INVARIANTS OF FINITE REFLECTION GROUPS

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To Richard Brauer on his 60th birthday

1. Let K be a field of characteristic zero. Let V be an n-dimensional vector space over K and let S be the graded ring of polynomial functions on V. If G is a group of linear transformations of V, then G acts naturally as a group of automorphisms of S if we define

$$(\gamma s)(v) = s(\gamma^{-1}v) \qquad \gamma \in G,\ s \in S,\ v \in V$$

The elements of S invariant under all $\gamma \in G$ constitute a homogeneous subring $I(S)$ of S called the ring of polynomial invariants of G.

A linear transformation of V is a reflection if it has finite order and leaves fixed an $n-1$ dimensional subspace, its reflecting hyperplane. If G has finite order and is generated by reflections we call it a finite reflection group. For such groups we know from work of Chevalley [2] and Coxeter [3] that the ring $I(S)$ is a polynomial ring generated by n algebraically independent forms $f_1, \ldots, f_n$. In fact, Shephard and Todd [4] have shown that this property of the ring of polynomial invariants characterizes the finite groups generated by reflections. It has been known for a long time, at least for the real orthogonal groups, that the degrees $m_1+1, \ldots, m_n+1$ of the forms $f_1, \ldots, f_n$ satisfy the product formula $(m_1+1)\cdots(m_n+1) = g$, where g is the order of G, and that the sum $m_1 + \cdots + m_n$ is equal to the number of reflections in the group. More recently, Shephard and Todd [4] discovered and verified the general formula

$$(1) \qquad (1+m_1t)\cdots(1+m_nt) = g_0 + g_1t + \cdots + g_nt^n$$

where g_r is the number of elements of G that fix some $n-r$ dimensional subspace of V but fix no subspace of higher dimension. If G is a crystallographic group then the Poincaré polynomial of the corresponding Lie group is known to be $(1+t^{2m_1+1})\cdots(1+t^{2m_n+1})$ so that the formula yields a method for com-

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puting the Betti numbers of a complex simple Lie group from the structure of its Weyl group.

In this paper we supply a proof of the formula. The idea in the argument is close to one that Brauer [1] has used to compute the Betti numbers of the classical groups. We use knowledge of the polynomial invariants of a group to deduce the nature of its invariant differential forms, and then a counting argument yields the result.

2. Let $E=\sum_{p=0}^{n}E_p$ be the Grassmann algebra of V. The homogeneous component E_p of degree p is for $p=1,\dots,n$ the K-space of all p-linear alternating functions on V, and we agree that $E_0=K$. Then G acts naturally on E_p if we define

$$(\gamma e)(v_1,\dots,v_p)=e(\gamma^{-1}v_1,\dots,\gamma^{-1}v_p)\qquad \gamma\in G,\ e\in E_p,\ v_i\in V$$

and thus G acts as a group of automorphisms of E. We form the tensor product $S\otimes E$ over K and give it the structure of algebra over K by defining

$$(s_1\otimes e_1)(s_2\otimes e_2)=s_1s_2\otimes e_1e_2.$$

Let G act on $S\otimes E$ by

$$\gamma(s\otimes e)=\gamma s\otimes\gamma e$$

Then G acts as a group of automorphisms of $S\otimes E$. Let us choose a fixed coordinate system in V and let $x_1,\dots,x_n$ be the coordinate linear functions. Let $d\colon S\otimes E\to S\otimes E$ be the K-linear map defined by

$$d\colon\ s\otimes x_{i_1}\cdots x_{i_p}\to\sum_{j=1}^{n}\frac{\partial s}{\partial x_j}\otimes x_jx_{i_1}\cdots x_{i_p}$$

The map d commutes with the action of G on $S\otimes E$. This is easily checked as follows. First verify that $d\gamma(x_i\otimes 1)=\gamma d(x_i\otimes 1)$ for all $\gamma\in G$, and then use the formula $d(st\otimes 1)=d(s\otimes 1)\cdot(t\otimes 1)+(s\otimes 1)\cdot d(t\otimes 1)$ to conclude by induction on the degree of s that $d\gamma(s\otimes 1)=\gamma d(s\otimes 1)$ for all homogeneous s and hence for all $s\in S$. Now $d(s\otimes e)=d(s\otimes 1)\cdot(1\otimes e)$ yields $d\gamma(s\otimes e)=\gamma d(s\otimes e)$ for all $s\in S$ and $e\in E$. If we identify S with $S\otimes K$, then $dx_i=d(x_i\otimes 1)=1\otimes x_i$ so that the elements of $S\otimes E_p$ may be written in the form

$$\sum_{i_1<\cdots<i_p}s_{i_1\dots i_p}dx_{i_1}\dots dx_{i_p}$$

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It is clear that $S \otimes E$ is just the Cartan algebra of diffential forms on V and that d is exterior differentiation.

If L is the quotient field of S, the field of rational functions on V, we may extend the action of G to L. We then imbed $S \otimes E$ in the vector space $L \otimes E$ over L and extend the action of G to $L \otimes E$.

3. If our group G acts on a vector space W over K we let $I(W)$ denote the subspace of invariant elements of W. Since d commutes with the action of G on $S \otimes E$ it follows that d carries $I(S \otimes E)$ into $I(S \otimes E)$. In particular the differentials $df_1, \ldots, df_n$ are invariant elements of $S \otimes E$. The structure of the ring $I(S \otimes E)$ of invariant differential forms is given by the following

THEOREM: *Let G be a finite reflection group and let $f_1, \ldots, f_n$ be algebraically independent polynomial forms which generate the ring $I(S)$ of polynomial invariants of G. Then every invariant differential p-form may be written uniquely as a sum*

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \ldots i_p} df_{i_1} \cdots df_{i_p}$$

with $a_{i_1 \ldots i_p} \in I(S)$. Thus the K-algebra $I(S \otimes E)$ of invariant differential forms is an exterior algebra over the K-algebra $I(S)$ of invariant polynomials. It is generated over $I(S)$ by the unit element and the differentials $df_1, \ldots, df_n$ of the basic invariants $f_1, \ldots, f_n$.

For the proof of this theorem we need a

LEMMA: *Let G be a finite reflection group and let j be the Jacobian (determinant) of its basic invariants $f_1, \ldots, f_n$. Then $\gamma j = det(\gamma) j$ for all $\gamma \in G$. If $u \in S$ is a form such that $\gamma u = det(\gamma) u$ for all $\gamma \in G$ then $u = fj$ where $f \in I(S)$ is an invariant.*

Proof of the lemma: Since $\gamma(dx_1 \cdots dx_n) = \det(\gamma^{-1}) dx_1 \cdots dx_n$ and since the action of G commutes with d, we have the string of equalities $j\, dx_1 \cdots dx_n = df_1 \cdots df_n = d(\gamma f_1) \cdots d(\gamma f_n) = \gamma(df_1) \cdots \gamma(df_n) = \gamma(df_1 \cdots df_n) = \gamma(j dx_1 \cdots dx_n) = \gamma j \det(\gamma^{-1}) dx_1 \cdots dx_n$. This proves that $\gamma j = \det(\gamma) j$. Let $W_1, \ldots, W_k$ be the hyperplanes of V which occur as reflecting hyperplanes of elements of G, and let $l_1, \ldots, l_k$ be the linear forms which define them. The subgroup of G fixing W_i is cyclic and generated by a reflection γ_i say of order r_i. It is

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known [4, 5] that

$$j = c l_1^{r_1 - 1} \cdots l_k^{r_k - 1} \tag{2}$$

where $c \in K$ is a non-zero constant. For the case of Euclidean reflections, each γ_i has order two and this says that the Jacobian of the basic invariants is the product of the linear forms which when equated to zero define the reflecting hyperplanes. Let us choose a fixed index i and write $\gamma = \gamma_i$, $l = l_i$, $r = r_i$. Choose a coordinate system in V so that the map γ becomes

$$\gamma: \ (t_1, \ldots, t_{n-1}, t_n) \to (t_1, \ldots, t_{n-1}, \varepsilon t_n)$$

where $\varepsilon = \det(\gamma)$. The condition $\gamma u = \det(\gamma) u$ is then

$$u(t_1, \ldots, t_{n-1}, \varepsilon^{-1} t_n) = \varepsilon u(t_1, \ldots, t_{n-1}, t_n)$$

Differentiate this q times partially with respect to t_n and set $t_n = 0$. This yields

$$\varepsilon^{-q} \frac{\partial^q u}{\partial t_n^q}(t_1, \ldots, t_{n-1}, 0) = \varepsilon \frac{\partial^q u}{\partial t_u^q}(t_1, \ldots, t_{n-1}, 0)$$

Since $\varepsilon^{-q} \neq \varepsilon$ for $q = 0, 1, \ldots, r-2$ it follows that $\frac{\partial^q u}{\partial t_n^q}(t_1, \ldots, t_{n-1}, 0) = 0$ for $q = 0, 1, \ldots, r-2$ and hence l^{r-1} divides u. If we apply this to each of the relatively prime forms $l_1, \ldots, l_k$ and keep (2) in mind we see that $u = fj$ for some $f \in S$. Then in the quotient field L we have $\gamma f = \frac{\gamma u}{\gamma j} = \frac{\det(\gamma) u}{\det(\gamma) j} = f$ so that $f \in I(S)$ is an invariant. This proves the lemma.

To prove the theorem we let G act on the vector space $L \otimes E$ over L. The $\binom{n}{p}$ differential p-forms $df_{i_1} \cdots df_{i_p}$, $1 \leq i_1 < \cdots < i_p \leq n$, are linearly independent over L. For given any relation $\sum k_{i_1 \ldots i_p} df_{i_1} \cdots df_{i_p} = 0$ with $k_{i_1 \ldots i_p} \in L$, choose a fixed set of indices $i_1, \ldots, i_p$ and let $i_{p+1}, \ldots, i_n$ be the complementary subset of $1, \ldots, n$. Multiplication by $df_{i_{p+1}} \cdots df_{i_n}$ shows that $k_{i_1 \ldots i_p} df_1 \cdots df_n = 0$. But $df_1 \cdots df_n = j \ dx_1 \cdots dx_n$ where j is the Jacobian of the forms $f_1, \ldots, f_n$. Since $f_1, \ldots, f_n$ are algebraically independent, $j \neq 0$, and hence $k_{i_1 \ldots i_p} = 0$. This proves the linear independence. Since $L \otimes E_p$ has dimension $\binom{n}{p}$ as vector space over L, the forms $df_{i_1} \cdots df_{i_p}$ span $L \otimes E_p$ over L. Thus given any invariant differential p-form ω we may write $\omega = \sum r_{i_1 \ldots i_p} df_{i_1} \cdots df_{i_p}$ with $r_{i_1 \ldots i_p} \in L$. Average this equation over the group

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G. The invariance of ω and the df_i implies

$$g\omega = \sum_{i_1<\cdots<i_p} \Big(\sum_{\gamma\in G} \gamma r_{i_1\ldots i_p}\Big)\, df_{i_1}\cdots df_{i_p}$$

Since the characteristic is zero we have a formula

$$\omega = \sum \frac{s_{i_1\ldots i_p}}{t_{i_1\ldots i_p}}\, df_{i_1}\cdots df_{i_p}$$

where $s_{i_1\ldots i_p}/t_{i_1\ldots i_p}$ is an invariant element of L. Again we choose a fixed set of indices $i_1, \ldots, i_p$ and multiply by $df_{i_{p+1}}\cdots df_n$. This yields a formula

$$u_{i_1\ldots i_p}\, dx_1\cdots dx_n = \frac{s_{i_1\ldots i_p}}{t_{i_1\ldots i_p}}\, j\, dx_1\cdots dx_n$$

with $u_{i_1\ldots i_p}\in S$. Since $s_{i_1\ldots i_p}/t_{i_1\ldots i_p}$ is an invariant element of L, so is $u_{i_1\ldots i_p}/j$. But $\gamma j = \det(\gamma)j$ and hence $\gamma u_{i_1\ldots i_p} = \det(\gamma)u_{i_1\ldots i_p}$ so by the lemma we conclude that j divides $u_{i_1\ldots i_p}$ in S. Thus $t_{i_1\ldots i_p}$ divides $s_{i_1\ldots i_p}$ and we have a formula

$$\omega = \sum v_{i_1\ldots i_p}\, df_{i_1}\cdots df_{i_p}$$

with $v_{i_1\ldots i_p}\in I(S)$. This proves the theorem. The argument shows for any finite linear group G, that the algebra $I(L\otimes E)$ is an exterior algebra over the field $I(L)$ of invariant rational functions, generated over $I(L)$ by the unit element and the differentials of any n algebraically independent invariant polynomial forms.

If $M = \sum_{q=0}^{\infty} M_q$ is a graded K-space in which all the M_q are finite dimensional, we let $P(M;\ t) = \sum_{q=0}^{\infty} \dim(M_q)t^q$ be the Poincaré series of M. We let $S = \sum_{q=0}^{\infty} S_q$ be graded in the natural way with S_q the space of polynomial forms of degree q. Then $I(S)$ inherits the grading $I(S) = \sum_{q=0}^{\infty} I(S_q)$ and, as Chevalley [2] has remarked

$$P(I(S);\ t) = \frac{1}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})} \tag{3}$$

For each $p = 1, \ldots, n$, $S\otimes E_p$ has the grading $S\otimes E_p = \sum_{q=0}^{\infty} S_q\otimes E_p$ and $I(S\otimes E_p)$ inherits the grading $I(S\otimes E_p) = \sum_{q=0}^{\infty} I(S_q\otimes E_p)$. Let $\sigma_p(t_1, \ldots, t_n)$ be the p-th elementary symmetric function of indeterminates $t_1, \ldots, t_n$. Then from the preceding theorem we conclude at once by counting that

$$P(I(S\otimes E_p);\ t)=\frac{\sigma_p(t^{m_1},\ .\ .\ .\ ,t^{m_n})}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})} \tag{4}$$

for all $p=1,\ .\ .\ .\ ,n$.

4. If our group G has a representation in a vector space W over K, then dim $I(W)$ is the average over G of the character of the representation. Let τ_q be the character of the representation of G in S_q and let σ_p be the character of the representation of G in E_p. We interpret both τ_0 and σ_0 as the principal character. Then $\tau_q\sigma_p$ is the character of the representation of G in $S_q\otimes E_p$ and hence

$$\dim\ I(S_q\otimes E_p)=\frac{1}{g}\sum_{\gamma\in G}\tau_q(\gamma)\sigma_p(\gamma)$$

We work now in the algebraic closure of K. Let $\omega_1(\gamma),\ .\ .\ .\ ,\omega_n(\gamma)$ be the eigenvalues of the linear transformation $\gamma\in G$. The formulas

$$\frac{1}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}=\sum_{q=0}^{\infty}\tau_q(\gamma^{-1})t^q$$

$$(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)=\sum_{p=0}^{n}(-1)^p\sigma_p(\gamma^{-1})t^p$$

were known to Frobenius and may be verified easily by assuming γ in diagonal form. The first of these yields at once the Poincaré series

$$\frac{1}{g}\sum_{\gamma\in G}\frac{\sigma_p(\gamma^{-1})}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}=P(I(S\otimes E_p);\ t)$$

and the second tells us that $\sigma_p(\gamma^{-1})=\sigma_p(\omega_1(\gamma),\ .\ .\ .\ ,\omega_n(\gamma))$. Thus, in view of (4) we have

$$\frac{1}{g}\sum_{\gamma\in G}\frac{\sigma_p(\omega_1(\gamma),\ .\ .\ .\ ,\omega_n(\gamma))}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}=\frac{\sigma_p(t^{m_1},\ .\ .\ .\ ,t^{m_n})}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})} \tag{5}$$

for $p=1,\ .\ .\ .\ ,n$ and even for $p=0$ if one interprets the numerator on the right hand side as 1. Thus one can average any function $\dfrac{\sigma_p(\omega_1(\gamma),\ .\ .\ .\ ,\omega_n(\gamma))}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}$ over the group G by simply replacing $\omega_i(\gamma)$ by t^{m_i} wherever it occurs. This formula can be put to good use. For each $p=1,\ .\ .\ .\ ,n$ one has an identity of the form

$$\sum_{i_1<\cdots<i_p}\frac{\omega_{i_1}(\gamma)\cdots\omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t)\cdots(1-\omega_{i_p}(\gamma)t)}=\frac{h_{p1}(t)\sigma_1(\gamma)+\cdots+h_{pn}(t)\sigma_n(\gamma)}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)} \tag{6}$$

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where the $h_{pq}(t)$ are polynomials in t. For example in case $p=1$ the identity is

$$\sum_i \frac{\omega_i(\gamma)}{1-\omega_i(\gamma)t} = \frac{\sigma_1(\gamma)-2\,t\sigma_2(\gamma)+\cdots\pm nt^{n-1}\sigma_n(\gamma)}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}$$

The important thing in (6) is not the exact nature of the $h_{pq}(t)$, they are in fact monomials, but the fact that the numerator on the right hand side is a linear combination of characters $\sigma_1(\gamma),\ldots,\sigma_n(\gamma)$ with coefficients that are independent of γ. Thus we may average (6) over the group by replacing $\omega_i(\gamma)$ by t^{m_i} wherever it occurs and this yields

$$(7)\qquad \frac{1}{g}\sum_{\gamma\in G}\sum_{i_1<\cdots<i_p}\frac{\omega_{i_1}(\gamma)\cdots\omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t)\cdots(1-\omega_{i_p}(\gamma)t)} = \sum_{i_1<\cdots<i_p}\frac{t^{m_{i_1}}\cdots t^{m_{i_p}}}{(1-t^{m_{i_1}+1})\cdots(1-t^{m_{i_p}+1})}$$

for all $p=1,\ldots,n$. The rational function on the right hand side of this formula has a pole of order p at $t=1$. Let us expand both sides in a series of powers of $1-t$ and equate the coefficients of $(1-t)^{-p}$. Let G_r be the set of elements of G that fix some $n-r$ dimensional subspace of V but fix no subspace of higher dimension, so that the number of elements in G_r is g_r. An element of G occurs in G_r if and only if the number 1 occurs as an eigenvalue with multiplicity $n-r$. Thus an element $\gamma\in G_r$ can contribute to the coefficient of $(1-t)^{-p}$ on the left hand side of (7) if and only if $n-r\geq p$ and in fact for $\gamma\in G_r$

$$\sum_{i_1<\cdots<i_p}\frac{\omega_{i_1}(\gamma)\cdots\omega_{i_p}(\gamma)}{(1-\omega_{i_1}(\gamma)t)\cdots(1-\omega_{i_p}(\gamma)t)} = \binom{n-r}{p}\frac{1}{(1-t)^p}+\cdots$$

Thus

$$\frac{1}{g}\sum_{r=0}^{n}\binom{n-r}{p}g_r = \sum_{i_1<\cdots<i_p}\frac{1}{(m_{i_1}+1)\cdots(m_{i_p}+1)}$$

But $g=(m_1+1)\cdots(m_n+1)$. As Shephard and Todd have remarked, this follows at once from (5) for the case $p=0$. Thus

$$\sum_{r=0}^{n}\binom{n-r}{p}g_r = \sum_{i_1<\cdots<i_{n-p}}(m_{i_1}+1)\cdots(m_{i_{n-p}}+1)$$

Let $G(t)=g_0+g_1t+\cdots+g_nt^n$. By direct computation we verify the formulas

$$\frac{d^p}{dt^p} t^n G\left(\frac{1}{t}\right)\Big|_{t=1} = \sum_{r=0}^{n} p!\binom{n-r}{p} g_r$$

$$\frac{d^p}{dt^p}(t+m_1)\cdots(t+m_n)\Big|_{t=1} = \sum_{i_1<\cdots<i_{n-p}} p!(m_{i_1}+1)\cdots(m_{i_{n-p}}+1)$$

Thus

$$t^n G\left(\frac{1}{t}\right) = (t+m_1)\cdots(t+m_n)$$

which amounts to

$$G(t) = (1+m_1 t)\cdots(1+m_n t)$$

This completes the proof.

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