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NON-REAL PERIODIC POINTS OF ENTIRE FUNCTIONS

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ABSTRACT. It is shown that if f is an entire transcendental function, l a straight line in the complex plane, and $n \geq 2$, then f has infinitely many repelling periodic points of period n that do not lie on l.

1. **Introduction and main result.** Let f be an entire function and denote by f^n its n-th iterate. We say that z_0 is a periodic point of f if $f^n(z_0) = z_0$ for some $n \in \mathbb{N}$. The smallest n with this property is called the *period* of z_0. A periodic point z_0 of period n is called *attracting*, *indifferent*, or *repelling* depending on whether $|(f^n)'(z_0)|$ is less than, equal to, or greater than 1. The following results were proved in [2] and [3].

THEOREM A. *Let f be an entire transcendental function and $n \geq 2$. Then f has infinitely many repelling periodic points of period n.*

THEOREM B. *Let f be an entire transcendental function, l a straight line in the complex plane, and $n \geq 2$. Then f^n has infinitely many fixpoints that do not lie on l.*

These results confirmed conjectures by I. N. Baker [5, Problems 2.20 and 2.23]. For a discussion of the background of these results we refer to [2] and [3]. The purpose of this paper is to prove a theorem which combines these two results.

THEOREM. *Let f be an entire transcendental function, l a straight line in the complex plane, and $n \geq 2$. Then f has infinitely many repelling periodic points of period n that do not lie on l.*

Among the tools used in [2] are the Wiman-Valiron theory about the behavior of entire functions near points of maximum modulus, Ahlfors' theory of covering surfaces, and results relating critical points to attracting and indifferent periodic points. These techniques are also used in [3], the essential new ingredient being estimates of the Fourier coefficients of $\log|h(re^{i\theta})|$ for entire h. In the present paper we use some ideas from [2] and [3], but replace the Fourier series method by more elementary considerations. We shall also use some Nevanlinna theory. We note that the method of [3] works only in the case where the order of f^n is greater than 2, the remaining case (and in fact the case that f^n has finite order) being covered by a result of Baker [1]. Our present approach does not require any restrictions on the growth of f or f^n.

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2. **Preliminaries.** In the first part of the proof of Theorem A in [2, Section 3] it is shown that if f has only finitely many repelling periodic points of period n, then for given $\varepsilon > 0$ there exists a sequence (r_ν) tending to ∞ and having the property that for all ν there is a Jordan curve Γ_0 contained in $\{z : r_\nu \leq |z| \leq (r_\nu)^{1+\varepsilon}\}$ which surrounds the origin and is such that $|f^n(z)| = M(r_\nu, f^n)$ for $z \in \Gamma_0$. Moreover, if G_0 is the interior of Γ_0 and $G_k = f^k(G_0)$ for $k = 1, \ldots, n$, then f is a proper map from G_{k-1} onto G_k. We denote its degree by p_k.

It is shown in [2, Section 6] that if $\bar{N}_{\mathrm{rep}}$ denotes the number of repelling periodic points of period n in G_0 and if $P_k = p_1 p_2 \cdots p_k$ for $k = 1, \ldots, n$, then

$$\bar{N}_{\mathrm{rep}} \geq P_n - \sum_{k<n, k|n} P_k - 3n(p_n - 1). \tag{1}$$

From this Theorem A immediately follows. (Note that the p_k depend also on ν and, in fact, we have $p_k \longrightarrow \infty$ as $\nu \longrightarrow \infty$ for each k.) We mention that we use the same terminology as in [2], except that what was called *primitive period* in [2] is now called *period*.

In [3, Section 3] it is described in detail how one can find such a sequence (r_ν) and corresponding $\Gamma_0, G_0, \ldots$ if $n = 2$ and if f^2 has only finitely many non-real fixpoints. It is mentioned that this also works if $n \geq 3$, and the minor modifications necessary to handle this case are left to the reader. Here we note that the methods of [2] and [3] do in fact also work under the weaker assumption that f has only finitely many non-real repelling periodic points of period n. Assuming this we again obtain a sequence (r_ν) with the properties described above. We omit the details.

3. **Results from Nevanlinna theory.** We shall apply Nevanlinna's theory on the distribution of values and thus briefly introduce the definitions and results that we shall use. For a detailed account of the theory we refer to [4] or [6].

For a meromorphic function h, we denote by $n(r, h)$ the number of poles of h in $|z| \leq r$, counted according to multiplicity. For $a \in \mathbb{C}$ the number of a-points of h in $|z| \leq r$ is thus given by $n\big(r, 1/(h-a)\big)$. Next we define $N(r, h) = \int_0^r n(t, h)/t \, dt$, omitting the slight modification necessary if $h(0) = \infty$. We note that

$$N(r, h) = \int_1^r \frac{n(t, h)}{t} \, dt + N(1, h) \leq n(r, h) \int_1^r \frac{dt}{t} + N(1, h) = n(r, h) \log r + N(1, h) \tag{2}$$

and

$$N(r^\alpha, h) \geq \int_r^{r^\alpha} \frac{n(t, h)}{t} \, dt \geq n(r, h) \int_r^{r^\alpha} \frac{dt}{t} = (\alpha - 1) n(r, h) \log r \tag{3}$$

if $r, \alpha \geq 1$.

The Nevanlinna characteristic $T(r, h)$ is defined by

$$T(r, h) = N(r, h) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ |h(re^{i\theta})| \, d\theta.$$

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Here $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ otherwise. It is easy to see that if h_1 and h_2 are meromorphic, then

$$T(r, h_1 + h_2) \leq T(r, h_1) + T(r, h_2) + O(1) \tag{4}$$

as $r \longrightarrow \infty$. Nevanlinna's first fundamental theorem says that if $a \in \mathbb{C}$, then $T\big(r, 1/(h-a)\big) = T(r, h) + O(1)$ as $r \longrightarrow \infty$. An immediate consequence is that

$$N\Big(r, \frac{1}{h-a}\Big) \leq T(r, h) + O(1) \tag{5}$$

as $r \longrightarrow \infty$. A simplified version of Nevanlinna's second fundamental theorem says that

$$T(r, h) \leq \big(1 + o(1)\big)\Big(N(r, h) + N\Big(r, \frac{1}{h}\Big) + N\Big(r, \frac{1}{h-1}\Big)\Big)$$

as $r \longrightarrow \infty$ outside some exceptional set of finite measure. If $\alpha > 1$ and r is large, then there exists $s \in [r, r^\alpha]$ which is not in the exceptional set and thus

$$\begin{aligned} T(r, h) &\leq T(s, h) \\ &\leq \big(1 + o(1)\big)\Big(N(s, h) + N\Big(s, \frac{1}{h}\Big) + N\Big(s, \frac{1}{h-1}\Big)\Big) \\ &\leq \big(1 + o(1)\big)\Big(N(r^\alpha, h) + N\Big(r^\alpha, \frac{1}{h}\Big) + N\Big(r^\alpha, \frac{1}{h-1}\Big)\Big). \end{aligned}$$

The argument shows that we could replace r^α by $r + \varepsilon$ for every $\varepsilon > 0$ here, but we do not need this. Note that $N(r, h) = 0$ for entire h so that the last inequality takes the form

$$T(r, h) \leq \big(1 + o(1)\big)\Big(N\Big(r^\alpha, \frac{1}{h}\Big) + N\Big(r^\alpha, \frac{1}{h-1}\Big)\Big) \tag{6}$$

in this case.

4. **Proof of the theorem.** As in [3] we may assume without loss of generality that l is the real axis. We assume that the theorem is false and denote the number of non-real repelling periodic points of period n by K. Then there exist a sequence (r_ν) and corresponding Γ_0, G_k, p_k, P_k with the properties described in Section 2, and (1) holds. By $\overline{N}_{\mathrm{rep},\mathbb{R}}$ we denote the number of real repelling periodic points of period n contained in G_0. By assumption we have $\overline{N}_{\mathrm{rep},\mathbb{R}} \geq \overline{N}_{\mathrm{rep}} - K$.

As in [3] we prove first that f is real on the real axis. To this end we note that if z_0 is a repelling periodic point of period n, then so is $f(z_0)$. Thus f is real at the repelling periodic points of period n, with at most K exceptions. Thus $g(z) = f(z) - \overline{f(\overline{z})}$ has at least $\overline{N}_{\mathrm{rep},\mathbb{R}} - K$ zeros in G_0. Assuming that g does not vanish identically and using the fact that $G_0 \subset \{z : |z| \leq (r_\nu)^{1+\varepsilon}\}$, as well as the Nevanlinna theory estimates (3), (5), (4), (6), and (2), we obtain for $r = r_\nu$

$$\begin{aligned}
\overline{N}_{\mathrm{rep}} - 2K &\le \overline{N}_{\mathrm{rep},\mathbb{R}} - K \\
&\le n\Bigl(r^{1+\varepsilon}, \frac{1}{g}\Bigr) \\
&\le \frac{1}{\varepsilon \log r} N\Bigl(r^{1+2\varepsilon}, \frac{1}{g}\Bigr) \\
&\le \frac{1}{\varepsilon \log r}\bigl(T(r^{1+2\varepsilon}, g) + O(1)\bigr) \\
&\le \frac{2}{\varepsilon \log r}\bigl(T(r^{1+2\varepsilon}, f) + O(1)\bigr) \\
&\le \frac{2+\varepsilon}{\varepsilon \log r}\Bigl(N\Bigl(r^{1+3\varepsilon}, \frac{1}{f}\Bigr) + N\Bigl(r^{1+3\varepsilon}, \frac{1}{f-1}\Bigr) + O(1)\Bigr) \\
&\le \frac{(2+\varepsilon)(1+3\varepsilon)}{\varepsilon}\Bigl(n\Bigl(r^{1+3\varepsilon}, \frac{1}{f}\Bigr) + n\Bigl(r^{1+3\varepsilon}, \frac{1}{f-1}\Bigr)\Bigr) + o(1) \\
&\le \frac{(2+\varepsilon)(1+3\varepsilon)}{\varepsilon} 2p_2 + o(1)
\end{aligned}$$

as $\nu \longrightarrow \infty$. The last inequality follows since $\{z : |z| \le (r_\nu)^{1+3\varepsilon}\} \subset G_1$ for large ν (*cf.* [2]) and f takes every value in G_2 (and thus for large ν in particular the values 0 and 1) exactly p_2 times in G_1, counted according to multiplicity. Clearly the above estimate contradicts (1) for large ν because $p_k \longrightarrow \infty$ for each k as $\nu \longrightarrow \infty$. Thus g vanishes identically and hence f is real on the real axis. This implies that we may assume that G_0 is symmetric with respect to the real axis. In particular, $G_0 \cap \mathbb{R} = (a_\nu, b_\nu)$ where $a_\nu < b_\nu$.

We denote the number of real fixpoints of f^n in (a_ν, b_ν) by Q and the number of real repelling fixpoints of f^n in (a_ν, b_ν) by Q_{rep}. Here and in the following we disregard multiplicities. Note that $Q \le P_n$ and $Q_{\mathrm{rep}} \ge \overline{N}_{\mathrm{rep},\mathbb{R}} \ge \overline{N}_{\mathrm{rep}} - K$. The real fixpoints of f^n bound $Q-1$ subintervals of (a_ν, b_ν). At most $2(Q - Q_{\mathrm{rep}})$ of these intervals have a non-repelling endpoint and thus $Q - 1 - 2(Q - Q_{\mathrm{rep}}) = 2Q_{\mathrm{rep}} - Q - 1$ of these intervals are bounded by two repelling fixpoints of f^n. For an interval bounded by two repelling fixpoints of f^n the derivative $(f^n)'$ must be greater than 1 at one endpoint and less than -1 at the other endpoint. In particular, such an interval must contain a zero of $(f^n)'$. Thus the number S of zeros of $(f^n)'$ in (a_ν, b_ν) satisfies

$$(7)\quad S \ge 2Q_{\mathrm{rep}} - Q - 1 \ge 2(\overline{N}_{\mathrm{rep}} - K) - P_n - 1 \ge P_n - 2\sum_{k<n, k|n} P_k - 6n(p_n - 1) - 2K - 1$$

by (1).

Next we show that f' has infinitely many real zeros. To do this we assume that f' has only finitely many, say N, real zeros. Then f takes every real value at most $N+1$ times on the real axis. Since $(f^n)' = (f^{n-1})'(f)f'$ and since $(f^{n-1})'$ has at most $p_2 p_3 \cdots p_n - 1$ zeros in G_1 we conclude that $S \le (p_2 p_3 \cdots p_n - 1)(N+1) + N$. Combining this with (7) we obtain a contradiction since $p_k \longrightarrow \infty$ as $\nu \longrightarrow \infty$. Thus f' has infinitely many real zeros.

We now fix an interval $[a,b]$ and denote by L the number of zeros of f' in $[a,b]$ and by M the number of zeros of $(f^n)'$ in $[a,b]$. For large ν we have $a_\nu < a < b < b_\nu$ and f' has at most $p_1 - 1 - L$ zeros in $(a_\nu, a) \cup (b, b_\nu)$. Thus f assumes every value in G_1 at most $p_1 - L + 1$ times in $(a_\nu, a) \cup (b, b_\nu)$. As before we conclude that $(f^n)'$ has at most $(p_2 p_3 \cdots p_n - 1)(p_1 - L + 1) + p_1 - 1 - L$ zeros in $(a_\nu, a) \cup (b, b_\nu)$. Thus

(8) $S \leq (p_2 p_3 \cdots p_n - 1)(p_1 - L + 1) + p_1 - 1 - L + M = P_n - (L-1) p_2 p_3 \cdots p_n - 2 + M.$

From (7) and (8) we obtain

$$P_n - 2 \sum_{k<n, k|n} P_k - 6n(p_n - 1) - 2K - 1 \leq P_n - (L-1) p_2 p_3 \cdots p_n - 2 + M$$

and hence

$$(L-1) p_2 p_3 \cdots p_n \leq 2 \sum_{k<n, k|n} P_k + 6n(p_n - 1) + 2K + M - 1.$$

Since $p_{k+1} \geq p_k$ and $p_k \longrightarrow \infty$ as $\nu \longrightarrow \infty$ we obtain a contradiction if we choose (a,b) such that L is sufficiently large. In fact, if $n \geq 3$ it is enough to take $L = 2$ and if $n = 2$ the choice $L = 16$ suffices. This contradiction completes the proof of the theorem.

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