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Preservation of the Index of p-Adic Linear Operators under Compact Perturbations

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Abstract. In this paper it is proved that the index of a Fredholm operator between p-adic Banach spaces is preserved under compact perturbations. A case of special interest is provided when the ground field is nonspherically complete. In this case the classical techniques are no longer valid and the relation between the kernels of a Fredholm operator and that of a small compact perturbation turn out to be in general much richer than in the complex context.

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1. Introduction

The problem of perturbations of p-adic linear operators has been long studied through several steps. A first approach was carried out by J. P. Serre in [11], where he dealt with compact perturbations of the identity on Banach spaces having an orthogonal base. A step further was taken by L. Gruson ([5]) for a more general class of Banach spaces, always working on perturbations of the identity. A complete study of perturbations of the identity was finally achieved by W. H. Schikhof in [10].

In [6], P. Robba dealt with perturbations of injective operators, introducing the index as a useful tool to study the theory of p-adic differential operators (see also [7]). This connection between p-adic differential operators and index theory still represents an important current matter of research, as recent papers like [1], [2] and [3], among others, are covering new trends. Also, in the latter paper, a remarkable fact is that the restrictions on the perturbed operators have arisen for a large class of spaces, but just working over locally compact fields.

In this paper, we aim at a general theory of the perturbations of continuous linear operators between non-Archimedean Banach spaces by compact operators, regardless of the base field, and extend previous results. In this way, even if in

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the case when $\mathbb{K}$ is spherically complete the proofs of the classical theorems can somehow be adapted, the non-Archimedean theory turns out to be in general much richer than the corresponding one given for real or complex, spaces. In particular when the base field is not spherically complete, we are able to assure that not only results about the preservation of the index hold, but also the geometrical structure of the kernels is in some way preserved, arriving at a surprising result when the vectors of the kernel are not topologically complemented: in this case, the kernels of the original operator and the perturbed one coincide. This is clearly far apart from the case when dealing with real or complex Banach spaces, where all finite-dimensional subspaces are topologically complemented, and no similar approach can be taken. These results suggest that the tools used to study these perturbations when working over the real or complex numbers are no longer valid in our theory, forcing us to seek a completely different way to attack the problem.

2. Preliminaries

Throughout this paper, $\mathbb{K}$ is a commutative field endowed with a nontrivial non-Archimedean valuation $|\cdot|$, and complete with respect to the metric induced by its valuation.

Let X be a vector space over $\mathbb{K}$. For any subset D of X, the linear hull of D is denoted by $\langle D\rangle$. Also, a linear subspace M of X is said to be algebraically complemented in X when there exists a linear projection from X onto M, or equivalently, when there exists a linear subspace N of X such that $X = M \oplus N$, where by this last equality we mean that $X = M + N$ and $M \cap N = \{0\}$. Such an N is called algebraic complement of M.

If X and Y are vector spaces over $\mathbb{K}$, $\mathcal{L}(X, Y)$ is the set of all linear operators from X to Y. Also, given $T \in \mathcal{L}(X, Y)$, $\mathrm{Ker}(T)$ and $\mathrm{R}(T)$ are the kernel and the range of T respectively. The identity map on X is denoted by I_X.

Now, let X be a non-Archimedean Banach space over $\mathbb{K}$. A set $A \subset X$ is called compactoid if for every $\epsilon > 0$ there exists a finite set $B \subset X$ such that $A \subset \mathrm{co}(B) + \{x \in X : \|x\| \leqslant \epsilon\}$, where $\mathrm{co}(B)$ denotes the absolutely convex hull of B. Also, a closed linear subspace M of X is said to be topologically complemented in X if there exists a continuous linear projection from X onto M which, by the Open Mapping Theorem ([8] Theorem 3.11), is equivalent to the existence of a closed linear subspace N of X such that $X = M \oplus N$. Such an N is called topological complement of M. On the other hand, for a real number $t \in (0, 1]$, a finite family $\{x_1, x_2, \ldots, x_n\}$ of elements in X is said to be t-orthogonal if for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$, $t \max_{1\leqslant i\leqslant n}\|\lambda_i x_i\| \leqslant \left\|\Sigma_{i=1}^n \lambda_i x_i\right\|$.

Finally, for non-Archimedean Banach spaces X, Y over $\mathbb{K}$, we denote by $L(X, Y)$ the (non-Archimedean) Banach space of all continuous linear operators from X to Y, endowed with the norm

$$\|T\| = \inf\{c > 0 : \|Tx\| \leqslant c\|x\| \text{ for all } x \in X\}.$$

We write X' and $L(X)$ instead of $L(X, \mathbb{K})$ and $L(X, X)$ respectively. If $T \in L(X, Y)$ and M is a linear subspace of X, $T|M$ is the restriction of T to M. Also, $T \in L(X, Y)$ is called a compact operator if $T(\{x \in X: \|x\| \leqslant 1\})$ is a compactoid subset of Y. Recall that a continuous linear operator is compact if and only if it is completely continuous, that is, the limit of a sequence of operators of finite, dimensional range (see [7] p. 87, [8] p. 142). The set of all compact operators from X to Y is denoted by $C(X, Y)$. Again, $C(X) := C(X, X)$.

For more basic facts on non-Archimedean Banach spaces, we refer to [8].

3. Some Basic Facts on Linear Operators with Index

In this section X, Y and Z are linear vector spaces over $\mathbb{K}$.

DEFINITION 3.1. We say that $T \in \mathcal{L}(X, Y)$ has an index when both $\eta(T) := \dim \mathrm{Ker}(T)$ and $\delta(T) := \dim Y/\mathrm{R}(T)$ are finite. In this case, the index of the linear operator T is defined as $\chi(T) := \eta(T) - \delta(T)$.

A well-known property of the index of a linear operator which is very useful when studying the index of p-adic differential operators (see e.g. [7]) and that will also be very useful in the sequel, is the following.

PROPOSITION 3.2 ([7] Proposition 7.1.6). *If two of the three linear operators, $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$ and $ST \in \mathcal{L}(X, Z)$ have indexes, then the third one also has an index, and $\chi(ST) = \chi(T) + \chi(S)$.*

The concept of pseudoinverse given in [12] p. 251 for continuous linear operators between real or complex Banach spaces admits the following algebraic version in our case.

DEFINITION 3.3. If $T \in \mathcal{L}(X, Y)$, we say that $S \in \mathcal{L}(Y, X)$ is an algebraic pseudoinverse of T if $TST = T$.

Also, on the other hand, considering only the algebraic aspect of the proof of [12] Theorem IV.12.9, we obtain the following result.

PROPOSITION 3.4. *Every linear operator from X to Y has an algebraic pseudoinverse.*

By using Proposition 3.4 and attending only to linear projections and algebraic direct sums, we can follow the same proof as the corresponding one given in [12] p. 257 for continuous linear operators, to conclude that

THEOREM 3.5. *If $T \in \mathcal{L}(X, Y)$ has an index and $F \in \mathcal{L}(X, Y)$ has finite dimensional range, then $T + F$ has an index and $\chi(T + F) = \chi(T)$.*

From now on, X, Y and Z will be infinite-dimensional non-Archimedean Banach spaces over $\mathbb{K}$.

4. Fredholm Operators and Pseudoinverses

A continuous linear operator T from X to Y is called a *Fredholm operator* if it has an index. The set of all Fredholm operators from X to Y is denoted by $\Phi(X, Y)$. When $X = Y$ we will write $\Phi(X)$ instead of $\Phi(X, X)$.

PROPOSITION 4.1. *If $T \in \Phi(X, Y)$, then $\mathrm{R}(T)$ is closed and for each $t \in (0, 1)$ there exists a continuous linear projection P from Y onto $\mathrm{R}(T)$ such that $\|P\| \leqslant t^{-1}$.*

Proof. The fact that $\mathrm{R}(T)$ is closed can be found in [7] Proposition 7.2.2. Now, let $t \in (0, 1)$ and choose $t' \in (0, 1)$ such that $t < t' < 1$. By [8] Theorem 3.15 we can take a t'-orthogonal family $\{Q(y_1), Q(y_2), \ldots, Q(y_n)\} \subset Y/\mathrm{R}(T)$, that forms a base of $Y/\mathrm{R}(T)$, where $Q: Y \to Y/\mathrm{R}(T)$ is the canonical quotient map. For every $i \in \{1, 2, \ldots, n\}$, take $x_i \in Q(y_i)$ such that $\|x_i\| \leqslant t' t^{-1} \|Q(y_i)\|$. Then, $J: Y/\mathrm{R}(T) \to Y$, $J(Q(y_i)) \mapsto x_i$, is a continuous linear operator with $QJ = I_{Y/\mathrm{R}(T)}$, and $\|J\| \leqslant t^{-1}$. Finally, $P := I_Y - JQ$ satisfies the required conditions.

DEFINITION 4.2. Given $T \in L(X, Y)$, a continuous algebraic pseudoinverse of T is called a pseudoinverse of T.

Notice that although every linear operator between $\mathbb{K}$-vector spaces has an algebraic pseudoinverse (Proposition 3.4), the same is not true in general when we consider continuous linear operators between Banach spaces (see the comments before Proposition 4.7). The situation in this last case is described in the next proposition whose proof follows as in [12] Theorem IV.12.9.

PROPOSITION 4.3. *For $T \in L(X, Y)$, the following statements are equivalent*

(a) *There exist linear projections $P \in L(X)$ and $Q \in L(Y)$ such that*

$$\mathrm{R}(P) = \mathrm{Ker}(T), \qquad \mathrm{R}(Q) = \mathrm{R}(T);$$

(b) *There exist closed subspaces $W \subset X$ and $Z \subset Y$ such that*

$$X = \mathrm{Ker}(T) \oplus W, \qquad Y = Z \oplus \mathrm{R}(T);$$

(c) *T has a pseudoinverse.*

In particular, by Proposition 4.1, for Fredholm operators we have

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COROLLARY 4.4. *$T \in \Phi(X, Y)$ has a pseudoinverse if and only if* $\mathrm{Ker}(T)$ *is topologically complemented in X.*

Remark. Observe that since X and Y are infinite dimensional spaces, when $T \in \Phi(X, Y)$ has a pseudoinverse $S \in \mathcal{L}(Y, X)$, necessarily $S \neq 0$.

Again, with the same proof as in [12] Theorem IV.13.5, we obtain the following lemma.

LEMMA 4.5. *Suppose that $T \in \Phi(X, Y)$ admits a pseudoinverse $S \in L(Y, X)$ (that is,* $\mathrm{Ker}(T)$ *is topologically complemented in X). If $B \in L(X, Y)$ is such that $I_X + SB \in \Phi(X)$ and $\chi(I_X + SB) = 0$, then $T + B \in \Phi(X, Y)$ and $\chi(T + B) = \chi(T)$.*

If in addition $I_X + SB$ is bijective, we also have $\delta(T + B) \leqslant \delta(T)$ and $\eta(T + B) \leqslant \eta(T)$.

THEOREM 4.6. *Let $T \in \Phi(X, Y)$ be such that there exists a closed subspace M of X with $X =$* $\mathrm{Ker}(T) \oplus M$*. Then $T|M$ has a pseudoinverse. Also, for each pseudoinverse S of $T|M$ and for each $B \in L(X, Y)$ with $\|B\| < \|S\|^{-1}$ we have*

(a) $T + B \in \Phi(X, Y)$,
(b) $\chi(T + B) = \chi(T)$,
(c) $\delta(T + B) \leqslant \delta(T)$,
(d) $\eta(T + B) \leqslant \eta(T)$.

In particular, if T is surjective, we have

(e) $T + B$ is surjective,
(f) $\eta(T + B) = \eta(T)$.

Proof. Clearly $T|M$ is injective and $\mathrm{R}(T|M) = \mathrm{R}(T)$. Hence, $T|M \in \Phi(M, Y)$ and, by Corollary 4.4, we obtain that $T|M$ has a pseudoinverse $S \in L(Y, M)$. Also, $\overline{S} := iS \in L(Y, X)$ (where $i\colon M \to X$ is the canonical inclusion from M into X) is a pseudoinverse of T with $\|\overline{S}\| = \|S\|$. Now, given $B \in L(X, Y)$ with $\|B\| < \|S\|^{-1} = \|\overline{S}\|^{-1}$, we have that $\|\overline{S}B\| < 1$ and so $I_X + \overline{S}B$ is bijective. Then, the conclusions follow from Lemma 4.5.

Remark. In the particular case when T is an injective Fredholm operator and S is a continuous left inverse of T, Theorem 4.6 appears in [7] Proposition 7.2.3, being this last result a useful tool to study the index of p-adic differential operators.

Recall that if $\mathbb{K}$ is not spherically complete, there are Banach spaces X over $\mathbb{K}$ for which $X' = \{0\}$ ([8] Corollary 4.3) and hence, by Corollary 4.4, every non injective Fredholm operator T from X into an arbitrary Banach space has not a pseudoinverse because no nontrivial finite-dimensional subspace of X is topologically complemented in X. The situation is in sharp contrast with the classical case: indeed, when X and Y are Banach spaces over the real or complex field, every $T \in \Phi(X, Y)$ has a pseudoinverse ([12] Theorem IV.13.2).

The next result, which will be useful later, characterizes the finite-dimensional subspaces of non-Archimedean Banach spaces that are topologically complemented.

PROPOSITION 4.7. *Let* $\{x_1, x_2, \dots, x_n\}$ *be a finite family of linearly independent vectors in* X. *For* $D := \langle x_1, x_2, \dots, x_n \rangle$ *the following statements are equivalent*

(a) *D is topologically complemented in X.*
(b) *For any $x \in D \setminus \{0\}$ there exists $f \in X'$ with $f(x) \neq 0$.*
(c) *There exist $f_1, f_2, \dots, f_n \in X'$ such that $f_i(x_j) = \delta_{i,j}$ (Kronecker's delta) for all $i, j \in \{1, 2, \dots, n\}$.*

Proof. $(a) \Rightarrow (b)$ follows from the fact that D' separates the points of D ([8] Theorem 3.15). For $(b) \Rightarrow (c)$ see the proof of [4] Theorem 2.2. Finally, for $(c) \Rightarrow (a)$, take $f_1, f_2, \dots, f_n$ as in (c). Then, notice that $P: X \to D$, $x \mapsto \sum_{i=1}^{n} f_i(x)x_i$, is a continuous linear projection from X onto D.

Remark. It follows from Proposition 4.7 that if X' separates the points of X (e.g. when $\mathbb{K}$ is spherically complete), every finite-dimensional subspace of X is topologically complemented. In this case, by Corollary 4.4, every $T \in \Phi(X, Y)$ has a pseudoinverse. As a consequence, Theorem 4.6 proves in particular that if $\mathbb{K}$ is spherically complete the index of a Fredholm operator under a small perturbation does not change. Unfortunately, as we saw above, when $\mathbb{K}$ is not spherically complete $\mathrm{Ker}(T)$ is not necessarily topologically complemented, which forces us to seek for new techniques to deal with the problem of small perturbations of Fredholm operators. This will be the purpose of the following section.

5. The Nonspherically Complete Case

If X and Y are non-Archimedean Banach spaces and $T \in \Phi(X, Y)$, we denote by $\hat{X}$ the Banach space $X/\mathrm{Ker}(T)$ endowed with the quotient norm, and by $\hat{Q}$ the canonical quotient map $\hat{Q}: X \to X/\mathrm{Ker}(T)$. Clearly $\hat{Q}$ is continuous and $\|\hat{Q}\| \leqslant 1$. Also, notice that $\hat{Q} \in \Phi(X, \hat{X})$ and $\chi(\hat{Q}) = \eta(T)$. Also, if $\hat{T}$ denotes the injective linear operator from $\hat{X}$ to Y associated to T, then $\hat{T} \in \Phi(\hat{X}, Y)$, $\mathrm{R}(\hat{T}) = \mathrm{R}(T)$ and, by Corollary 4.4, $\hat{T}$ has a pseudoinverse. This pseudoinverse plays an important role in the next result which will be crucial to our purpose.

THEOREM 5.1. *Let $T \in \Phi(X, Y)$ and let S be a pseudoinverse of $\hat{T}$. For any $B \in L(X, Y)$ with* $\mathrm{Ker}(T) \subset \mathrm{Ker}(B)$ *and* $\|B\| < \|S\|^{-1}$ *we have*

(a) $T + B \in \Phi(X, Y)$,
(b) $\chi(T + B) = \chi(T)$,
(c) $\delta(T + B) = \delta(T)$ (*in particular, if T is surjective, then so is $T + B$*),

(d) $\mathrm{Ker}(T+B)=\mathrm{Ker}(T)$.

Proof. Let $\hat{B}: \hat{X} \to Y$ be given by $\hat{B}(\hat{Q}(x)) = Bx$, $x \in X$. Clearly $\hat{B}$ is a well defined continuous linear operator such that $\|\hat{B}\| = \|B\| < \|S\|^{-1}$. Applying Theorem 4.6 we conclude that

$$\text{(a}'\text{)}\quad \hat{T}+\hat{B} \in \Phi(\hat{X}, Y), \qquad \text{(b}'\text{)}\quad \chi(\hat{T}+\hat{B}) = \chi(\hat{T}),$$

$$\text{(c}'\text{)}\quad \delta(\hat{T}+\hat{B}) = \delta(\hat{T}) = \delta(T), \quad \text{(d}'\text{)}\quad \eta(\hat{T}+\hat{B}) = \eta(\hat{T}) = 0.$$

By Proposition 3.2, $T+B = (\hat{T}+\hat{B})\hat{Q} \in \Phi(X, Y)$, and

$$\begin{aligned}\chi(T+B) &= \chi(\hat{T}+\hat{B}) + \chi(\hat{Q})\\ &= \chi(\hat{T}) + \chi(\hat{Q})\\ &= \chi(T).\end{aligned}$$

Then (a) and (b) follow.

To prove (d), observe that by (d′), $\hat{T}+\hat{B}$ is injective and so $\mathrm{Ker}(T+B) = \mathrm{Ker}(\hat{Q}) = \mathrm{Ker}(T)$.

Finally, (c) is a direct consequence of (b) and (d).

In the rest of the section, we will assume that $\mathbb{K}$ is not spherically complete.

PROPOSITION 5.2. *If D is a one-dimensional subspace of X which is not topologically complemented in X, then for each Banach space Y and each $K \in C(X, Y)$ we have that $D \subset \mathrm{Ker}(K)$.*

Proof. Assuming the contrary, suppose that $K(x) \neq 0$ for some $x \in D$. Since K is a compact operator, it follows from [8] Theorems 3.16 and 4.40 that $\mathrm{R}(K)'$ separates the points of $\mathrm{R}(K)$ and consequently there exists $f \in \mathrm{R}(K)'$ with $f(K(x)) \neq 0$. Now, $fK \in X'$ and satisfies $fK(x) \neq 0$ which, by Proposition 4.7, is in contradiction with the fact that $D = \langle x \rangle$ is not topologically complemented in X.

Remarks. (1) Notice that, by Proposition 5.2, given $T \in L(X, Y)$, any $x \in \mathrm{Ker}(T)$ such that $\langle x \rangle$ is not topologically complemented in X is also contained in $\mathrm{Ker}(T+K)$ for every compact operator K, a result which does not have a real or complex counterpart.

(2) On the other hand, Proposition 5.2 does not hold in general for finite dimensional subspaces D with $\dim D > 1$, as the following example shows.

Take $X = \mathbb{K} \times (\ell^\infty/c_0)$ and $D = \langle x_1, x_2 \rangle$, where $x_1 = (1, 0)$ and $x_2 = (1, \pi(1, 1, \ldots, 1, \ldots))$, being $\pi: \ell^\infty \to \ell^\infty/c_0$ the canonical quotient map.

Since $(\ell^\infty/c_0)' = \{0\}$ ([8] Corollary 4.3) we have that D is not topologically complemented in X.

Now, if Proposition 5.2 above was true, for this D and this X, taking $Y = \mathbb{K}$, we obtain that $\langle x_1 \rangle, \langle x_2 \rangle \subset \mathrm{Ker}(f)$ for any $f \in X'$, which contradicts Proposition 4.7, because $M = \{(\lambda, \pi(a_n)) \in X : \lambda = 0\}$ is a closed algebraic complement of both of $\langle x_1 \rangle$ and $\langle x_2 \rangle$ in X (and hence topological, by the Open Mapping Theorem).

Now, Theorem 5.1 and Proposition 5.2 allow us to state the following result.

THEOREM 5.3. *Assume $T \in \Phi(X, Y)$ satisfies* $\mathrm{Ker}(T) = \langle x_1, x_2, \ldots, x_n \rangle$, *where $\{x_1, x_2, \ldots, x_n\}$ is a finite family of linearly independent vectors in X such that, for any $i \in \{1, 2, \ldots, n\}$, $\langle x_i \rangle$ is not topologically complemented in X. Let S be a pseudoinverse of $\hat{T}$. Then, for any $K \in C(X, Y)$ with $\|K\| < \|S\|^{-1}$ we have*

(a) $T + K \in \Phi(X, Y)$,

(b) $\chi(T + K) = \chi(T)$,

(c) $\delta(T + K) = \delta(T)$ (*in particular, if T is surjective, then so is $T + K$*),

(d) $\mathrm{Ker}(T + K) = \mathrm{Ker}(T)$.

Remark. Now we are going to give an example of continuous linear operators T, K satisfying the hypotheses of Theorem 5.3, for which $\mathrm{R}(T + K) \neq \mathrm{R}(T)$ (although, by property (c) of this theorem, we always have that $\delta(T + K) = \delta(T)$).

Being $\mathbb{K}$ not spherically complete, there exists a Banach space X and an $a \in X \setminus \{0\}$ such that $(X/\langle a \rangle)'$ separates the points of $X/\langle a \rangle$, but X' does not separate the points of X (see [9] p. 214).

Take $Y = \mathbb{K} \times (X/\langle a \rangle)$ and $T : X \to Y, x \mapsto (0, Q(x))$, where $Q : X \to X/\langle a \rangle$ is the canonical quotient map. Then, T is a (non surjective) Fredholm operator from X to Y for which $\mathrm{Ker}(T) = \langle a \rangle = \cap_{f \in X'} \mathrm{Ker}(f)$ and so, by Proposition 4.7, $\mathrm{Ker}(T)$ is not topologically complemented in X.

Let S be a pseudoinverse of $\hat{T}$ and choose $g \in X' \setminus \{0\}$ with $\|g\| < \|S\|^{-1}$. Then, $K : X \to Y, x \mapsto (g(x), 0)$, is a compact operator from X to Y for which $\|K\| = \|g\| < \|S\|^{-1}$. Hence, T and K satisfy the hypotheses of Theorem 5.3.

However, for every $x \in X$ with $g(x) \neq 0$, we have that $(g(x), Q(x))$ is an element of $\mathrm{R}(T + K)$ which does not belong to $\mathrm{R}(T)$.

Observe that this example also shows the existence of continuous linear operators T and K under the hypotheses of Theorem 5.1, for which $\mathrm{R}(T + K) \neq \mathrm{R}(T)$.

The example given in Remark 2 after Proposition 5.2 also shows the existence of non-Archimedean Banach spaces X and finite-dimensional subspaces $D = \langle x_1, x_2, \ldots, x_n \rangle$ such that $\langle x_i \rangle$ is topologically complemented in X for all $i \in \{1, 2, \ldots, n\}$, but D is not (obviously, by [8] Theorem 3.15, topological complementation of D implies the same for each $\langle x_i \rangle$, $i \in \{1, 2, \ldots, n\}$). This fact makes the proof of the following result be not so straightforward as one could expect at a first glance.

LEMMA 5.4. *Let D be a finite-dimensional subspace of X with* $\dim D = n$. *Then, there exist D_0 and D_1 subspaces of D with $D = D_1 \oplus D_0$, D_1 topo-*

logically complemented in X, and $\langle x\rangle$ not topologically complemented in X for any $x \in D_0 \setminus \{0\}$.

Proof. We can assume that $n > 0$ and that D is not topologically complemented in X.

By Proposition 4.7, $D_0 := \{x \in D\colon f(x) = 0 \text{ for any } f \in X'\}$ is a not trivial subspace of D and $D_0 \setminus \{0\}$ coincides with the set of all $x \in D \setminus \{0\}$ such that $\langle x\rangle$ is not topologically complemented in X. Call $\{x_{r+1}, x_{r+2}, \dots, x_n\}$, $(0 \leqslant r < n)$, a base of D_0.

If $r = 0$ the conclusion obviously follows.

Now, suppose that $r \geqslant 1$ and extend $\{x_{r+1}, x_{r+2}, \dots, x_n\}$ to a base $\{x_1, \dots, x_r, x_{r+1}, \dots, x_n\}$ of D.

To finish, it will be enough to see that $D_1 := \langle x_1, x_2, \dots, x_r\rangle$ is topologically complemented in X. But this is a consequence of Proposition 4.7 and the fact that $D_0 \cap D_1 = \{0\}$.

Given $T \in \Phi(X, Y)$ with $\dim \mathrm{Ker}(T) = n$, by Lemma 5.4, there exist subspaces D_0 and D_1 of $\mathrm{Ker}(T)$ such that

(α) $\mathrm{Ker}(T) = D_1 \oplus D_0$,
(β) D_1 has a topological complement M in X,
(γ) For each $x \in D_0 \setminus \{0\}$, $\langle x\rangle$ is not topologically complemented in X.

THEOREM 5.5. *Let $T \in \Phi(X, Y)$ with $\dim \mathrm{Ker}(T) = n$ and let D_0,$D_1 \subset \mathrm{Ker}(T)$ and $M \subset X$ be as in (α), (β) and (γ) above. Then the restriction, T^*, of T to M is a Fredholm operator. Also, if S is a pseudoinverse of $\widehat{T^*}$, then for every $K \in C(X, Y)$ satisfying $\|K\| < \|S\|^{-1}$ we have*

(a) $T + K \in \Phi(X, Y)$,
(b) $\chi(T + K) = \chi(T)$,
(c) $\delta(T + K) \leqslant \delta(T)$,
(d) $\eta(T + K) \leqslant \eta(T)$.

In particular, if T is surjective, we have

(e) *$T + K$ is surjective,*
(f) $\eta(T+K) = \eta(T)$.

Proof. When $D_0 = \{0\}$ or $D_1 = \{0\}$, then the conclusions follow from Theorems 4.6 and 5.3 respectively. So, we suppose that both D_0 and D_1 are not equal to $\{0\}$.

Take $x_1, \dots, x_r, x_{r+1}, \dots, x_n \in \mathrm{Ker}(T) \setminus \{0\}$, $0 < r < n$, such that

$$D_1 = \langle x_1, x_2, \dots, x_r\rangle \quad \text{and} \quad D_0 = \langle x_{r+1}, x_{r+2}, \dots, x_n\rangle.$$

Since, by (β), $X = \langle x_1, x_2, \dots, x_r\rangle \oplus M$, then for every $i \in \{r+1, r+2, \dots, n\}$, there exist $\lambda_1^i, \lambda_2^i, \dots, \lambda_r^i \in \mathbb{K}$ and $b_i \in M$ such that

$$x_i = \sum_{j=1}^{r} \lambda_j^i x_j + b_i. \tag{1}$$

Applying (α) and (1) we easily deduce that

$$\operatorname{Ker}(T) = \langle x_1, \dots, x_r, b_{r+1}, \dots, b_n\rangle \tag{2}$$

and consequently $b_{r+1}, b_{r+2}, \dots, b_n$ are linearly independent vectors in X.

Now, we are going to prove the following properties, (i), (ii) and (iii), on T^*.

(i) $\mathrm{R}(T^*) = \mathrm{R}(T)$.

This fact follows easily from (α) and (β).

(ii) $\operatorname{Ker}(T^*) = \langle b_{r+1}, b_{r+2}, \dots, b_n\rangle$.

By (2) it is clear that $\langle b_{r+1}, b_{r+2}, \dots, b_n\rangle \subset \operatorname{Ker}(T) \cap M = \operatorname{Ker}(T^*)$.

Conversely, applying (2) again, every $m \in \operatorname{Ker}(T^*)$ can be written as

$$m = \sum_{i=1}^{r} \lambda_i x_i + \sum_{j=r+1}^{n} \mu_j b_j$$

for some $\lambda_i, \mu_j \in \mathbb{K}$, $i \in \{1, 2, \dots, r\}$, $j \in \{r+1, r+2, \dots, n\}$. Then $m - \sum_{j=r+1}^{n} \mu_j b_j \in \langle x_1, x_2, \dots, x_r\rangle$ and, on the other hand, it belongs to M, which by (β) implies that it is zero. Hence, $m = \sum_{j=r+1}^{n} \mu_j b_j \in \langle b_{r+1}, b_{r+2}, \dots, b_n\rangle$.

(iii) For every $j \in \{r+1, r+2, \dots, n\}$, $\langle b_j\rangle$ is not topologically complemented in M.

We suppose, on the contrary, that there exists $j_0 \in \{r+1, r+2, \dots, n\}$ such that $\langle b_{j_0}\rangle$ is topologically complemented in M. Then, there exists a closed subspace M_{j_0} of M such that $M = M_{j_0} \oplus \langle b_{j_0}\rangle$. By (β), $X = \langle x_1, x_2, \dots, x_r\rangle \oplus M_{j_0} \oplus \langle b_{j_0}\rangle$. Now by (1), $\langle x_1, x_2, \dots, x_r\rangle \oplus \langle b_{j_0}\rangle = \langle x_1, \dots, x_r, x_{j_0}\rangle$. Hence, M_{j_0} is an algebraic complement of $\langle x_1, \dots, x_r, x_{j_0}\rangle$ in X (and hence topological, by the Open Mapping Theorem). This implies (see the comments before Lemma 5.4) that $\langle x_{j_0}\rangle$ is topologically complemented in X, which is in contradiction with (γ).

Observe that (i) and (ii) imply that $T^* \in \Phi(M, Y)$ and $\chi(T^*) = n - r - \delta(T)$.

On the other hand, if we consider the compact operator $K^* := K|M \in C(M, Y)$, it is clear that $\|K^*\| \leqslant \|K\| < \|S\|^{-1}$. Then, Theorem 5.3 together with properties (i), (ii) and (iii), allow us to conclude that

(a*) $T^* + K^* \in \Phi(M, Y)$,
(b*) $\chi(T^* + K^*) = \chi(T^*) = n - r - \delta(T)$,
(c*) $\delta(T^* + K^*) = \delta(T^*) = \delta(T)$.

Next, we are going to prove (a) and (b).

According to (β), there exists a continuous linear projection P from X onto M such that $\mathrm{Ker}(P) = \langle x_1, x_2, \dots, x_r\rangle$. Clearly $P \in \Phi(X, M)$ and $\chi(P) = r$.

Then, applying Proposition 3.2 in combination with (a^*) and (b^*), we deduce that $(T^* + K^*)P \in \Phi(X, Y)$ and

$$\chi((T^* + K^*)P) = \chi(T). \tag{3}$$

Also, observe that

$$T + K = (T^* + K^*)P + K(I_X - P), \tag{4}$$

where $\dim \mathrm{R}(K(I_X - P)) < \infty$. Now (a) and (b) are direct consequences of (3), (4) and Theorem 3.5.

To finish we will prove (d) (observe that (c) follows directly from (b) and (d)).

Since $\|K\| < \|S\|^{-1}$, there exists $t \in (0, 1)$ such that $\|K\| < t\|S\|^{-1}$. Also, by Proposition 4.1, there exists a continuous linear projection Q from Y onto $\mathrm{R}(T)$ with $\|Q\| \leqslant t^{-1}$. Then, $\overline{T} := QT \in L(X, \mathrm{R}(T))$ is a Fredholm operator from X onto $\mathrm{R}(T)$ for which $\mathrm{Ker}(\overline{T}) = \mathrm{Ker}(T)$. Further, it is straightforward to verify that $\overline{S} := S|\mathrm{R}(T)$ is a pseudoinverse of $\widehat{\overline{T}^*}$, where $\overline{T}^* = \overline{T}|M$.

On the other hand, $\overline{K} := QK$ is a compact operator from X to $\mathrm{R}(T)$ for which

$$\|\overline{K}\| \leqslant \|Q\|\|K\| < \|S\|^{-1} \leqslant \|\overline{S}\|^{-1}.$$

Therefore, we can apply properties (a), (b) and (c*) above, by taking $Y = \mathrm{R}(T)$, $T = \overline{T}$ and $K = \overline{K}$, to conclude that $\overline{T} + \overline{K}$ is a Fredholm operator from X to $\mathrm{R}(T)$ such that

$$\chi(\overline{T} + \overline{K}) = \chi(\overline{T}) \tag{5}$$

and

$$\delta(\overline{T}^* + \overline{K}^*) = \delta(\overline{T}) = 0, \tag{6}$$

where $\overline{K}^* = \overline{K}|M$.

By (6), we have that $\overline{T} + \overline{K}$ is surjective which, together with (5), imply that

$$\eta(\overline{T} + \overline{K}) = \eta(\overline{T}) = \eta(T). \tag{7}$$

Finally, observe that since $\mathrm{Ker}(T + K) \subset \mathrm{Ker}(Q(T + K)) = \mathrm{Ker}(\overline{T} + \overline{K})$, it follows from (7) that $\eta(T + K) \leqslant \eta(T)$ which proves (d).

Remarks. (1) Although the structure of the kernel is in some way preserved, as we saw in Remark 1 after Proposition 5.2, there exist T and K as in Theorem 5.5 for which $\mathrm{Ker}(T + K) \neq \mathrm{Ker}(T)$, even when T is surjective, as we can see in the following example (compare with Theorem 5.3).

First of all, note that $R: \ell^\infty/c_0 \to \ell^\infty/c_0$, $\pi(a_n) \mapsto \pi(a_{n+1} - a_n)$ (where $\pi: \ell^\infty \to \ell^\infty/c_0$ is the canonical quotient map) is a well defined linear and continuous operator such that R is surjective and $\mathrm{Ker}(R) = \langle \pi(1, 1, \ldots, 1, \ldots)\rangle$.

Now, take $X = \mathbb{K} \times (\ell^\infty/c_0)$, $Y = \ell^\infty/c_0$ and $T: X \to Y$ defined as $T((\lambda, \pi(a_n))) := R(\pi(a_n))$. Then, T is a surjective Fredholm operator with $\mathrm{Ker}(T) = \langle (1, 0), (0, \pi(1, 1, \ldots, 1, \ldots))\rangle$.

Choose $\alpha \in \mathbb{K} \setminus \{0\}$ with $|\alpha| < \| S\|^{-1}$, being S a pseudoinverse of $\widehat{T^*}$ (where $\widehat{T^*}$ is defined as in Theorem 5.5). Then, $K: X \to Y$, $(\lambda, \pi(a_n)) \mapsto -\alpha\pi(\lambda, \lambda, \ldots, \lambda, \ldots)$ is a compact operator with $\|K\| < \|S\|^{-1}$.

Hence, T and K satisfy the hypotheses of Theorem 5.5.

On the other hand, we can easily see that $\mathrm{Ker}(T + K) = \langle (1, \alpha\pi(1, 2, 3, \ldots, n, \ldots)), (0, \pi(1, 1, \ldots, 1, \ldots))\rangle$, and so $\mathrm{Ker}(T + K) \neq \mathrm{Ker}(T)$, because $(1, \alpha\pi(1, 2, 3, \ldots, n, \ldots)) \notin \mathrm{Ker}(T)$.

(2) Also we provide an example of T and K as in Theorem 5.5 for which $\delta(T + K) < \delta(T)$ and $\eta(T + K) < \eta(T)$ (compare again with Theorem 5.3). In particular, this implies that, when T is not surjective, Statements (e) and (f) of Theorem 5.5 are not true in general.

Take $X = Y = \mathbb{K} \times \mathbb{K} \times (\ell^\infty/c_0)$ and define $T: X \to Y$ as $T((\lambda, \mu, \pi(a_n))) := (0, 0, R(\pi(a_n)))$, with R and π as above. We have that $\mathrm{Ker}(T) = \langle (1, 0, 0), (0, 1, 0), (0, 0, \pi(1, 1, \ldots, 1, \ldots))\rangle$, and $\mathrm{R}(T) = \{(\lambda, \mu, \pi(a_n)) \in X: \lambda = \mu = 0\}$. Hence, $T \in \Phi(X)$, $\eta(T) = 3$, and $\delta(T) = 2$.

As in Remark 1, choose $\alpha \in \mathbb{K} \setminus \{0\}$ with $|\alpha| < \|S\|^{-1}$, being S a pseudoinverse of $\widehat{T^*}$. Then $K: X \to Y$ defined as $K((\lambda, \mu, \pi(a_n))) := (\alpha\lambda, 0, 0)$ is a compact operator with $\|K\| < \|S\|^{-1}$.

Hence, we again have that T and K satisfy the hypotheses of Theorem 5.5.

But, on the other hand, we can easily see that

$$\mathrm{Ker}(T + K) = \langle (0, 1, 0), (0, 0, \pi(1, 1, \ldots, 1, \ldots))\rangle$$

and

$$\mathrm{R}(T + K) = \{(\lambda, \mu, \pi(a_n)) \in X: \mu = 0\}.$$

So we have that

$$1 = \delta(T + K) < \delta(T) = 2,$$

$$2 = \eta(T + K) < \eta(T) = 3.$$

6. The Main Result

Finally, we are in a position to prove a final theorem which is an extension of the corresponding one given in [3] Theorem 8.1.1 for a locally compact ground field $\mathbb{K}$.

THEOREM 6.1. *For $T \in \Phi(X, Y)$ and $K \in C(X, Y)$ we have that $T + K \in \Phi(X, Y)$ and $\chi(T + K) = \chi(T)$.*

Proof. When $\mathbb{K}$ is spherically complete $\mathrm{Ker}(T)$ is topologically complemented in X (see the remark after Proposition 4.7) and in this case we take S to be as in Theorem 4.6; otherwise, S is taken as in Theorem 5.5. By [8] Theorem 4.39, compactness of K implies the existence of an $F \in L(X, Y)$ with $\dim \mathrm{R}(F) < \infty$ and such that $\|K - F\| < \|S\|^{-1}$. Obviously $K - F \in C(X, Y)$. Now $T + K - F \in \Phi(X, Y)$ and $\chi(T + K - F) = \chi(T)$: this follows directly from Theorem 4.6 when $\mathbb{K}$ is spherically complete and from Theorem 5.5 when $\mathbb{K}$ is not. Next, since $\dim \mathrm{R}(F) < \infty$, we can apply Theorem 3.5 to conclude that $T + K = (T + K - F) + F \in \Phi(X, Y)$, and $\chi(T + K) = \chi(T + K - F) = \chi(T)$.

Remark. As an application of Theorem 6.1 we obtain that if $K \in C(X)$, then $I_X + K \in \Phi(X)$ and $\chi(I_X + K) = 0$, which was already proved in p-adic Fredholm theory (see e.g. [5], [10] and [11]).

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