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A CONSTRUCTION OF ASYMPTOTIC SOLUTIONS AND THE EXISTENCE OF SMOOTH NULL-SOLUTIONS FOR A CLASS OF NON-FUCHSIAN PARTIAL DIFFERENTIAL OPERATORS

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§1. Introduction

Consider a partial differential operator

$$P = \sum_{j+|\alpha| \le m} a_{j,\alpha}(t, x)\partial_t^j \partial_x^\alpha, \quad a_{m,0}(t, x) \equiv t^\kappa, \tag{1.1}$$

where κ is a non-negative integer and $a_{j,\alpha}$ are real-analytic in a neighborhood of $(0,0) \in \mathbf{R}_t \times \mathbf{R}_x^n$.

M. S. Baouendi and C. Goulaouic [1] defined *Fuchsian partial differential operators*, and proved the unique solvability of the characteristic Cauchy problems in the category of real-analytic (or holomorphic) functions, which is a generalization of the classical Cauchy-Kowalevsky theorem. They also proved a generalization of the Holmgren uniqueness theorem. Especially, from their results it easily follows that if P is a Fuchsian operator with real-analytic coefficients, then there exist no sufficiently smooth null-solutions. Here, a Schwartz distribution u in a neighborhood of (0,0) is called a *null-solution* for P at (0,0), if $Pu = 0$ in a neighborhood of (0,0) and $(0,0) \in \operatorname{supp} u \subset \{t \ge 0\}$, where $\operatorname{supp} u$ denotes the support of u.

The author considered the characteristic Cauchy problems for a class of operators wider than the Fuchsian operators in [3]. In that result, he showed the unique solvability of the characteristic Cauchy problems in the category of functions which are of class C^∞ with respect to t and real-analytic with respect to x. He also showed the non-existence of sufficiently smooth null-solutions. (As for

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distribution null-solutions, see [4]). This class of operators is defined in terms of four conditions. He gave a conjecture that if the third condition is violated, then there exists a C^∞ null-solution.

In this article, we construct an asymptotic solution of $Pu = 0$ in the form

$$u(t, x) := \exp\Big(- \sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\Big) \cdot t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x), \tag{1.2}$$

where

(i) M is a non-negative integer, and q is a positive integer,

(ii) $\mu[j]$ $(j = 0,1, \ldots, M)$ are positive rational numbers such that $\mu[0] > \mu[1] > \cdots > \mu[M] > 0$.

(iii) $\lambda[j]$ $(j = 0,1, \ldots, M+1)$ and $v_{l,p}$ $(l \geq 0 ; 0 \leq p \leq lm)$ are real-analytic in a fixed open neighborhood of $0 \in \mathbf{R}^n$,

for a class of operators wider than that considered in [3].

Further, using these asymptotic solutions, we prove the conjecture in [3] mentioned above under an additional assumption. The C^∞ null-solution constructed here is one of the most fastly decaying nontrivial solutions of $Pu = 0$.

In Section 2, we give the statements of the main theorems. After giving some preliminaries in Section 3, we prove the main theorems in Sections 4 and 5.

NOTATIONS:

(i) The set of all integers (resp. nonnegative integers) is denoted by $\mathbf{Z}$ (resp. $\mathbf{N}$). Put $\mathbf{N}/q := \{p/q : p \in \mathbf{N}\}$ for a positive integer q, and put $\mathbf{Z}/q$ similarly.

(ii) Put $\vartheta := t\partial_t$.

(iii) For a bounded domain Ω in $\mathbf{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω.

(iv) The space of the Schwartz distributions on U is denoted by $\mathcal{D}'(U)$.

(v) For a complete locally convex topological vector space E, put

$$C^N_{flat}([0, T] ; E) := \{f \in C^N([0,T] ; E) : \frac{d^j f}{dt^j}\Big|_{t=0} = 0 \text{ for } 0 \leq j \leq N-1\}.$$

(vi) Put $(\lambda)_j := \Pi_{l=0}^{j-1}(\lambda - l)$ for $\lambda \in \mathbf{C}$ and $j \in \mathbf{N}$.

(vii) For a commutative ring R, the ring of polynomials of λ with the coefficients belonging to R is denoted by $R[\lambda]$. The degree of $F \in R[\lambda]$ is denoted by $\deg_\lambda F$.

§2. Statement of the main result

Let q be a positive integer, Ω be a bounded domain in $\boldsymbol{C}^n$ that includes the origin 0, and T be a positive real number. Consider a linear partial differential operator of the form (1.1). We assume only the following weaker condition on the coefficients.

(A-0) $$a_{j,\alpha} \in \widehat{\mathscr{F}_q}([0, T]; \mathscr{O}(\Omega)) \quad (j + |\alpha| \leq m),$$

where

$$\mathscr{F}_q([0, T]; \mathscr{O}(\Omega)) := \{\phi \in C^\infty((0, T]; \mathscr{O}(\Omega)) : [s \mapsto \phi(s^q)] \in C^\infty([0, T^{1/q}]; \mathscr{O}(\Omega))\},$$

$$\widehat{\mathscr{F}_q}([0, T]; \mathscr{O}(\Omega)) := \{\phi \in C^\infty((0, T]; \mathscr{O}(\Omega)) : t^M \phi(t) \in \mathscr{F}_q([0, T]; \mathscr{O}(\Omega)) \text{ for some } M \in \mathbf{N}\}.$$

Let $r(j, \alpha)$ be the *generalized vanishing order* of $a_{j,\alpha}$ on the hypersurface $\Sigma := \{(0, x) : x \in \Omega\}$, that is

(2.1) $$r(j, \alpha) := \sup\{r \in \mathbf{Z}/q : t^{-r} a_{j,\alpha} \in \mathscr{F}_q([0, T]; \mathscr{O}(\Omega))\}.$$

If $r(j, \alpha) = \infty$, then we redefine $r(j, \alpha) := R$ for a sufficiently large R ($R := \max\{r(j, \alpha) : r(j, \alpha) < \infty\} + 1$ will suffice). Put

(2.2) $$\tilde{a}_{j,\alpha}(t, x) := t^{-r(j,\alpha)} a_{j,\alpha}(t, x) \quad (\in \mathscr{F}_q([0, T]; \mathscr{O}(\Omega))).$$

Note that if $r(j, \alpha) < R$, then $\tilde{a}_{j,\alpha}(0, x) \not\equiv 0$.

Associating a *weight* $\omega(j, \alpha) := j - r(j, \alpha)$ to each differential monomial $a_{j,\alpha}(t, x) \partial_t^j \partial_x^\alpha$, we draw a Newton polygon $\Delta(P)$ using the points $(j + |\alpha|, -\omega(j, \alpha))$ $(j + |\alpha| \leq m)$ in (u, v)-plane as follows.

Definition 2.1 ([3]). (1) Put

$$\Delta(P) := ch\Big(\bigcup_{j+|\alpha| \leq m} \{(u, v) \in \mathbf{R}^2 : u \leq j + |\alpha|, v \geq -\omega(j, \alpha)\}\Big),$$

where $ch(A)$ denotes the convex hull of A. This is called the *Newton polygon* of P.

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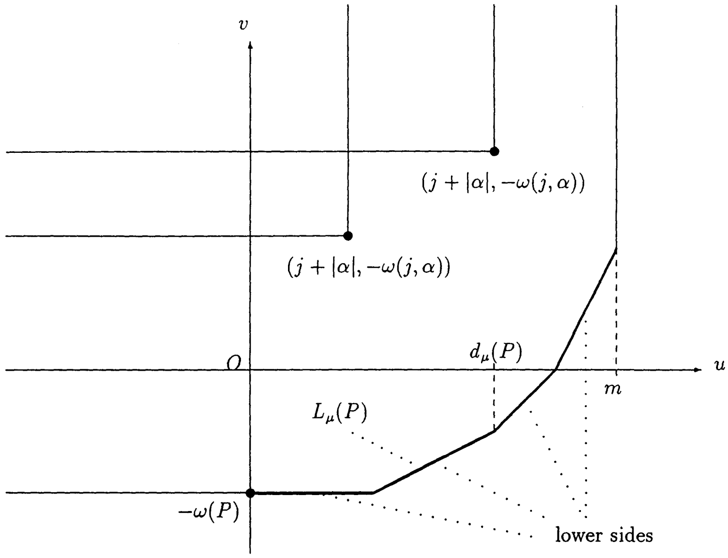

FIGURE 1. Newton polygon of $P : \Delta(P)$

(2) Put

$$\hat{V} = \hat{V}(P) := \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \text{ is a vertex of } \Delta(P)\}.$$

(3) Put

$$\omega = \omega(P) := \max\{\omega(j, \alpha) \in \mathbf{R} : j + |\alpha| \leq m\},$$

which is the *maximum weight* of P.
(4) The boundary of $\Delta(P) \cap ([0, \infty) \times \mathbf{R})$ is the union of two vertical half-lines and a finite number of compact line segments with distinct slopes. Each of these compact line segments is called a *lower side* of $\Delta(P)$. The set of the slopes of the lower sides of $\Delta(P)$ is denoted by $S = S(P) (\subset \mathbf{Q})$. For $\mu \in S(P)$, the lower side of $\Delta(P)$ with slope μ is denoted by $L_\mu = L_\mu(P)$. Put

$$I_\mu = I_\mu(P) := \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \in L_\mu(P)\}.$$

Let the right end points of $L_\mu(P)$ be (u_1, v_1). We put $d_\mu(P) := u_1$, and call it the *degree* of the slope μ.

If $0 \notin S$, we put $L_0(P) := \{(0, -\omega(0,0))\} = \{(0, -\omega(P))\} \subset \mathbf{R}^2$, $I_0(P) := \{(0,0)\} \subset \mathbf{N} \times \mathbf{N}^n$, and $d_0(P) := 0$.

By the use of these notions, Fuchsian operators in the sense of M. S. Baouendi and C. Goulaouic [1] are characterized as follows. (In fact, they assumed that the coefficients belong to $C^m([0, T]; \mathcal{O}(\Omega))$. This difference is, however, not essential and hence we ignore the difference of the classes of coefficients.)

PROPOSITION 2.2. *The operator* P *is Fuchsian if and only if* $\omega(P) \geq 0$, $S(P) = \{0\}$, *and there exist no* $(j, \alpha) \in I_0(P)$ *such that* $\alpha \neq 0$.

We consider a class of operators wider than the class of Fuchsian operators. First, we assume the following condition.

(A-1) For all $\mu \in S(P)$, there exist no $(j, \alpha) \in I_\mu(P)$ such that $\alpha \neq 0$.

DEFINITION 2.3. For $\mu \in S(P)$ with $\mu > 0$, we put

$$\mathscr{C}_\mu[P](x; \lambda) := \sum_{(j,0) \in I_\mu(P)} \tilde{a}_{j,0}(0, x)\lambda^j \in \mathcal{O}(\Omega)[\lambda].$$

We also put

$$\mathscr{C}_0[P](x; \lambda) := \sum_{(j,0) \in I_0(P)} \tilde{a}_{j,0}(0, x)(\lambda)_j \in \mathcal{O}(\Omega)[\lambda].$$

The polynomial $\mathscr{C}_\mu[P]$ of λ is called the *indicial polynomial of* P *associated with the slope* $\mu \in S(P) \cup \{0\}$. Note that $d_\mu(P) = \deg_\lambda \mathscr{C}_\mu[P]$.

For $\mu \in S(P) \cup \{0\}$, we consider the following condition.

(A-2; μ) If $(j, 0) \in \hat{V}(P)$ and $j \geq d_\mu(P)$, then $\tilde{a}_{j,0}(0,0) \neq 0$.

This is equivalent to the following.

(A-2; μ) For every $\nu \in S(P)$ with $\nu \geq \mu$, the coefficient of the top order term of $\mathscr{C}_\nu[P](x; \lambda) \in \mathcal{O}(\Omega)[\lambda]$ does not vanish at $x = 0$.

Remark 2.4. Note that if $(j, 0) \in \hat{V}(P)$, then $\tilde{a}_{j,0}(0, x) \not\equiv 0$. Thus, the condition (A-2; μ) is a kind of non-degeneracy at $x = 0$. Further, the condition (A-2; μ) for $\mu > 0$ is weaker than the condition (A-2; 0), and (A-2; 0) is equivalent to (A-2) in [3].

Now, the following is one of the three main theorems in this article.

THEOREM 2.5. *Assume that* P *satisfies* (A-0) *and* (A-1). *Let* $\mu_0 \in S(P) \cap \mathbf{N}/q$, $\mu_0 > 0$, *and assume the condition* (A-2; μ_0). *If* λ_0 *is a simple root of* $\mathscr{C}_{\mu_0}[P](0; \lambda) = 0$, *then there exist*

(i) $M \in \mathbf{N}$,

(ii) $\mu[j] \in \mathbf{N}/q (j = 0, 1, \ldots, M)$, *where* $\mu_0 = \mu[0] > \mu[1] > \cdots > \mu[M] > 0$,

(iii) *a subdomain* Ω_0 *of* Ω *including* 0,

(iv) $\lambda[j] \in \mathcal{O}(\Omega_0)$ $(j = 0, 1, \ldots, M+1)$, *where* $\lambda[0](0) = \lambda_0$,

such that the following holds.

For an arbitrarily given $v_{0,0}(x) \in \mathcal{O}(\Omega_0)$, *there exists* $v_{l,p}(x) \in \mathcal{O}(\Omega_0)$ $(l \geq 0; 0 \leq p \leq lm)$ *such that a formal series*

$$u(t, x) := \exp\Big(-\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\Big) \cdot t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x) \tag{2.3}$$

is an asymptotic solution of $Pu = 0$. *That is, for every* $N \in \mathbf{N}$ *there holds*

$$t^{-\lambda[M+1](x)} \cdot \exp\Big(\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\Big) \cdot P\Big(\exp\Big(-\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\Big) \times t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{N} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)\Big) = o(t^{N/q - r_0}), \tag{2.4}$$

with some $r_0 \in \mathbf{N}$.

This theorem shall be proved in Section 4. We shall also give a proposition which corresponds to the case of $\mu_0 = 0$ and $M = -1$.

Remark 2.6. Even if $\mu_0 \in S(P)$ but $\mu_0 \notin \mathbf{N}/q$, we can retake another q such that $\mu_0 \in \mathbf{N}/q$ and (A-0) is satisfied. Hence, we can always apply this theorem with this new q.

Next, we consider the following condition for $\mu \in S(P)$.

(A-6; μ) If $\nu \in S(P)$ and $\nu > \mu$, then all non-zero roots λ of $\mathscr{C}_\nu[P](0; \lambda) = 0$ satisfy $\mathrm{Re}\,\lambda < 0$. Further, there exists $\lambda_0 \in \mathbf{C}$ which satisfies the following.

(i) $\mathrm{Re}\,\lambda_0 > 0$,

(ii) λ_0 is a simple root of $\mathscr{C}_\mu[P](0; \lambda) = 0$ and the other roots λ satisfy $\mathrm{Re}\,\lambda < \mathrm{Re}\,\lambda_0$.

Remark 2.7. In this section, we define only the conditions (A-0), (A-1), (A-2; μ), and (A-6; μ). This apparently strange numbering is for the consistency with [3]. We shall introduce another condition (A-3) in Section 5.

Using the theorem above, we can show the existence theorem of smooth null-solutions, which is the second of the main theorems.

THEOREM 2.8. *Assume the conditions* (A-0), (A-1), (A-2; μ_0), *and* (A-6; μ_0) *for some* $\mu_0 \in S(P)$ *with* $\mu_0 > 0$. *Then,* P *has a* C^∞ *null-solution at* (0,0).

The C^∞ null-solution given in this theorem is one of the most fastly decaying nontrivial solutions as $t \to +0$. In fact, we have the following theorem, which is the last of the main theorems.

THEOREM 2.9. *Assume the conditions* (A-0), (A-1), (A-2; μ_0), *and* (A-6; μ_0) *for some* $\mu_0 \in S(P)$ *with* $\mu_0 > 0$. *Assume that* u *is a* C^0 *solution of* $Pu = 0$ *for* $t > 0$. *If there exist* $\delta > \mathrm{Re}\,\lambda_0$ *and* $C_0 > 0$ *such that the inequality*

$$|u(t, x)| \leq C_0 \exp\left(-\frac{\delta}{\mu_0} t^{-\mu_0}\right)$$

holds for $t > 0$ *in a neighborhood of* (0,0), *then* $u = 0$ for $t > 0$ *in a neighborhood of* (0,0).

Theorems 2.8 and 2.9 shall be proved in Section 5.

Finally, let us consider a typical example.

EXAMPLE 2.10. First, we consider the following ordinary differential operator decomposed into first order operators.

$$P_0 := t^d(t^{k_1}\vartheta - \lambda_1(t, x)) \cdots (t^{k_r}\vartheta - \lambda_r(t, x))(\partial_t - \tilde{\lambda}_{r+1}(t, x)) \cdots (\partial_t - \tilde{\lambda}_m(t, x)),$$

where $m, r, d \in \mathbf{N}$, $0 \leq r \leq m$, $k_j \in \mathbf{N}(1 \leq j \leq r)$ and $\lambda_j, \tilde{\lambda}_l \in C^\infty([0, T]; \mathcal{O}(\Omega))$ $(1 \leq j \leq r; r+1 \leq l \leq m)$. Assume that $\lambda_j(0, x) \not\equiv 0$ $(1 \leq j \leq r)$ and $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$. For this operator, $S(P_0) = \{k_1, \ldots, k_r, 0\}$ if $r < m$, and $S(P_0) = \{k_1, \ldots, k_m\}$ if $r = m$. The condition (A-1) is trivially satisfied, and the condition (A-2; μ) is " if $k_j > \mu$ then $\lambda_j(0, 0) \neq 0$". We can also show that

$$\mathscr{C}_\mu[P_0](x; \lambda) = \prod_{j:k_j>\mu} (-\lambda_j(0, x)) \cdot \prod_{j:k_j=\mu} (\lambda - \lambda_j(0, x)) \cdot \lambda^{h(\mu)+m-r}$$

for $\mu \in S(P_0)$ with $\mu > 0$, where $h(\mu)$ is the number of k_j's that satisfy $k_j < \mu$. Thus, the condition (A-6; μ_0) for $\mu_0 > 0$ is the following.

If $k_j > \mu_0$ then $\mathrm{Re}\,\lambda_j(0, 0) < 0$. Further, there exists j_0 such that
(i) $k_{j_0} = \mu_0$,
(ii) $\mathrm{Re}\,\lambda_{j_0}(0, 0) > 0$,
(iii) If $k_j = \mu_0$ and $j \neq j_0$, then $\mathrm{Re}\,\lambda_j(0, 0) < \mathrm{Re}\,\lambda_{j_0}(0, 0)$.

Next, we consider a partial differential operator. Put $\mu_j := 0$ $(1 \leq j \leq m-r)$ and $\mu_{m-r+j} := k_{r+1-j}$ $(1 \leq j \leq r)$. Also put $\omega_j := d + \sum_{l=1}^{j} \mu_l$ $(0 \leq j \leq m)$. Consider an operator

$$P = P_0 + \sum_{j=0}^{m} t^{\omega_j+1} B_j(t, x; \vartheta, \partial_x),$$

where $B_j(t, x; \vartheta, \partial_x) = \sum_{|\alpha| \le j} b_{j,\alpha}(t, x) \partial_x^\alpha \vartheta^{j-|\alpha|}$ and $b_{j,\alpha} \in C^\infty([0, T]; \mathscr{O}(\Omega))$. Then, P satisfies the condition (A-1), and there hold $\Delta(P) = \Delta(P_0)$, $S(P) = S(P_0)$, $\mathscr{C}_\mu[P] = \mathscr{C}_\mu[P_0]$. (See Lemma 3.1.) Hence, P satisfies the condition (A-2; μ_0) (resp. (A-6; μ_0)), if and only if P_0 satisfies (A-2; μ_0) (resp. (A-6; μ_0)).

§3. Preliminaries

In this section, we give some preliminaries for the proofs of the main theorems.

Let P be an operator (1.1) satisfying (A-0). By $t^j \partial_t^j = \vartheta(\vartheta - 1) \dots (\vartheta - j + 1) = (\vartheta)_j$, we can easily show the following lemma, which is useful in our arguments.

LEMMA 3.1. *We can rewrite P as*

$$P = \sum_{j+|\alpha| \le m} b_{j,\alpha}(t, x) \vartheta^j \partial_x^\alpha, \tag{3.1}$$

with $b_{j,\alpha} \in \widehat{\mathscr{F}_q}([0, T]; \mathscr{O}(\Omega))$. For this $b_{j,\alpha}$, we define the generalized vanishing order

$$r'(j, \alpha) := \sup\{r \in \mathbf{Z}/q : t^{-r} b_{j,\alpha} \in \mathscr{F}_q([0, T]; \mathscr{O}(\Omega))\}.$$

For $\mu \ge 0$, we put $\omega_\mu(P) := \max\{-r'(j, \alpha) + \mu(j + |\alpha|) : j + |\alpha| \le m\}$. Then, we have

$$\Delta(P) = ch\Big(\bigcup_{j+|\alpha| \le m} \{(u, v) \in \mathbf{R}^2 : u \le j + |\alpha|, v \ge r'(j, \alpha)\}\Big),$$
$$\hat{V}(P) = \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, r'(j, \alpha)) \text{ is a vertex of } \Delta(P)\},$$
$$\omega(P) = \max\{-r'(j, \alpha) \in \mathbf{R} : j + |\alpha| \le m\} = \omega_0(P),$$
$$I_\mu(P) = \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : -r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P)\}.$$

Further, the condition (A-1) *is stated as follows*:

(A-1) *For every $\mu \in S(P)$, if $-r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P)$, then $\alpha = 0$.*

Under (A-1), *there holds*

$$\mathscr{C}_\mu[P](x; \lambda) = \sum_{j=0}^{m} \{b_{j,0}(t, x) t^{\omega_\mu(P) - \mu j}\}|_{t=0} \lambda^j \tag{3.2}$$
$$= \begin{cases} [t^{\omega_\mu(P)} e^{\lambda t^{-\mu}/\mu} P(e^{-\lambda t^{-\mu}/\mu})]|_{t=0} & (\mu > 0), \\ [t^{\omega(P)} t^{-\lambda} P(t^\lambda)]|_{t=0} & (\mu = 0), \end{cases}$$

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and the condition (A-2; μ) *is stated as follows*:

(A-2; μ) *If* $(j, 0) \in \hat{V}(P)$ *and* $j \geq d_\mu(P)$, *then* $\{b_{j,0}(t, 0)t^{-r'(j,0)}\}|_{t=0} \neq 0$.

It is convenient to consider the operator in the form (3.1) rather than the form (1.1).

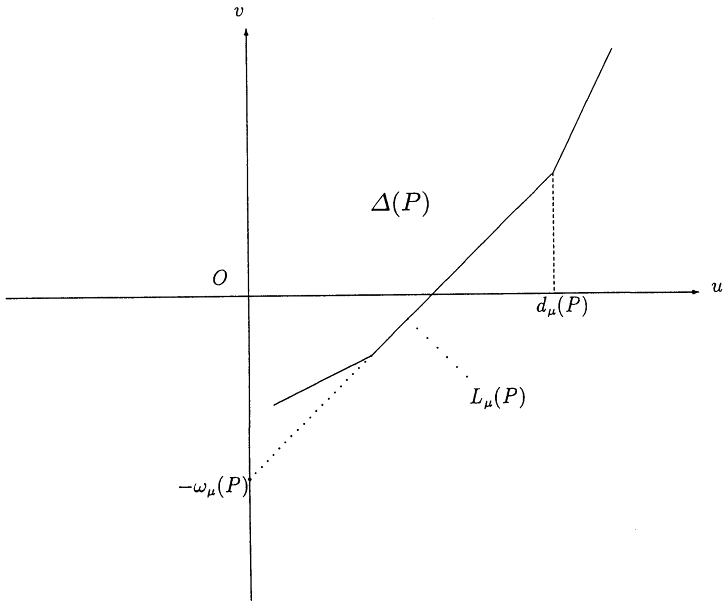


FIGURE 2. $\omega_\mu(P)$

Remark 3.2. For $\mu \geq 0$, we can define $\mathscr{C}_\mu[P]$ by (3.2), even if $\mu \notin S$. If $\mu \in S$ and $\mu > 0$, then $\mathscr{C}_\mu[P]$ has more than one term as a polynomial of λ. If $\mu \notin S$ and $\mu > 0$, then $\mathscr{C}_\mu[P]$ has only one term.

The key tool for the proofs of main thorems is the following type of transformation of operators.

LEMMA 3.3. *Assume that an operator* P *of the form* (1.1) (*or* (3.1)) *satisfies the conditions* (A-0) *and* (A-1). *Let* $\mu \in S(P) \cap \mathbf{N}/q$, $\mu > 0$, *and assume* (A-2; μ). *Let* λ_1 *be a simple root of* $\mathscr{C}_\mu[P](0; \lambda) = 0$. *Take a subdomain* Ω' *of* Ω *including* 0 *and* $\lambda(x) \in \mathscr{O}(\Omega')$ *so that they satisfy* $\lambda(0) = \lambda_1$ *and* $\mathscr{C}_\mu[P](x; \lambda(x)) \equiv 0$ *on* Ω'. *If we put*

$$P' := \exp\Big(\frac{\lambda(x)}{\mu}\, t^{-\mu}\Big) \circ P \circ \exp\Big(-\frac{\lambda(x)}{\mu}\, t^{-\mu}\Big),$$

then P' is an operator on $[0, T] \times \Omega'$ of the form (1.1) *and satisfies the following*:

(a) *The operator P' satisfies* (A-0) *and* (A-1).

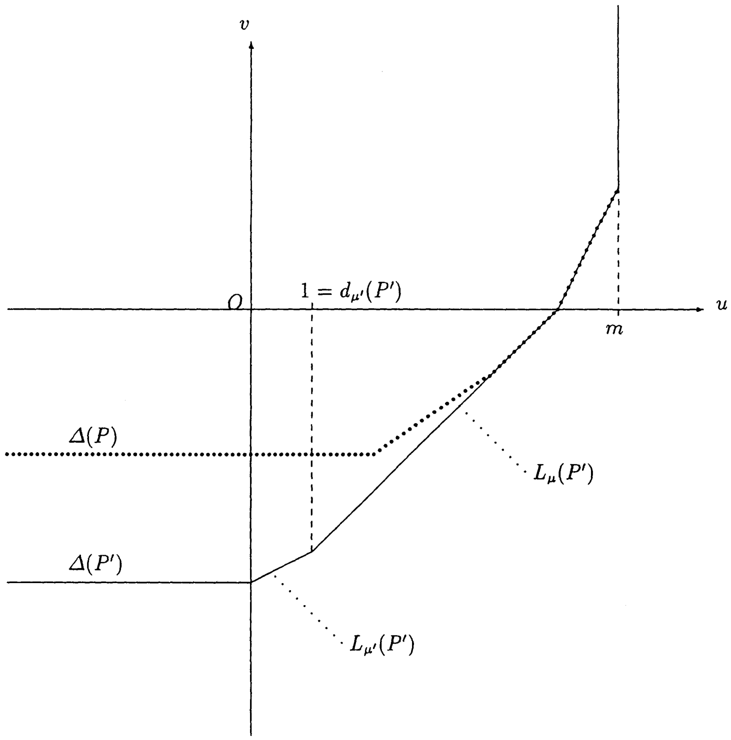
(b) $S(P') \cap (\mu, \infty) = S(P) \cap (\mu, \infty)$.

(c) $\mathscr{C}_\nu[P'](x\,;\cdot) = \mathscr{C}_\nu[P](x\,;\cdot)$ *for every $\nu > \mu$ and $x \in \Omega'$.*

(d) *There holds $\mathscr{C}_\mu[P'](x\,;\lambda) = \mathscr{C}_\mu[P](x\,;\lambda + \lambda(x))$. Further, if $d_\mu(P) > 1$, then $\mu \in S(P')$; if $d_\mu(P) = 1$, then $\mu \notin S(P')$.*

(e) *There exists $\mu' < \mu$ such that $\mu' \in \mathbf{N}/q$ and $S(P') \cap [0, \mu) = \{\mu'\}$.*

(f) $d_{\mu'}(P') = 1$ *and P' satisfies* (A-2; μ').



The upper part of the dotted line is $\Delta(P)$.
The upper part of the real line is $\Delta(P')$.
FIGURE 3. $\Delta(P')$ and $\Delta(P)$

Proof. First, note that

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$$\text{(3.3)}\qquad \begin{aligned} &\exp\Big(\frac{\lambda(x)}{\mu}t^{-\mu}\Big)\circ\vartheta\circ\exp\Big(-\frac{\lambda(x)}{\mu}t^{-\mu}\Big) = \vartheta + \lambda(x)t^{-\mu},\\ &\exp\Big(\frac{\lambda(x)}{\mu}t^{-\mu}\Big)\circ\partial_x\circ\exp\Big(-\frac{\lambda(x)}{\mu}t^{-\mu}\Big) = \partial_x + \frac{-\lambda_x(x)}{\mu}t^{-\mu}. \end{aligned}$$

From these, it is easy to see that P' is an operator of the form (3.1) and satisfies the conditions (A-0), (A-1), and (A-2; μ). It is also easy to see that there hold the conclusions (b), (c). Further, we have $\mathscr{C}_\mu[P'](x;\lambda) = \mathscr{C}_\mu[P](x;\lambda+\lambda(x))$. Since $\mathscr{C}_\mu[P'](x;0)\equiv 0$ and since $(\partial_\lambda\mathscr{C}_\mu[P'])(0;0) = (\partial_\lambda\mathscr{C}_\mu[P])(0;\lambda_1)\neq 0$, we have $(1,0,\dots,0)\in\hat{V}(P')$ $(\subset\mathbf{N}\times\mathbf{N}^n)$. Hence, if $d_\mu(P)>1$, then $\mu\in S(P')$; if $d_\mu(P)=1$, then $\mu\notin S(P')$. Further, there exists $\mu'\in\mathbf{N}/q$ such that $\mu'<\mu$, $S(P')\cap[0,\mu)=\{\mu'\}$, and $d_{\mu'}(P')=1$. The condition (A-2; μ) and the fact that $(\partial_\lambda\mathscr{C}_\mu[P'])(0;0)\neq 0$ imply (A-2; μ'). □

By an iterative use of this lemma, we have the following.

PROPOSITION 3.4. *Assume that P satisfies* (A-0) *and* (A-1). *Let $\mu_0\in S(P)\cap\mathbf{N}/q$, $\mu_0>0$, and assume* (A-2; μ_0). *Let λ_0 be a simple root of $\mathscr{C}_{\mu_0}[P](0;\lambda)=0$. Then, there exist*

(i) $M\in\mathbf{N}$,

(ii) $\mu[j]\in\mathbf{N}/q$ $(j=0,1,\dots,M)$, *where* $\mu_0=\mu[0]>\mu[1]>\cdots>\mu[M]>0$,

(iii) *a subdomain Ω_{M+1} of Ω including* 0,

(iv) $\lambda[j]\in\mathscr{O}(\Omega_{M+1})$ $(j=0,1,\dots,M)$, *where* $\lambda[0](0)=\lambda_0$,

such that the operator

$$P^{(M+1)} := \exp\Big(\sum_{j=0}^{M}\frac{\lambda[j](x)}{\mu[j]}t^{-\mu[j]}\Big)\circ P\circ\exp\Big(-\sum_{j=0}^{M}\frac{\lambda[j](x)}{\mu[j]}t^{-\mu[j]}\Big)$$

is an operator on $[0,T]\times\Omega_{M+1}$ *of the form* (1.1) *and satisfies the following*:

(a) *The operator $P^{(M+1)}$ satisfies* (A-0) *and* (A-1).

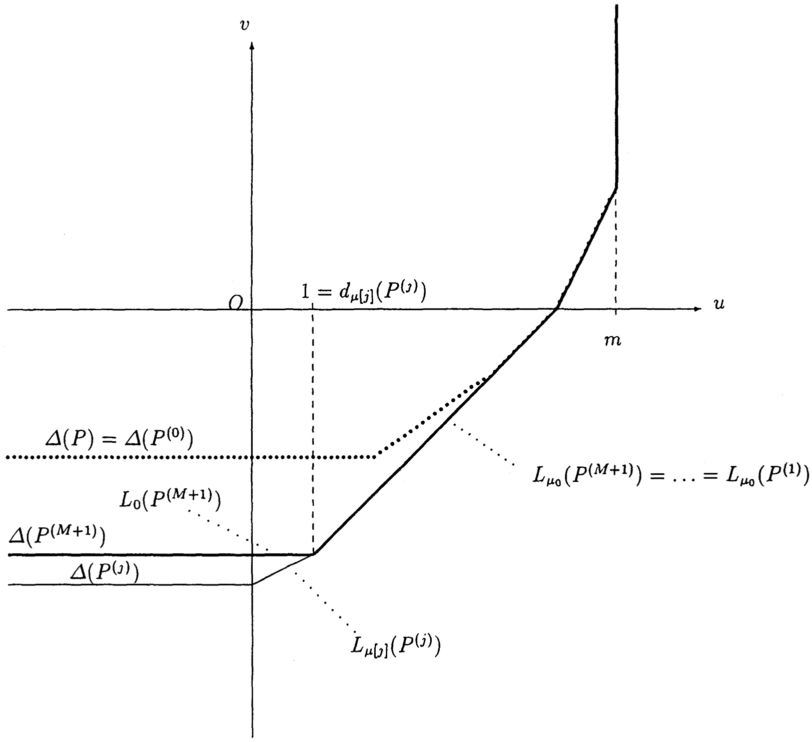
(b) $S(P^{(M+1)})\cap(\mu_0,\infty) = S(P)\cap(\mu_0,\infty)$.

(c) $\mathscr{C}_\nu[P^{(M+1)}](x;\cdot) = \mathscr{C}_\nu[P](x;\cdot)$ *for every* $\nu>\mu_0$ *and* $x\in\Omega_{M+1}$.

(d) *There holds* $\mathscr{C}_{\mu_0}[P^{(M+1)}](x;\lambda) = \mathscr{C}_{\mu_0}[P](x;\lambda+\lambda[0](x))$. *If* $d_{\mu_0}(P)>1$, *then* $\mu_0\in S(P^{(M+1)})$; *if* $d_{\mu_0}(P)=1$, *then* $\mu_0\notin S(P^{(M+1)})$.

(e) $S(P^{(M+1)})\cap[0,\mu_0)=\{0\}$.

(f) $d_0(P^{(M+1)})=1$ *and $P^{(M+1)}$ satisfies* (A-2; 0).



The upper part of the dotted line is $\Delta(P) = \Delta(P^{(0)})$.
The upper part of the real line is $\Delta(P^{(j)})$ $(1 \leq j \leq M)$.
The upper part of the bold real line is $\Delta(P^{(M+1)})$.

FIGURE 4. $\Delta(P) = \Delta(P^{(0)})$ and $\Delta(P^{(M+1)}) \subset \cdots \subset \Delta(P^{(1)})$

Proof. Since λ_0 is a simple root, we can take a subdomain Ω_1 of Ω including 0 and $\lambda[0](x) \in \mathcal{O}(\Omega_1)$ such that they satisfy $\lambda[0](0) = \lambda_0$ and $\mathscr{C}_{\mu_0}[P](x\,;\lambda[0](x)) \equiv 0$ on Ω_1.

Put $P^{(0)} := P$ and $\mu[0] := \mu_0$. If we put

$$P^{(1)} := \exp\Big(\frac{\lambda[0](x)}{\mu[0]}\,t^{-\mu[0]}\Big) \circ P^{(0)} \circ \exp\Big(-\frac{\lambda[0](x)}{\mu[0]}\,t^{-\mu[0]}\Big),$$

then by Lemma 3.3, the operator $P^{(1)}$ is also an operator of the form (1.1) on $[0, T] \times \Omega_1$ and satisfies the following:

(a) The operator $P^{(1)}$ satisfies (A-0) and (A-1).

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(b) $S(P^{(1)}) \cap (\mu[0], \infty) = S(P^{(0)}) \cap (\mu[0], \infty)$.

(c) $\mathscr{C}_\nu[P^{(1)}](x;\cdot) = \mathscr{C}_\nu[P^{(0)}](x;\cdot)$ for every $\nu > \mu[0]$ and $x \in \Omega_1$.

(d) There holds $\mathscr{C}_{\mu[0]}[P^{(1)}](x;\lambda) = \mathscr{C}_{\mu[0]}[P^{(0)}](x;\lambda + \lambda[0](x))$. If $d_{\mu[0]}(P^{(0)}) > 1$, then $\mu[0] \in S(P^{(1)})$; if $d_{\mu[0]}(P^{(0)}) = 1$, then $\mu[0] \notin S(P^{(1)})$.

(e) There exists $\mu[1] < \mu[0]$ such that $\mu[1] \in \mathbf{N}/q$ and $S(P^{(1)}) \cap [0, \mu[0]) = \{\mu[1]\}$.

(f) $d_{\mu[1]}(P^{(1)}) = 1$ and $P^{(1)}$ satisfies (A-2; $\mu[1]$).

By (f), we have $\mathscr{C}_{\mu[1]}[P^{(1)}](x;\lambda) = a[1](x)\lambda - b[1](x)$ for some $a[1], b[1] \in \mathscr{O}(\Omega_1)$ with $a[1](0) \neq 0$.

If $\mu[1] = 0$, then put $M = 0$. Consider the case when $\mu[1] > 0$. We can take a subdomain Ω_2 of Ω_1 including 0 such that $a[1](x) \neq 0$ on Ω_2, and hence we can take $\lambda[1] \in \mathscr{O}(\Omega_2)$ such that $\mathscr{C}_{\mu[1]}[P^{(1)}](x;\lambda[1](x)) \equiv 0$ on Ω_2.

If we put

$$P^{(2)} := \exp\Big(\frac{\lambda[1](x)}{\mu[1]} t^{-\mu[1]}\Big) \circ P^{(1)} \circ \exp\Big(-\frac{\lambda[1](x)}{\mu[1]} t^{-\mu[1]}\Big),$$

then by Lemma 3.3 and by $d_{\mu[1]}(P^{(1)}) = 1$, the operator $P^{(2)}$ is also an operator of the form (1.1) and satisfies the following:

(a) The operator $P^{(2)}$ satisfies (A-0) and (A-1).

(b) $S(P^{(2)}) \cap (\mu[1], \infty) = S(P^{(1)}) \cap (\mu[1], \infty)$.

(c) $\mathscr{C}_\nu[P^{(2)}](x;\cdot) = \mathscr{C}_\nu[P^{(1)}](x;\cdot)$ for every $\nu > \mu[1]$ and $x \in \Omega_2$.

(d) There holds $\mathscr{C}_{\mu[1]}[P^{(2)}](x;\lambda) = \mathscr{C}_{\mu[1]}[P^{(1)}](x;\lambda + \lambda[1](x)) = a[1](x)\lambda$, and $\mu[1] \notin S(P^{(2)})$.

(e) There exists $\mu[2] < \mu[1]$ such that $\mu[2] \in \mathbf{N}/q$ and $S(P^{(2)}) \cap [0, \mu[1]] = \{\mu[2]\}$.

(f) $d_{\mu[2]}(P^{(2)}) = 1$ and $P^{(2)}$ satisfies (A-2; $\mu[2]$).

We can continue this procedure unless $\mu[j] = 0$. Since $\mu[j] \in \mathbf{N}/q$ and $\mu[0] > \mu[1] > \cdots \geq 0$, we necessarily reach $\mu[M+1] = 0$. □

The following lemma is used to construct each term of infinite series in asymptotic solutions.

LEMMA 3.5. *Let $Q(x;\lambda) \in \mathscr{O}(\Omega)[\lambda]$ and $\Lambda \in \mathscr{O}(\Omega)$. Assume that $Q(x;\Lambda(x)) \neq 0$ on Ω. Then, we can solve the equation*

$$Q(x;\vartheta)v = t^{\Lambda(x)} \sum_{p=0}^{L} g_p(x)(\log t)^p, \quad g_p \in \mathscr{O}(\Omega)\ (0 \leq p \leq L) \tag{3.4}$$

as $v = t^{\Lambda(x)} \sum_{p=0}^{L} v_p(x)(\log t)^p, \quad v_p \in \mathcal{O}(\Omega)\ (0 \le p \le L)$.

Proof. By an easy calculation, we have

$$Q(x\,;\vartheta)(t^{\Lambda(x)}(\log t)^p) = \sum_{j=0}^{p} \binom{p}{j} (\partial_\lambda^j Q)(x\,;\Lambda(x)) \cdot t^{\Lambda(x)}(\log t)^{p-j}.$$

Hence, (3.4) is equivalent to

$$Q(x\,;\Lambda(x)) \cdot v_p(x) + \sum_{j=1}^{L-p} \binom{p+j}{j} (\partial_\lambda^j Q)(x\,;\Lambda(x)) \cdot v_{p+j}(x) = g_p(x) \quad (p = 0,1,\dots, L).$$

Thus, by $Q(x\,;\Lambda(x)) \neq 0$, we can uniquely determine $v_L, v_{L-1},\dots, v_0$. □

§4. Proof of Theorem 2.5

In this section, we prove Theorem 2.5. First, we give the existence of an asymptotic solution with no exponential factor, which corresponds to the case $\mu_0 = 0$ and $M = -1$ in Theorem 2.5. Although we use only the case when $\deg_\lambda \mathscr{C}_0[P] = 1$ in the proof of main theorems, this proposition has its own value.

PROPOSITION 4.1. *Assume that* P *satisfies* (A-0), (A-1), *and* (A-2; 0). *Let* $\lambda(x) \in \mathcal{O}(\Omega_0)$ *satisfy*

(i) $\mathscr{C}_0[P](x\,;\lambda(x)) \equiv 0$ *on* Ω_0,
(ii) $\mathscr{C}_0[P](x\,;\lambda(x) + l/q) \neq 0$ *on* Ω_0 *for* $l \in \mathbf{N} \setminus \{0\}$,

for some subdomain Ω_0 *of* Ω *including* 0. *Then, for an arbitrarily given* $v_{0,0}(x) \in \mathcal{O}(\Omega_0)$, *there exist* $v_{l,p}(x) \in \mathcal{O}(\Omega_0)$ $(l \ge 0; 0 \le p \le lm)$ *such that*

$$u(t, x) := t^{\lambda(x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x) \tag{4.1}$$

is an asymptotic solution of $Pu = 0$. *That is*

$$t^{-\lambda(x)} P\Big(t^{\lambda(x)} \cdot \sum_{l=0}^{N} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)\Big) = o(t^{N/q-\omega(P)}),$$

for every $N \in \mathbf{N}$.

Proof. We can formally expand P with respect to t as

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$$P = t^{-\omega}\Big(\mathscr{C}_0[P](x;\vartheta) + \sum_{h=1}^{\infty} B_h(x, \partial_x; \vartheta)\, t^{h/q}\Big),$$

where $B_h(x, \partial_x; \vartheta) = \sum_{j+|\alpha| \le m} b_{h,j,\alpha}(x)\, \partial_x^{\alpha} \vartheta^j$ with $b_{h,j,\alpha} \in \mathscr{O}(\Omega)$ and $\omega := \omega(P)$. Hence, we have only to find $v_{l,p}$ that satisfy

$$\begin{aligned}&\mathscr{C}_0[P](x;\vartheta)\Big(t^{\lambda(x)+l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)\Big) \\ &\quad = -\sum_{h=0}^{l-1} B_{l-h}(x, \partial_x; \vartheta)\Big(t^{\lambda(x)+l/q} \sum_{p=0}^{hm} (\log t)^p v_{h,p}(x)\Big) \quad (l \in \mathbf{N}).\end{aligned}$$

Since

$$\mathscr{C}_0[P](x;\vartheta)(t^{\lambda(x)} v_{0,0}(x)) = \mathscr{C}_0[P](x;\lambda(x)) \cdot t^{\lambda(x)} v_{0,0}(x) \equiv 0,$$

$$\begin{aligned}&\partial_x(t^{\lambda(x)+l/q}(\log t)^p v(x)) \\ &\quad = t^{\lambda(x)+l/q}(\log t)^p (\partial_x v)(x) + t^{\lambda(x)+l/q}(\log t)^{p+1}(\partial_x \lambda)(x) v(x),\end{aligned}$$

and since

$$\mathscr{C}_0[P](x;\lambda(x) + l/q) \ne 0 \quad (l \ge 1) \quad \text{on } \Omega_0,$$

we can get $v_{l,p}$ with an arbitrarily given $v_{0,0}$ by applying Lemma 3.5. □

Proof of Theorem 2.5. We can apply Proposition 3.4 to P. By (f) of the proposition, we have

$$\mathscr{C}_0[P^{(M+1)}](x;\lambda) = a[M+1](x)\lambda - b[M+1](x)$$

for some $a[M+1]$, $b[M+1] \in \mathscr{O}(\Omega_{M+1})$ with $a[M+1](0) \ne 0$. Hence, we can take a subdomain Ω_0 of Ω_{M+1} including 0 such that $a[M+1](x) \ne 0$ on Ω_0. We can take $\lambda[M+1] \in \mathscr{O}(\Omega_0)$ such that $\mathscr{C}_0[P^{(M+1)}](x;\lambda[M+1](x)) \equiv 0$ and $\mathscr{C}_0[P^{(M+1)}](x;\lambda[M+1](x) + l/q) \ne 0$ on Ω_0 for $l \in \mathbf{N} \setminus \{0\}$.

By applying Proposition 4.1 to $P^{(M+1)}$, we can construct an asymptotic solution

$$v = t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x) \tag{4.2}$$

of $P^{(M+1)}v = 0$ for an arbitrarily given $v_{0,0} \in \mathscr{O}(\Omega_0)$.

Thus, the proof of Theorem 2.5 is completed. □

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§5. Proof of Theorems 2.8 and 2.9

In this section, we prove Theorems 2.8 and 2.9.

First, we introduce another condition (A-3).

(A-3) If $\mu \in S(P)$ and $\mu > 0$, then all the non-zero roots λ of $\mathscr{C}_\mu[P](0;\lambda) = 0$ satisfy $\operatorname{Re}\lambda < 0$.

From the results in [3], we easily get the following theorem, which shall be used later.

THEOREM 5.1. *Assume the conditions* (A-0), (A-1), (A-2; 0), *and* (A-3). *Then, there exist $N_0 \in \mathbf{N}$, $T_0 > 0$, and a domain Ω_0 including* 0 *for which the following holds*:

(1) *For every $N \geq N_0$ and every $f \in C_{flat}^{N-\omega(P)}([0, T]; \mathcal{O}(\Omega))$, there exists a unique $u \in C_{flat}^{N}([0, T_0]; \mathcal{O}(\Omega_0))$ such that $Pu = f$ on $[0, T_0] \times \Omega_0$.*

(2) *If $u \in t^{N_0} \times C^0([0, T]; \mathscr{D}'(\Omega \cap \mathbf{R}^n))$ and $Pu = 0$ for $t > 0$ in a neighborhood of* (0,0), *then $u = 0$ for $t > 0$ in a neighborhood of* (0,0). *Especially, there exists no sufficiently smooth null-solution for P at* (0,0).

In (2) of this theorem, the domain where $u = 0$ may depend not only on the domain where $Pu = 0$ but also on u itself. As for solutions in $C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$, however, we can show the existence of a common domain of uniqueness, by a standard argument as follows.

COROLLARY 5.2. *Assume the same assumptions as in the theorem above. Then there exists $N_0 \in \mathbf{N}$ such that for every $T' \in (0, T)$ and every open neighborhood U' of $0 \in \mathbf{R}^n$, there exist $T'' \in (0, T')$ and an open neighborhood U'' of* 0 *for which the following holds. If $u \in t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$ and $Pu = 0$ on $(0, T') \times U'$, then $u = 0$ on $(0, T'') \times U''$.*

Proof. Put $K := \{u \in t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n)) : Pu = 0 \text{ on } (0, T') \times U'\}$. This is a closed subspace of a Fréchet space $t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$, and hence it is also a Fréchet space. Let $\{T_n\}_{n\in N}$ be a decreasing sequence of positive real numbers converging to 0 and let $\{U_n\}_{n\in N}$ be a fundamental system of open neighborhoods of 0. Put $L_n := \{u \in K : u = 0 \text{ on } (0, T_n) \times U_n\}$, which are closed subspaces of K. By Theorem 5.1-(2), there holds $K = \cup_{n=0}^{\infty} L_n$. Since a Fréchet space is a Baire space, there exists an n such that L_n has an inner point, that is $L_n = K$. □

Now, we give a proof of Theorem 2.8.

Proof of Theorem 2.8. We may assume that $\mu_0 \in \mathbf{N}/q$ without loss of generality, and we can apply Proposition 3.4 to P. The operator $P^{(M+1)}$ satisfies (A-0), (A-1) and (A-2; 0). By the assumption (A-6; μ_0) for P and by the conditions (c), (d), (e) in Proposition 3.4, the operator $P^{(M+1)}$ satisfies (A-3). Further, as we have shown in the proof of Theorem 2.5, the operator $P^{(M+1)}$ has a formal solution (4.2) with $v_{0,0} \equiv 1$.

If we put

$$v_N := t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{qN} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)$$

and $g_N := P^{(M+1)}(v_N)$ for sufficiently large $N \in \mathbf{N}$, then we have

$$g_N \in C_{flat}^{N-r_0}([0, T]; \mathscr{O}(\Omega_0)),$$

where Ω_0 is a subdomain of Ω including 0 and $r_0 \in \mathbf{N}$, both independent of N. By Theorem 5.1, we get $w_N \in C_{flat}^{N+\omega(P^{(M+1)})-r_0}([0, T_0]; \mathscr{O}(\Omega_0'))$ such that $P^{(M+1)}(w_N) = -g_N$, where $T_0 > 0$ and Ω_0' is a subdomain of Ω_0 including 0. Thus, $v := v_N + w_N$ satisfies $P^{(M+1)}(v) = 0$ and $t^{-\lambda[M+1](x)} v(t, x) \to 1 (t \to +0)$. Note that Corollary 5.2 implies that v is independent of N for sufficiently large N in a neighborhood of (0,0).

Since $\mathrm{Re}\,\lambda[0](0) > 0$ by the assumption, we can easily show that

$$u(t, x) := \exp\left(-\sum_{j=0}^{M} \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot v(t, x)$$

belongs to $C_{flat}^{\infty}([0, T_0]; \mathscr{O}(\Omega_0'))$. Thus, u is a C^{∞} null-solution for P. □

Next, we give a proof of Theorem 2.9.

Proof of Theorem 2.9. If we take δ' as $\delta > \delta' > \mathrm{Re}\lambda_0$, and if we put $v := \exp(\delta' t^{-\mu_0}/\mu_0) u$, then we have $v \in t^N \times C^0([0, T]; \mathscr{D}'(\Omega_0 \cap \mathbf{R}^n))$ for every $N \in \mathbf{N}$ with some domain Ω_0 and $T > 0$. We also have

$$0 = P\left(\exp\left(-\frac{\delta'}{\mu_0} t^{-\mu_0}\right) v\right) = \exp\left(-\frac{\delta'}{\mu_0} t^{-\mu_0}\right) \tilde{P} v,$$

that is, $\tilde{P} v = 0$, where $\tilde{P} := \exp(\delta' t^{-\mu_0}/\mu_0) \circ P \circ \exp(-\delta' t^{-\mu_0}/\mu_0)$. We have only to show that $v = 0$ for $t > 0$ in a neighborhood of (0,0).

By an argument similar to and easier than that in the proof of Lemma 3.3, the operator $\tilde{P}$ is an operator of the form (1.1) and satisfies the following:

(a) The operator $\tilde{P}$ satisfies (A-0), (A-1), and (A-2; μ_0).
(b) $S(\tilde{P}) \cap (\mu_0, \infty) = S(P) \cap (\mu_0, \infty)$.
(c) $\mathscr{C}_\nu[\tilde{P}](x\,;\cdot) = \mathscr{C}_\nu[P](x\,;\cdot)$ for every $\nu > \mu_0$ and $x \in \Omega_0$.
(d) $\mathscr{C}_{\mu_0}[\tilde{P}](x\,;\lambda) = \mathscr{C}_{\mu_0}[P](x\,;\lambda + \delta')$.
(e) $S(\tilde{P}) \cap [0, \mu_0] = \{\mu_0\}$.

By (d) and the condition (A-6; μ_0) for P, all the roots λ of $\mathscr{C}_{\mu_0}[\tilde{P}](0\,;\lambda) = 0$ satisfy $\mathrm{Re}\,\lambda < 0$. This and the conditions (c), (e) imply that the operator $\tilde{P}$ satisfies (A-3). Further, also by (d), we have $\mathscr{C}_{\mu_0}[\tilde{P}](0\,;0) \neq 0$. This and the assumption (A-2; μ_0) imply (A-2; 0). Thus, we can apply Theorem 5.1 to $\tilde{P}$, and hence, we have $v = 0$ for $t > 0$ in a neighborhood of (0,0). □

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