$$

§ 5. Base-change for abundant central extensions

In this section let K/k be a Galois extension of algebraic number fields of finite degree. Put $\mathfrak{g} = \mathrm{Gal}(K/k)$. The maximal central extension $MC(K/k)$ of K/k contains $K\cdot k^{\mathrm{ab}}$. There exists a canonical isomorphism

$$\iota_{K/k}\colon \mathfrak{M}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Gal}(MC(K/k)/K\cdot k^{\mathrm{ab}})$$

(see e.g. [Mi]). Since $MC(K/k)$ is a finite extension of $K\cdot k^{\mathrm{ab}}$, there is a finite central extension L of K/k such that $MC(K/k)$ is equal to the composite field $L\cdot k^{\mathrm{ab}}$. Such an L is called an abundant central extension of K/k. Put $L^* = L \cap K \cdot k^{\text{ab}}$. Then the isomorphism $\iota_{K/k}$ induces an isomorphism of $\mathfrak{M}(\mathfrak{g})$ onto $\operatorname{Gal}(L/L^*)$ if L is abundant. Suppose that another finite Galois extension K_1/k, $K_1 \supset K$, is given, and put $G = \operatorname{Gal}(K_1/k)$ and $N = \operatorname{Gal}(K_1/K)$. Then $\mathfrak{g} = G/N$. The homomorphism γ of (1) in Section 3 gives a basic relation between $\mathfrak{M}(G)$ and $\mathfrak{M}(\mathfrak{g})$. Let $\hat{K}_1$ be the maximal central extension of K/k in K_1, i.e. $\hat{K}_1 = K_1 \cap MC(K/k)$, and let K_1^* be the genus field, i.e. $K_1^* = K_1 \cap K \cdot k^{\text{ab}}$. Then by the definition we see that $\operatorname{Gal}(\hat{K}_1/K_1^*)$ is isomorphic to $N \cap [G, G]/[N, G]$, the third term of the exact sequence (1). Now let us denote the composite field $L \cdot K_1$ by L_1, and put $L_1^{**} = L_1 \cap K_1 \cdot k^{\text{ab}}$. Since L_1 is a central extension of K_1/k, $\operatorname{Gal}(L_1/L_1^{**})$ is a homomorphic image of $\mathfrak{M}(G)$ under the map induced by the canonical homomorphism $\iota_{K_1/k}$ for the Galois extension K_1/k.

THEOREM 4. *Let the notation and the assumptions be as above. Then L_1 is a central extension of K_1/k with the following properties*:

(i) $\operatorname{Gal}(L_1/L_1^{**})$ *is canonically isomorphic to* $\operatorname{Gal}(L/L \cap L_1^{**})$;

(ii) $\operatorname{Gal}(L \cap L_1^{**}/L^*)$ *is canonically isomorphic to* $\operatorname{Gal}(\hat{K}_1/K_1^*)$;

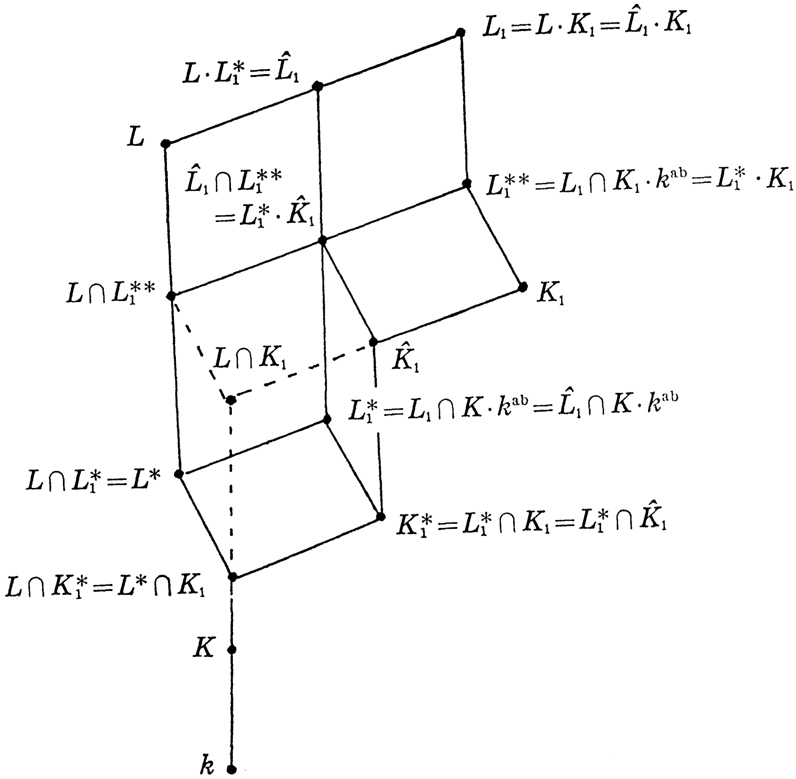
(iii) *We have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} \mathfrak{M}(G) & \longrightarrow & \mathfrak{M}(\mathfrak{g}) & \longrightarrow & N \cap [G, G]/[N, G] & \longrightarrow & 1 \\ \downarrow & & \downarrow \wr & & \downarrow \wr & & \\ \operatorname{Gal}(L_1/L_1^{**}) & \longrightarrow & \operatorname{Gal}(L/L^*) & \longrightarrow & \operatorname{Gal}(L \cap L_1^{**}/L^*) & \longrightarrow & 1. \end{array}$$

Proof. Put $\mathfrak{H} = \operatorname{Gal}(L_1/k)$ and $\mathfrak{N} = \operatorname{Gal}(L_1/K)$; $\mathfrak{N}$ is a normal subgroup with quotient $\mathfrak{g} = \mathfrak{H}/\mathfrak{N}$. Put $\mathfrak{A} = \operatorname{Gal}(L_1/K_1)$ and $\mathfrak{B} = \operatorname{Gal}(L_1/L)$; then $\operatorname{Gal}(L_1/L \cap K_1)$ is a direct product $\mathfrak{B} \times \mathfrak{A}$ because $L_1 = L \cdot K_1$. Let $\hat{L}_1$ be the subfield of L_1 determined by the condition $\operatorname{Gal}(L_1/\hat{L}_1) = [\mathfrak{N}, \mathfrak{H}]$; this is the maximal central extension of K/k in L_1, i.e. $\hat{L}_1 = L_1 \cap MC(K/k)$; it contains L. Therefore we have $[\mathfrak{N}, \mathfrak{H}] \subset \mathfrak{B}$. Since $\mathfrak{A}$ is a normal subgroup of $\mathfrak{H}$ contained in $\mathfrak{N}$, the commutator $[\mathfrak{A}, \mathfrak{H}]$ is contained in $\mathfrak{A} \cap [\mathfrak{N}, \mathfrak{H}]$; hence we have $[\mathfrak{A}, \mathfrak{H}] = 1$ because $\mathfrak{A} \cap \mathfrak{B} = 1$; this shows that $\mathfrak{A}$ lies in the center of $\mathfrak{H}$, which means that L_1 is a central extension of K_1/k.

Now let us see (i). We have $\operatorname{Gal}(L_1/L_1^{**}) = \mathfrak{A} \cap [\mathfrak{H}, \mathfrak{H}]$ because $L_1^{**} = L_1 \cap K_1 \cdot k^{\text{ab}} = K_1 \cdot (L_1 \cap k^{\text{ab}})$ and $\operatorname{Gal}(L_1/L_1 \cap k^{\text{ab}}) = [\mathfrak{H}, \mathfrak{H}]$. Therefore $\operatorname{Gal}(L_1/L \cap L_1^{**}) = \mathfrak{B} \times \operatorname{Gal}(L_1/L_1^{**})$ because $\operatorname{Gal}(L_1/L \cap K_1) = \mathfrak{B} \times \mathfrak{A}$. Hence it is obvious that the projection of $\operatorname{Gal}(L_1/L \cap L_1^{**})$ onto $\operatorname{Gal}(L/L \cap L_1^{**})$ maps $\operatorname{Gal}(L_1/L_1^{**})$ isomorphically onto $\operatorname{Gal}(L/L \cap L_1^{**})$. This proves (i). Note that $\operatorname{Gal}(L_1/L_1^{**})$ is a homomorphic image of $\mathfrak{M}(G)$ because L_1^{**} is the genus field of the central extension L_1 of K_1/k. Put $L_1^* = L_1 \cap K \cdot k^{\text{ab}}$; this

Diagram



is the genus field of the central extension $\hat{L}_1 = L_1 \cap MC(K/k)$ of K/k. Since $L_1^* = K \cdot (L_1 \cap k^{ab})$, we have $\mathrm{Gal}(L_1/L_1^*) = \mathfrak{N} \cap [\mathfrak{H}, \mathfrak{H}]$. By the assumption, L is an abundant central extension of K/k. Therefore $\hat{L}_1$ is contained in $L \cdot k^{ab}$ and hence equal to $L \cdot L_1^*$. Then $\mathrm{Gal}(\hat{L}_1/L_1^*)$ is naturally isomorphic to $\mathrm{Gal}(L/L^*)$ because $L \cap L_1^* = L \cap K \cdot k^{ab} = L^*$. Under this isomorphism the intermediate field $L \cap L_1^{**}$ of L/L^* corresponds to $(L \cap L_1^{**}) \cdot L_1^*$. Furthermore, we also have natural isomorphisms

$$\mathrm{Gal}(\hat{L}_1/L) \cong \mathrm{Gal}((L \cap L_1^{**}) \cdot L_1^*/L \cap L_1^{**}) \cong \mathrm{Gal}(L_1^*/L^*) .$$

Since there is a natural isomorphism of $\mathrm{Gal}(L_1/L)$ onto $\mathrm{Gal}(L_1^{**}/L \cap L_1^{**})$, we conclude that the intermediate field $(L \cap L_1^{**}) \cdot L_1^*$ coincides with $\hat{L}_1 \cap L_1^{**}$. Therefore $\mathrm{Gal}(L \cap L_1^{**}/L^*)$ is naturally isomorphic to $\mathrm{Gal}(\hat{L}_1 \cap L_1^{**}/L_1^*)$. Next let us look at the extension L_1^{**}/K_1^*. By definition we easily see $L_1^* \cap K_1 = K_1^*$. Hence $\mathrm{Gal}(L_1/K_1^*) = (\mathfrak{N} \cap [\mathfrak{H}, \mathfrak{H}]) \cdot \mathfrak{A}$ because $\mathrm{Gal}(L_1/L_1^*) = \mathfrak{N} \cap [\mathfrak{H}, \mathfrak{H}]$ and $\mathrm{Gal}(L_1/K_1) = \mathfrak{A}$. Therefore we obtain

$$\begin{aligned}\mathrm{Gal}(L_1^*/K_1^*) &= (\mathfrak{N}\cap[\mathfrak{H},\mathfrak{H}])\cdot\mathfrak{A}/(\mathfrak{N}\cap[\mathfrak{H},\mathfrak{H}])\\ &\cong \mathfrak{A}/\mathfrak{A}\cap(\mathfrak{N}\cap[\mathfrak{H},\mathfrak{H}])\\ &= \mathfrak{A}/\mathfrak{A}\cap[\mathfrak{H},\mathfrak{H}]\\ &= \mathrm{Gal}(L_1^{**}/K_1).\end{aligned}$$

In particular we have $[L_1^{**}: K_1] = [L_1^*: K_1^*]$. It is clear that L_1^{**} contains $L_1^*\cdot K_1$. Since $[L_1^*\cdot K_1: K_1] = [L_1^*: L_1^*\cap K_1] = [L_1^*: K_1^*] = [L_1^{**}: K_1]$, we conclude $L_1^{**} = L_1^*\cdot K_1$. This implies that $\mathrm{Gal}(\hat{L}_1\cap L_1^{**}/L_1^*)$ is naturally isomorphic to $\mathrm{Gal}(\hat{K}_1/K_1^*)$ because $\hat{K}_1 = (\hat{L}_1\cap L_1^{**})\cap K_1$. Combining this with the result obtained above, we have shown that $\mathrm{Gal}(L\cap L_1^{**}/L^*)$ is naturally isomorphic to $\mathrm{Gal}(\hat{K}_1/K_1^*)$ as is claimed in (ii).

The last Galois group is isomorphic to

$$\mathrm{Gal}(K_1/K_1^*)/\mathrm{Gal}(K_1/\hat{K}_1) = N\cap[G, G]/[N, G].$$

Since all the isomorphisms of Galois groups discussed above are natural and group-theoretic, (iii) is also clear.

§ 6. The proof of Theorem 3

We use the same notation as in Section 4. Let K/k be a finite normal subextension of $k^{(c)}/k$, $c \geqq 1$; its Galois group is denoted by $\mathfrak{g} = \mathrm{Gal}(K/k)$; $U = \mathrm{Gal}(k^{(c)}/K)$ is an open normal subgroup of G^c. We fix an abundant central extension L of K/k; $MC(K/k)\cdot k^{(c)}$ is equal to $L\cdot k^{(c)}$. Put $K_0 = L\cap k^{(c)}$; then $\mathrm{Gal}(L/K_0)$ is canonically isomorphic to $\mathrm{Gal}(MC(K/k)\cdot k^{(c)}/k^{(c)})$. Put $\tilde{U} = \mathrm{Gal}(k^{(c)}/K_0)$, $G = \mathrm{Gal}(K_0/k)$ and $N = \mathrm{Gal}(K_0/K)$. We use Theorem 4 for $L/K/k$ and $K_1 = K_0$; in this case we have $L\supset K_1\supset L^*$; N lies in the center of G, and $[N, G] = 1$; therefore $K_1 = \hat{K}_1$, $K_1^* = L^*$; moreover we have $L_1 = L$, $L_1^* = L^*$ and $L_1^{**} = K_1 = L\cap L_1^{**}$; hence the image of the homomorphism

$$\gamma_N^G\colon \mathfrak{M}(G) \longrightarrow \mathfrak{M}(\mathfrak{g})$$

is mapped isomorphically onto $\mathrm{Gal}(L/K_0)$ by the isomorphism

$$\mathrm{res}\circ\iota_{K/k}\colon \mathfrak{M}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Gal}(MC(K/k)/K\cdot k^{\mathrm{ab}}) \xrightarrow{\sim} \mathrm{Gal}(L/L^*).$$

Let us express this using U and $\tilde{U}$; we have $\mathfrak{M}(G) = \mathfrak{M}(G^c/\tilde{U})$, $\mathfrak{M}(\mathfrak{g}) = \mathfrak{M}(G^c/U)$ and $\gamma_N^G = \gamma_{U,\tilde{U}}$ in the notation of Section 3; furthermore, the image of $\gamma_{U,\tilde{U}}$, $\mathrm{Im}(\gamma_{U,\tilde{U}})$, is mapped isomorphically onto the subgroup of $\mathrm{Gal}(MC(K/k)/K\cdot k^{\mathrm{ab}})$ by $\iota_{K/k}$ which is canonically isomorphic to $\mathrm{Gal}(L/K_0)$ and hence also to $\mathrm{Gal}(MC(K/k)\cdot k^{(c)}/k^{(c)})$. Next let V be an open normal subgroup of G^c contained in U; put $\tilde{V} = V \cap \tilde{U}$; then we have

$$\gamma_{U,\tilde{V}} = \gamma_{U,V} \circ \gamma_{V,\tilde{V}} = \gamma_{U,\tilde{U}} \circ \gamma_{\tilde{U},\tilde{V}};$$

hence, in order to show $\operatorname{Im}\gamma_U = \operatorname{Im}(\gamma_{U,\tilde{U}})$ for $\gamma_U(=\gamma_K) = \lim_{V\subset U}\gamma_{U,V}$, it is sufficient to prove that $\operatorname{Im}(\gamma_{U,\tilde{V}})$ in $\mathfrak{M}(\mathfrak{g})$ is mapped isomorphically onto $\operatorname{Gal}(L/K_0)$ by $\operatorname{res}\circ\iota_{K/k}$. This time we use Theorem 4 for $L/K/k$ and the extension K_1/k determined by the condition $\operatorname{Gal}(K_1/k) = G^c/\tilde{V}$. We have $K_1 \supset K_0$ by the choice of V; since K_1 is a subfield of $k^{(c)}$, we also have $L \cap K_1 = K_0$ and hence $L^* \subset L \cap K_1$. For $c \geqq 1$, $L_1^{**} = L_1^* \cdot K_1$ is contained in $k^{(c)}$ because $k^{(c)} \supset K \cdot k^{ab}$ and $L_1^* = L_1 \cap K \cdot k^{ab}$; therefore $L \cap L_1^{**} \subset L \cap k^{(c)} = K_0$; conversely, it is clear that $L \cap L_1^{**} \supset K_0 = L \cap K_1$; thus we have $L \cap L_1^{**} = K_0$. It now follows from Theorem 4 that the image of the homomorphism

$$\gamma_{U,\tilde{V}}\colon \mathfrak{M}(G^c/\tilde{V}) \longrightarrow \mathfrak{M}(G^c/U)$$

is isomorphically mapped onto $\operatorname{Gal}(L/K_0)$ by $\operatorname{res}\circ\iota_{K/k}$. This proves that $\operatorname{Im}\gamma_U$, $\gamma_U = \gamma_K$, coincides with $\operatorname{Im}(\gamma_{U,\tilde{U}})$ and also that there exists a canonical isomorphism

$$\iota_K\colon \operatorname{Im}\gamma_K \longrightarrow \operatorname{Gal}(MC(K/k)\cdot k^{(c)}/k^{(c)}).$$

The rest of Theorem 3 will easily be seen in a straightforward way by dualizing the diagram of Theorem 4.

§ 7. The canonical 2-cohomology classes

We fix an algebraic number field k of finite degree. Let K be a finite Galois extension of k, $K_{\mathbf{A}}^{\times}$ be the idele group of K, $K_{\infty+}^{\times}$ be the connected component of the identity element of the Archimedean part of $K_{\mathbf{A}}^{\times}$ and $K^{\#}$ be the closure of $K^{\times}\cdot K_{\infty+}^{\times}$ in $K_{\mathbf{A}}^{\times}$. We have the Artin map of K,

$$\alpha_K\colon K_{\mathbf{A}}^{\times}/K^{\#} \longrightarrow \operatorname{Gal}(K^{ab}/K),$$

which is a topological isomorphism, and the natural exact sequence

$$E(K/k)\colon\quad 1 \longrightarrow K_{\mathbf{A}}^{\times}/K^{\#} \longrightarrow \operatorname{Gal}(K^{ab}/k) \longrightarrow \operatorname{Gal}(K/k) \longrightarrow 1.$$

The structure of $\operatorname{Gal}(K^{ab}/k)$ is then determined by the canonical 2-cohomology class $\xi_{K/k}$ of $\operatorname{Gal}(K/k)$ with values in $K_{\mathbf{A}}^{\times}/K^{\#}$. More specifically, the cohomology group $H^2(\operatorname{Gal}(K/k), K_{\mathbf{A}}^{\times}/K^{\#})$ is a cyclic group generated by $\bar{\xi}_{K/k}$; its order is either $\frac{1}{2}\cdot[K\colon k]$ if there exists a ramified real Archimedian prime in K/k or $[K\colon k]$ otherwise (cf. Katayama [Kt] and also

Iyanaga [Iy]). In this sense $\mathrm{Gal}(K^{ab}/k)$ is determined by K/k.

Let F/k, $F \supset K$, be another finite Galois extension, $F_{\mathbf{A}}^{\times}/F^{\#}$ be as above for F and

$$N_{F/K}\colon F_{\mathbf{A}}^{\times}/F^{\#} \longrightarrow K_{\mathbf{A}}^{\times}/K^{\#}$$

be the norm map of F/K. Then we have a commutative diagram

$$\begin{array}{ccc} F_{\mathbf{A}}^{\times}/F^{\#} & \xrightarrow{\alpha_F} & \mathrm{Gal}(F^{ab}/F) \\ {\scriptstyle N_{F/K}}\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\ K_{\mathbf{A}}^{\times}/K^{\#} & \xrightarrow[\alpha_K]{} & \mathrm{Gal}(K^{ab}/K)\,. \end{array}$$

Therefore $N_{F/K}$ and the homomorphisms defined by restricting automorphisms of F^{ab} or of F to K^{ab} or to K, respectively, give a homomorphism of the exact sequence $E(F/k)$ to $E(K/k)$.

Now suppose that an infinite Galois extension $\tilde{k}/k$ is given. Put $G = \mathrm{Gal}(\tilde{k}/k)$. If we make K/k run over all finite Galois subextensions of $\tilde{k}/k$, we have projective systems $\{K_{\mathbf{A}}^{\times}/K^{\#}, N_{F/K}\}$, $\{\mathrm{Gal}(K^{ab}/K)\}$, $\{\mathrm{Gal}(K^{ab}/k)\}$, $\{\mathrm{Gal}(K/k)\}$ and $\{E(K/k)\}$, and also

$$\mathrm{Gal}(\tilde{k}^{ab}/k) = \varprojlim_K \mathrm{Gal}(K^{ab}/k)\,,$$

$$\mathrm{Gal}(\tilde{k}^{ab}/\tilde{k}) = \varprojlim_K \mathrm{Gal}(K^{ab}/K)\,,$$

$$G = \mathrm{Gal}(\tilde{k}/k) = \varprojlim_K \mathrm{Gal}(K/k)\,.$$

We put

$$\mathfrak{A}(\tilde{k}) = \varprojlim_K K_{\mathbf{A}}^{\times}/K^{\#}\,.$$

It is clear that $\mathfrak{A}(\tilde{k})$ depends only on $\tilde{k}$. Each $K_{\mathbf{A}}^{\times}/K^{\#}$ is naturally considered as a G-module. Therefore $\mathfrak{A}(\tilde{k})$ has a G-module structure. Through inner automorphisms of $\mathrm{Gal}(\tilde{k}^{ab}/k)$, $\mathrm{Gal}(\tilde{k}^{ab}/\tilde{k})$ becomes a G-module.

PROPOSITION 5. *Let the notation and the assumptions be as above. The Artin maps α_K for finite Galois subextensions K/k of $\tilde{k}/k$ give a G-isomorphism*

$$\alpha_{\tilde{k}} = \lim_K \alpha_K\colon\ \mathfrak{A}(\tilde{k}) \longrightarrow \mathrm{Gal}(\tilde{k}^{ab}/\tilde{k})\,.$$

The exact sequence

$$E(\tilde{k}/k)\colon\quad 1 \longrightarrow \mathfrak{A}(\tilde{k}) \longrightarrow \mathrm{Gal}(\tilde{k}^{ab}/k) \longrightarrow \mathrm{Gal}(\tilde{k}/k) \longrightarrow 1$$

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determined naturally by $\alpha_{\tilde{k}}$ *is the projective limit of* $\{E(K/k)\}$. *Therefore the canonical classes* $\bar{\xi}_{K/k}$ *determine the canonical* 2-*cohomology class* $\bar{\xi}_{\tilde{k}/k}$ *in* $H^2(G, \mathfrak{A}(\tilde{k}))$ *which gives the extension* $\mathrm{Gal}(\tilde{k}^{\mathrm{ab}}/k)$ *of* $\mathrm{Gal}(\tilde{k}/k)$ *by* $\mathfrak{A}(\tilde{k})$.

The proof is almost obvious because $\varprojlim$ is an exact functor in the category of compact groups (e.g. [E-S], Theorem 5.6, p. 226, or [Ko], Satz 1.9, p. 6).

COROLLARY. *Let* $MC(\tilde{k}/k)$ *be the maximal central extension of* $\tilde{k}/k$. *Then we have*

$$\mathrm{Gal}(MC(\tilde{k}/k)/\tilde{k}) = \mathfrak{A}(\tilde{k})/\mathfrak{A}(\tilde{k})^{\Delta G}$$

where

$$\mathfrak{A}(\tilde{k})^{\Delta G} = \langle x^{1-\sigma} \,|\, x \in \mathfrak{A}(\tilde{k}),\ \sigma \in G \rangle$$

(*the right-hand side means the topologically generated closed subgroup*).

Let us apply these results to the case where $\tilde{k}/k$ is the nilpotent extension $k^{(c)}/k$, $c \geqq 1$, and $G = G^c$. Then since $k^{(c+1)} = MC(k^{(c)}/k)$ we obtain from Theorem 2 the following result:

THEOREM 5. (i) *For each* $c \geqq 1$, *there exists a surjective homomorphism* $\alpha^c\colon \mathfrak{A}(k^{(c)}) \to \mathfrak{M}(G^c)$ *with* $\operatorname{Ker} \alpha^c = \mathfrak{A}(k^{(c)})^{\Delta G^c}$ *such that the homomorphism*

$$\iota^c \circ \alpha^c\colon \mathfrak{A}(k^{(c)}) \longrightarrow N^{c+1} = \mathrm{Gal}(k^{(c+1)}/k^{(c)})$$

coincides with the homomorphism induced naturally from the Artin map

$$\alpha_{k^{(c)}}\colon \mathfrak{A}(k^{(c)}) \longrightarrow \mathrm{Gal}(k^{(c),\mathrm{ab}}/k^{(c)}) .$$

(ii) *The group extension*

$$1 \longrightarrow \mathfrak{M}(G^c) \xrightarrow{\iota^c} G^{c+1} \longrightarrow G^c \longrightarrow 1$$

determined by $\iota^c\colon \mathfrak{M}(G^c) \to \mathrm{Gal}(k^{(c+1)}/k^{(c)})$ *corresponds to the image of the canonical class* $\bar{\xi}_{k^{(c)}/k}$ *under the induced homomorphism*

$$(\alpha^c)^*\colon H^2(G^c, \mathfrak{A}(k^{(c)})) \longrightarrow H^2(G^c, \mathfrak{M}(G^c)) .$$

Remark. For g and $h \in G^c$, take $\tilde{g}$ and $\tilde{h} \in G^{c+1}$ over g and h, respectively. Then the commutator $[\tilde{g}, \tilde{h}]$ depends only on g and h because the group extension of Theorem 5, (ii), is central. Hence from the definition we obtain an epimorphism

$$\varphi^c\colon N^c \otimes G^c \longrightarrow \mathfrak{M}(G^c)$$

such that

$$\iota^c(\varphi(n, g)) = [\tilde{n}, \tilde{g}]$$

for $n \in N^c$ and $g \in G^c$. Let f be a 2-cocycle which belongs to the cohomology class $(\alpha^c)^*(\xi_{k^{(c)}/k}) \in H^2(G^c, \mathfrak{M}(G^c))$. Then it is easily seen that for $n \in N^c$ and $g \in G^c$ we have

$$\varphi(n, g) = f(n, g) \cdot f(g, n)^{-1}$$

because N^c lies in the center of G^c. For a finite abelian extension K/k with $G = N = \mathrm{Gal}\,(K/k)$, a detailed analysis of the analogous map is given by Furuta [Fu].

Remark. The main body of our present results relies on two facts; one of them is that Schur multipliers of profinite groups and their duals have good functorial properties; the other is a result of Tate, $H^2(\mathfrak{G}, \mathbf{Q}/\mathbf{Z}) = 0$ for $\mathfrak{G} = \mathrm{Gal}\,(\bar{\mathbf{Q}}/k)$, from which we not only deduce Proposition 1 but also the existence of abundant central extensions of a finite Galois subextension of $\bar{\mathbf{Q}}/k$. It is, therefore, easy to see that parallel results also hold for a local number field k_v and its algebraic closure $\bar{k}_v$, and for an algebraic number field k and its maximal p-ramified p-extension $\tilde{k}^{(p)}$ when the Leopoldt conjecture holds for k and p, because we have $H^2(\mathfrak{G}, \mathbf{Q}/\mathbf{Z}) = 0$ for $\mathfrak{G} = \mathrm{Gal}\,(\bar{k}_v/k_v)$ and for $\mathfrak{G} = \mathrm{Gal}\,(\tilde{k}^{(p)}/k)$, e.g. [He] or [Ng] for the latter case.

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