SOME ELEMENTARY CONVERSE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS*

D. E. Seminar

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1. Introduction. In studying differential equations, the usual task is to determine properties of the solutions of such equations from a knowledge of the coefficient functions. The converse question, namely, of determining the coefficient functions from properties of solutions, also has significance. It has been studied especially in the case of Sturm-Liouville equations.

A discussion of the inverse Sturm-Liouville problem can be found in [8, Chapter 8], where references are given to the work of W.A. Ambarzumiam, G. Borg, I.M. Gelfand, M.G. Krein, B.M. Levitan, N. Levinson and W.A. Marchenko on this problem. Work of a quite different character, but dealing also with questions of a converse type arising from Sturm-Liouville equations, has been done by O. Borůvka and his colleagues and students [2].

Here we are concerned with far more elementary considerations than the foregoing. The problems discussed arise from [6] and [7]. There it was necessary to prove [6, § 9] that two linearly independent solutions, $y_1(x)$, $y_2(x)$, of a Sturm-Liouville equation $y'' + f(x)y = 0$ could have $y_1^2(x) + y_2^2(x)$ equal to a constant only if $f(x)$ were also constant** . The proof given [6, p. 72] was based on the general method used throughout that paper and was quite brief.

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*The results arose from discussions in a seminar on differential equations at Aarhus University, Denmark, during 1964-65. The participants were Jytte Bretlau, Villy K. Christensen, Jens Jørgen Holst, Margrethe Jørgensen, Tove Lund Jørgensen, Karen Skov Larsen, Lee Lorch, Niels Wendell Pedersen, Per Amdal Steffensen, Leif Hautop Sørensen, and Preben Dahl Vestergaard.

**In [6, p. 72] it is stated that if $y_1^2 + y_2^2$ equals a (non-zero) constant χ, then $f(x) = \chi^{-2}$. Actually, it should be said instead that $f(x) = (W/\chi)^2$, where $W = y_1 y_2' - y_2 y_1'$, the Wronskian of y_1, y_2, is a constant [3, p. 16]. See § 2, Remark 3.

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A still simpler proof can be based on a different approach, namely Appell's differential equation [1; 9, p. 298, example 10]* . This differential equation is satisfied, under certain conditions (see Theorem 2 below), by $z(x) = Ay_1^2 + By_1y_2 + Cy_2^2$, where A, B, C are arbitrary constants. These conditions are satisfied trivially in the applications of the result involved to [6; 7], so that the Appell equation can be used to show in an obvious fashion that $f(x)$ is constant when $z(x)$ is constant (see § 2, Remark 2 following Theorem 2).

This approach suggests the additional problems which are investigated here, and in which we consider partly the general second order linear homogeneous differential equation rather than only the Sturm-Liouville equation.

We establish first (§2) circumstances under which the Appell equation is valid (Appell assumed, apparently, that all derivatives used exist; less stringent hypotheses suffice), and apply this equation to the case of constant z. The case of polynomial z is discussed for Sturm-Liouville equations in § 3. For quite general z, further converse questions are considered in § 4.

2. The Appell differential equation. Consider the differential equation

$$y'' + p_1y' + p_2y = 0, \tag{1}$$

in an interval I, where the real functions $p_1(x)$ and $p_2(x)$ are continuous. For A, B and C arbitrary (real) numbers, and y_1, y_2 solutions of (1), we define

$$z(x) = Ay_1^2 + By_1y_2 + Cy_2^2, \tag{2}$$

and get our first remark:

THEOREM 1. *In the notation of* (1) *and* (2) *we have*

$$p_2z = A(y_1')^2 + By_1'y_2' + C(y_2')^2 - \tfrac{1}{2}p_1z' - \tfrac{1}{2}z'' \tag{3}$$

and

$$p_2z^2 = \tfrac{1}{4}(z')^2 - \tfrac{1}{2}p_1zz' - \tfrac{1}{2}zz'' - \tfrac{1}{4}(B^2 - 4AC)W^2, \tag{4}$$

* We became aware of the Appell equation from a reference to it by P. Hartman [4, p. 182].

where W is the Wronskian of y_1 and y_2.

Proof. Clearly, z'' exists. Computing it from (2) and replacing y_1'' and y_2'' by means of (1), we get (3). Furthermore, direct calculation shows that

$$4z[A(y_1')^2 + By_1'y_2' + C(y_2')^2] = (z')^2 - (B^2 - 4AC)W^2,$$

and, using (3) we get (4).

Remarks. 1. The function $p_2(x)$ is completely determined by $z(x)$ and $p_1(x)$, except possibly for the term $-\frac{1}{4}(B^2 - 4AC)(W/z)^2$, provided $z(x)$ is not identically zero.

2. In a Sturm-Liouville equation (i.e., where $p_1(x) \equiv 0$ for $x \in I$), $p_2(x)$ is completely determined, except for a constant, by $z(x)$, provided $z(x)$ is not identically zero.

3. When $z(x) \equiv 0$, nothing can be inferred concerning $p_2(x)$.

These remarks are obvious from (4), except for values of x for which $z(x) = 0$. Suppose that $z(\xi) = 0$, $\xi \in I$. It will be shown that ξ is not a limit-point of zeros of $z(x)$. Suppose it were. Then $z(\xi) = z'(\xi) = 0$ and, from (4), $(B^2 - 4AC)W^2 = 0$. Thus, either (i) $B^2 - 4AC = 0$, or (ii) $W = 0$.

In case (i), where $B^2 - 4AC = 0$, $z(x)$ is a perfect square, say $(\alpha_1 y_1 + \alpha_2 y_2)^2$, and so equals the square of a solution $Y(x)$ of (1). Clearly, the zeros of $Y(x)$ coincide with those of $z(x)$ so that ξ is a limit-point also of zeros of $Y(x)$, and so $Y(\xi) = Y'(\xi) = 0$. Hence $Y(x)$, as a solution of (1), is identically zero [3, p. 13, § 6], and so also is $z(x)$, contrary to the hypothesis.

In case (ii), where $W = 0$, the solutions $y_1(x)$, $y_2(x)$ of (1) are linearly dependent. Hence $z(x)$ is a multiple of the square of one of them. Thus it follows, as in case (i), that ξ is an isolated zero of $z(x)$.

Knowing now that any zero, ξ, in I, of $z(x)$ is isolated, we observe that $p_2(x)$ is determined by $z(x)$ (up to a constant) for all x sufficiently close to ξ, $x \neq \xi$. Thus, $p_2(\xi)$, being equal to $\lim_{x \to \xi} p_2(x)$, is determined as well.

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From (4) we obtain:

COROLLARY 1. (a) *If* $p_2'(x)$ *and* $\{p_1(x)z'\}'$ *exist, then* $z'''(x)$ *exists.*

(b) *If* $p_2'(x)$ *and* $z'''(x)$ *exist, then* $\{p_1(x)z'\}'$ *exists.*

(c) *At points where* $z(x) \neq 0$, *the existence of* $\{p_1(x)z'\}'$ *and of* $z'''(x)$ *jointly imply the existence of* $p_2'(x)$.

In the important special case of Sturm-Liouville equations, where $p_1(x) \equiv 0$, $x \in I$, Part (b) of Corollary 1 becomes vacuous. Parts (a) and (c) become

COROLLARY 2. *In a Sturm-Liouville equation, the existence of* $p_2'(x)$ *implies that of* $z'''(x)$ *and, when* $z(x) \neq 0$, *conversely.*

It is natural to ask if the requirement $z(x) \neq 0$ can be eliminated from Part (c) of Corollary 1 and from the converse part of Corollary 2. The answer is *no*:

Example. The differential equation

$$y'' - \tfrac{1}{4}\{25x^3 + |x|^{\frac{1}{2}}\} y = 0, \quad -\infty < x < +\infty,$$

in which $p_2'(0)$ does not exist, has linearly independent solutions

$$y_1(x) = \exp\left\{|x|^{\frac{5}{2}}\right\}, \quad y_2(x) = y_1(x) \int_0^x \left[\exp\left\{-2|t|^{\frac{5}{2}}\right\}\right] dt.$$

If we put $z(x) = y_2^2(x)$, we see that $z'''(x)$ and $\{p_1(x)z'\}'$ both exist for all x, so that the failure of $p_2'(0)$ to exist must be blamed on the fact that $z(0) = 0$.

COROLLARY 3. *If the function* $z(x)$ *is a non-zero constant, say* χ, *then*

$$p_2(x) = -\tfrac{1}{4}\chi^{-2}(B^2 - 4AC)W^2(x_0)\exp\left\{-2\int_{x_0}^{x} p_1(t)dt\right\}, \quad x_0 \in I.$$

In particular, for the Sturm-Liouville equation where $p_1(x) \equiv 0$, $x \in I$,

$$p_2(x) = -\tfrac{1}{4}\ \chi^{-2}(B^2 - 4AC)W^2 = \text{constant}.$$

Next follow conditions under which the Appell differential equation is valid:

THEOREM 2. *If* $p_2'(x)$ *exists and if either* $(p_1 z')'$ *or* z''' *exists, then* $z(x)$ *satisfies the Appell equation in the form*

$$(5) \qquad z''' + 2p_1 z'' + (p_1 z')' + (2p_1^2 + 4p_2)z' + (2p_2' + 4p_1 p_2)z = 0 .$$

If, in addition, $p_1'(x)$ *exists, then* (5) *may be written in the more usual form*

$$(6) \qquad z''' + 3p_1 z'' + (p_1' + 2p_1^2 + 4p_2)z' + (2p_2' + 4p_1 p_2)z = 0.$$

Proof. From Corollary 3 it follows that, if either $(p_1 z')'$ or z''' exists, so too must the other.

Differentiating (3), then replacing y_1'', y_2'' from (1) gives

$$z''' + 2p_2[2Ay_1 y_1' + B(y_1 y_2)' + 2Cy_2 y_2'] + 4p_1[A(y_1')^2 + By_1' y_2' + C(y_2')^2] + (p_1 z_1')' + 2p_2' z + 2p_2 z' = 0.$$

The first bracket equals z'. The second is seen, using (3), to be $\tfrac{1}{2}z'' + \tfrac{1}{2}p_1 z' + p_2 z$. These substitutions made, (5) and (6) follow.

Remarks 1. For the Sturm-Liouville differential equation $y'' + p_2(x)y = 0$, Corollary 3 shows that if $z(x)$ is a constant $\neq 0$, then so too is $p_2(x)$. This provides an alternative proof of the result in [6, §9, p. 72], which treats the special case of $z(x)$ in which $A = C = 1$, $B = 0$.

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2. For the cases to which the result of [6, § 9] is actually applied in [6, 7], a very simple direct proof of that result can be given *via* Appell's equation, since, in these cases, $p_2'(x) = f'(x)$ is known *a priori* to exist. Direct calculation then shows that $z(x) = y_1^2(x) + y_2^2(x)$ satisfies (6). With $z(x)$ constant $\neq 0$ and $p_1(x) \equiv 0$, it is then obvious that $p_2'(x) \equiv 0$, and the proof is complete.

3. The value of the constant c, to which $p_2(x)$ of the previous remark is equal, is clearly seen by Corollary 3 to be $(W/\chi)^2$ (cf. footnote **). This constant value of $p_2(x)$ is positive. But $p_2(x)$ could also be either (identically) zero or a negative constant, depending on the sign of the discriminant $B^2 - 4AC$. If $z(x) = y_1^2$ = constant, then $p_2(x) \equiv 0$. If $z(x) = y_1 y_2$ = constant, then $p_2(x)$ is a negative constant.

3. *The Sturm-Liouville equation with polynomial z.* Here we consider the special case of (1) with $p_1(x) \equiv 0$ and determine the form of $p_2(x)$ when $z(x)$ is a prescribed polynomial. This is a natural generalization of the case in which $z(x)$ is constant, discussed in § 2.

THEOREM 3. *Suppose that* $p_1(x) \equiv 0$, $x \in I$, *and that* z *is a polynomial, i.e.*, $z(x) = a_o + a_1 x + \ldots + a_n x^n$, *where* $a_o^2 + \ldots + a_n^2 > 0$. *Then*

$$p_2(x) = z^{-2}[d + c_o x + \ldots + c_{2n-3} x^{2n-2}] \tag{7}$$

for those x *for which* $z(x) \neq 0$, *where the constant* d *has the value* $-\frac{1}{4}(B^2 - 4AC)W^2 + \frac{1}{4} a_1^2 - \frac{1}{2} a_o a_2$, *and*

$$c_p = -\tfrac{1}{2}(p+1)^{-1} \sum_{i=0}^{p} (p-i+3)(p-i+2)(p-i+1) a_i a_{p-i+3},$$

where $a_i = 0$ *for* $i > n$, $p = 0, \ldots, 2n-3$.

Proof. This is a straightforward consequence of Theorem 1. The coefficients c_p can be determined by noting

$$d + c_o x + \ldots + c_{2n-3} x^{2n-2} = -\tfrac{1}{4}(B^2 - 4AC)W^2 - \tfrac{1}{2} z z'' + \tfrac{1}{4}(z')^2.$$

Differentiating both sides gives

$$c_o + 2c_1 x + \ldots + (2n-2)c_{2n-3}x^{2n-3} = -\tfrac{1}{2}zz'''$$

$$= -\tfrac{1}{2}[a_o + a_1 x + \ldots + a_n x^n][6a_3 + \ldots + n(n-1)(n-2)a_n x^{n-3}]$$

and c_p can now be obtained by equating coefficients, following (Cauchy) multiplication of the last member.

Remarks. 1. If $z(x)$ is quadratic or less, i.e., if $a_3 = a_4 = \ldots = a_n = 0$, then $p_2(z) = dz^{-2}$, i.e., $c_o = c_1 = \ldots = c_{2n-3} = 0$. This shows again that $p_2(z)$ is a constant when $z(x)$ is.

2. For $z(x)$ of degree n, with n at least 3, then the numerator of $p_2(x)$ is of degree $2n-2$. In fact, $c_{2n-3} = -\tfrac{1}{2}n(n-1)(n-2)a_n^2 \neq 0$.

4. Possible forms of z. Here we consider the relations between solutions of (1) and (6) and note that $z(x)$ can range over the entire class of non-vanishing, twice differentiable functions. But first a lemma is needed; it can be verified by direct calculation.

LEMMA 1. *Let $W(q_1, \ldots, q_n; x)$ denote, as usual, the Wronskian of the n functions $q_1, \ldots, q_n$, each of which is assumed to be differentiable $n-1$ times. If $u(x)$, $v(x)$ are arbitrary twice differentiable functions, then*

$$W(u^2, uv, v^2; x) = 2[W(u, v; x)]^3 .$$

THEOREM 4. *Let $z(x)$ be a given twice differentiable function, such that $z(x) \neq 0$, $x \in I$. Then there exists a function $p_2(x)$ such that for any pair $u(x)$, $v(x)$ of linearly independent solutions of the Sturm-Liouville differential equation*

$$y'' + p_2(x)y = 0, \quad x \in I, \tag{8}$$

there exist constants A, B, C such that

$$z = Au^2 + Buv + Cv^2, \quad x \in I .$$

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Proof. Define

(9) $$p_2(x) = -\frac{1}{2} z^{-2}[zz'' - \frac{1}{2}(z')^2] .$$

The (non-vanishing) function $z(x)$ may clearly be assumed to be positive, so we may define $w(x) = [z(x)]^{\frac{1}{2}}$. Thus $w(x)$ is a solution of (8) with $p_2(x)$ defined by (9) and therefore $w(x) = A_1 u(x) + B_1 v(x)$. Hence

$$z = A_1^2 u^2 + 2A_1 B_1 uv + B_1^2 v^2 = Au^2 + Buv + Cv^2 . \qquad \text{q.e.d.}$$

Remark. Obviously $A > 0$, $C > 0$ and $B = 2(AC)^{\frac{1}{2}}$ with z positive.

A converse to Theorem 2 is also valid:

THEOREM 5. *If $u(x)$, $v(x)$ are three-times differentiable functions such that u^2, uv and v^2 are linearly independent solutions of the Appell equation* (6), *then u and v are linearly independent solutions of* (1), $x \in I$.

Proof. In (6), put first $z = u^2$. On suitable rearrangement of the resulting expression, this becomes

(10) $$u[(u'' + p_1 u' + p_2 u)'] + (3u' + 2p_1 u)(u'' + p_1 u' + p_2 u) = 0 .$$

Similarly, we obtain also

(11) $$v[(v'' + p_1 v' + p_2 v)'] + (3v' + 2p_1 v)(v'' + p_1 v' + p_2 v) = 0 .$$

Putting $z = uv$ in (6) gives

(12) $$\begin{cases} v[(u'' + p_1 u' + p_2 u)'] + u[(v'' + p_1 v + p_2 v)'] \\ + (3v' + 2p_1 v)(u'' + p_1 u' + p_2 u) + (3u' + 2p_1 u)(v'' + p_1 v' + p_2 v) = 0. \end{cases}$$

Multiplying (12) by $u \cdot v$, and using (10) and (11) gives

(13) $$3(uv' - u'v)[v(u'' + p_1 u' + p_2 u) - u(v'' + p_1 v' + p_2 v)] = 0 .$$

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From the lemma

$$2[W(u,v;x)]^3 = W(u^2, uv, v^2; x) \neq 0 ,$$

$$v(u'' + p_1 u' + p_2 u) = u(v'' + p_1 v' + p_2 v) .$$

Assume now that the zeros of, say, u have a limit point x_o inside I. That is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x_o$, as $n \to \infty$, and $u(x_n) = 0$ for all n.

Then $u(x_o) = u'(x_o) = 0$, and $W(u,v;x_o) = u(x_o)v'(x_o) - u'(x_o)v(x_o) = 0$ and so $W(u^2, uv, v^2; x_o) = 0$. But this is impossible, since u^2, uv and v^2 are linearly independent solutions of the differential equation (6).

Thus, the zeros of both $u(x)$ and $v(x)$ are isolated.

Consider now those intervals where u and v are different from zero. In these intervals we define the function $k(x)$ as follows:

$$(14) \qquad k(x) = \frac{u''(x)+p_1(x)u'(x)+p_2(x)u(x)}{u(x)} = \frac{v''(x)+p_1(x)v'(x)+p_2(x)v(x)}{v(x)} .$$

Clearly $k(x)$ is differentiable in these intervals. We show now that $k(x) \equiv 0$ for all x for which it is defined.

From (14) we get:

$$u'' + p_1 u' + p_2 u = ku ,$$

$$v'' + p_1 v' + p_2 v = kv .$$

Substitute this in (10), (11) and (12):

$$(15) \qquad \begin{cases} 2k(u^2)' + (k' + 2p_1 k)u^2 = 0 \\ 2k(v^2)' + (k' + 2p_1 k)v^2 = 0 \\ 2k(uv)' + (k' + 2p_1 k)uv = 0 . \end{cases}$$

There are two cases to consider:

1^{o}: There exists an x_o such that $k(x_o) \neq 0$ and

$k'(x_o) + 2p_1(x_o)k(x_o) \neq 0$.

2^{o}: For all x , either $k(x) = 0$ or $k'(x) + 2p_1(x)k(x) = 0$.

In case 1^{o}:

From (15) we get:

$$2k(x_o)[u^2(x_o)]' = -[k'(x_o) + 2p_1(x_o)k(x_o)]u^2(x_o)$$

$$2k(x_o)[v^2(x_o)]' = -[k'(x_o) + 2p_1(x_o)k(x_o)]v^2(x_o)$$

so that

$$\frac{[u^2(x_o)]'}{[v^2(x_o)]'} = \frac{u^2(x_o)}{v^2(x_o)} ,$$

$$\frac{u'(x_o)}{v'(x_o)} = \frac{u(x_o)}{v(x_o)} .$$

But this says that $v'(x_o)u(x_o) - u'(x_o)v(x_o) = W(u, v; x_o) = 0$, which, as the lemma shows, contradicts the linear independence of u^2, uv, v^2 .

In case 2^{o}:

Let us suppose that there exists an x_1 such that $k(x_1) \neq 0$. Then

$$k'(x_1) + 2p_1(x_1)k(x_1) = 0 .$$

Then from (15),

$$2k(x_1)[u^2(x_1)]' = 0 \quad \text{or} \quad [u^2(x_1)]' = 0 ,$$

$$2k(x_1)[v^2(x_1)]' = 0 \quad \text{or} \quad [v^2(x_1)]' = 0 ,$$

$$2k(x_1)[u(x_1)v(x_1)]' = 0 \quad \text{or} \quad [u(x_1)v(x_1)]' = 0 .$$

Thus, we have again that $W(u^2, uv, v^2; x_1) = 0$, a contradiction. Thus, $k(x) = 0$ for all x such that $u(x)v(x) \neq 0$, that is, for such x we have

$$(16) \qquad u''(x) + p_1(x)u'(x) + p_2(x)u(x) = v''(x) + p_1(x)v'(x) + p_2(x)v(x) = 0.$$

By continuity (u'' and v'' are both continuous, since they are both differentiable) we see now that (16) is true for all x. But this shows that u and v are solutions of (1), as asserted.

Remark. In supposing that u^2, uv, v^2 are solutions of (6) we have, of course, assumed that p_1', p_2' exist. Theorem 5 remains valid if we assume somewhat less and work with equation (5) instead of (6). Therefore our calculations remain valid if we require only that $(p_1u)'$, $(p_1v)'$ and p_2' exist.

Finally, we establish a converse to Theorem 1:

THEOREM 6. *Let* $u(x) \in C'$ *for* x *in the open interval* I. *Suppose that* $u''(x)$ *exists for* $x \in I$ *whenever* $u(x) \neq 0$ *and that* $z(x) = u^2(x)$ *satisfies the differential equation*

$$p_2(x)z^2 = \tfrac{1}{4}(z')^2 - \tfrac{1}{2}p_1(x)zz' - \tfrac{1}{2}zz'', \quad x \in I,$$

whenever $u(x) \neq 0$, *where* $p_1(x)$, $p_2(x)$ *are continuous for* $x \in I$. *Then* $u(x)$ *satisfies the differential equation* (1) *for all* $x \in I$.

Proof. A straightforward calculation shows that $u^3[u'' + p_1(x)u' + p_2(x)u] = 0$, so that the assertion is established for those $x \in I$ for which $u(x) \neq 0$. To show that it holds also for those $x \in I$ for which $u(x) = 0$, we must demonstrate that $u''(x)$ exists and equals $-p_1(x)u'(x) - p_2(x)u(x)$ for such x.

Two lemmas are needed.

LEMMA 2. *Suppose that* $g(x)$ *is continuous*, $a \leq x \leq b$, *that* $g'(x)$ *exists for* $a < x < b$ *except possibly for* $x = \xi$, $a < \xi < b$, *and that* $g'(\xi +) = \lim_{x \to \xi +} g'(x)$ *and* $g'(\xi -) = \lim_{x \to \xi -} g'(x)$ *both exist and are equal*. *Then* $g'(\xi)$ *exists and equals this common value*.

A proof of this lemma is given in [5, Theorem 190, pp. 132-133].

The next lemma can be established for a class of differential equations broader than (1), as A. Meir remarked in conversation, and is phrased in a general form.

LEMMA 3. *Let* $u(x) \in C'$, $x \in I$ *and suppose that* $u''(x)$ *exists and equals* $f(x, u, u')$ *whenever* $u(x) \neq 0$. *Suppose further that* $f(x, 0, 0) \equiv 0$, $x \in I$; *that* $f(x, w, w')$ *is continuous in* x, w, w' *and such that the differential equation* $w'' = f(x, w, w')$ *has a unique solution when* $w(\xi)$, $w'(\xi)$ *are specified for a fixed* $\xi \in I$. *Then* $u''(x)$ *exists and equals* $f(x, u, u')$ *for all* $x \in I$.

Remark. The uniqueness condition obtains, e.g., when $f(x, w, w')$ satisfies a Lipschitz condition in w and w' separately [3, p. 12, Theorem 3], and, all the more, when, as in our intended application, $f(x, w, w') = -p_1(x)w' - p_2(x)w$, with $p_1(x)$, $p_2(x)$ continuous [3, p. 13, § 6].

Proof of Lemma 3. The trivial case $u(x) \equiv 0$, $x \in I$, aside, it will be shown (i) that each zero of $u(x)$ in I is isolated and (ii), that at an isolated zero of $u(x)$, the function has a second derivative and satisfies the differential equation.

(i) There exists $x_o \in I$ such that $u(x_o) \neq 0$. Define

$$\xi_o = \text{l.u.b.}\{x \mid x > x_o,\ u(x) \neq 0,\ x \in I\} .$$

If ξ_o is an endpoint of I, then nothing more need be proved for the subinterval $x \geq x_o$, $x \in I$. Let ξ_o be an interior point of I. Then $u(\xi_o) = 0$, since $u(x)$, being differentiable, is continuous. If $u'(\xi_o)$ were also zero, then it would follow from our uniqueness assumption that $u(x) \equiv 0$, $x_o \leq x \leq \xi$. But $u(x) \not\equiv 0$, $x_o \leq x < \xi_o$. Hence $u'(\xi_o) \neq 0$.

Thus, ξ_o is not a limit-point of zeros of $u(x)$. It can therefore be surrounded by a neighbourhood throughout which $u(x) \neq 0$ for $x \neq \xi_o$. In this neighbourhood we can select $x_1 > \xi_o$, $u(x_1) \neq 0$, and, repeating the above construction, arrive at $\xi_1 > x_1 > \xi_o$, where ξ_1 is either the right-hand endpoint of I or is an isolated zero. Thus, we see that any zero of $u(x)$ greater than x_o in I is isolated.

A similar argument establishes the same property for any zero of $u(x)$ less than x_o in I.

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(ii) Let $\xi \in I$ be an isolated zero of $u(x)$. Then

$$\lim_{x \to \xi+} \{u''(x)\} = \lim_{x \to \xi+} f(x, u, u') = f(\xi, u(\xi), u'(\xi))$$

$$= \lim_{x \to \xi-} f(x, u, u') = \lim_{x \to \xi-} \{u''(x)\} ,$$

since $u'' = f(x, u, u')$ when $u(x) \neq 0$, and $f(x, u, u')$, $u(x)$ and $u'(x)$ are all continuous.

Applying Lemma 2 now, with $g(x) = u'(x)$, shows that $u''(\xi)$ exists and equals $f(\xi, u(\xi), u'(\xi))$. This proves Lemma 3.

The proof of Theorem 6 is completed on noting that the differential equation (1) possesses the uniqueness property hypothesized in Lemma 3, [3, p. 13, § 6].

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