6
Algebras

In this chapter we recall basic definitions related to algebras, especially C^*- and W^**-algebras.*

Operator algebras are often used in mathematical formulations of quantum theory to describe observables of quantum systems. This is especially useful if we consider infinitely extended systems. They are also convenient to express the Einstein causality properties of relativistic quantum fields.

It is also common to express canonical commutation and anti-commutation relations in terms of algebras. This is especially natural in the case of the CAR. In fact, we will use algebras to treat the CAR in a representation-independent way in Chap. 14. Algebras are less useful in the case of the CCR. We will discuss various choices of CCR algebras in Sect. 8.3.

The theory of W^*-algebras, including elements of the *modular theory*, will be especially needed in Chap. 17, devoted to quasi-free states.

6.1 Algebras

6.1.1 Associative algebras

Let $\mathfrak{A}$ be a vector space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

Definition 6.1 $\mathfrak{A}$ *is called an* algebra over $\mathbb{K}$ *if it is equipped with a multiplication satisfying*

$$A(B+C) = AB + AC, \ \ (B+C)A = BA + CA,$$
$$(\alpha\beta)(AB) = (\alpha A)(\beta B), \ \ \alpha, \beta \in \mathbb{K}, \quad A, B, C \in \mathfrak{A}.$$

If in addition

$$A(BC) = (AB)C, \quad A, B, C \in \mathfrak{A},$$

then we say that it is an associative algebra*.*

Unless indicated otherwise, by an *algebra* we will mean an *associative algebra.*

Definition 6.2 *A subspace* $\mathfrak{I}$ *of an algebra* $\mathfrak{A}$ *is called a* (two-sided) ideal *of* $\mathfrak{A}$ *if* $A \in \mathfrak{A}$ *and* $B \in \mathfrak{I}$ *implies* $AB, BA \in \mathfrak{I}$.

If $\mathfrak{I}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{I}$ is naturally an algebra.

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Definition 6.3 *An algebra* $\mathfrak{A}$ *is called* simple *if* $\mathfrak{A}$ *has no ideals except for* $\{0\}$ *and itself, and* $\mathfrak{A} \neq \mathbb{K}$ *with the multiplication given by* $AB = 0$ *for all* $A, B \in \mathfrak{A}$.

For every subset $\mathfrak{T}$ of an algebra $\mathfrak{A}$ there exists the smallest ideal containing $\mathfrak{T}$.

Definition 6.4 *This ideal is called the* ideal generated by $\mathfrak{T}$ *and is denoted by* $\mathfrak{I}(\mathfrak{T})$.

Definition 6.5 *If* $\mathfrak{A}$, $\mathfrak{B}$ *are algebras, then a linear map* $\pi : \mathfrak{A} \to \mathfrak{B}$ *satisfying* $\pi(A_1 A_2) = \pi(A_1)\pi(A_2)$ *is called a* homomorphism. *It is called an* anti-homomorphism *if* $\pi(A_1 A_2) = \pi(A_2)\pi(A_1)$. *(In the well-known way, we also define* isomorphisms, automorphisms *etc.)*

6.1.2 $*$-algebras

Definition 6.6 *We say that an algebra* $\mathfrak{A}$ *is a* $*$-algebra *if it is equipped with an anti-linear involution* $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ *such that* $(AB)^* = B^* A^*$.

Let $\mathfrak{A}$ be a $*$-algebra. If $\mathfrak{I}$ is a $*$-invariant ideal of $\mathfrak{A}$, then $\mathfrak{A}/\mathfrak{I}$ is naturally a $*$-algebra.

Definition 6.7 *If* $\mathfrak{A}$, $\mathfrak{B}$ *are* $*$*-algebras, then a homomorphism* $\pi : \mathfrak{A} \to \mathfrak{B}$ *satisfying* $\pi(A^*) = \pi(A)^*$ *is called a* $*$-homomorphism. *(We also define* $*$-isomorphisms, $*$-automorphisms *etc.)* $\mathrm{Aut}(\mathfrak{A})$ *will denote the group of* $*$*-automorphisms of* $\mathfrak{A}$.

6.1.3 Algebras generated by symbols and relations

Suppose that $\mathcal{A}$ is a set.

Recall that $c_{\mathrm{c}}(\mathcal{A}, \mathbb{K})$ denotes the vector space over $\mathbb{K}$ consisting of finite linear combinations of elements indexed by the set $\mathcal{A}$. We adopt the convention that the element of $c_{\mathrm{c}}(\mathcal{A}, \mathbb{K})$ corresponding to $A \in \mathcal{A}$ is denoted simply by A. Recall also that $\overset{\mathrm{al}}{\otimes}\mathcal{Y}$ denotes the algebraic tensor algebra over the vector space $\mathcal{Y}$.

Definition 6.8 (1) *The* unital universal algebra over $\mathbb{K}$ with generators $\mathcal{A}$ *is defined as*

$$\mathfrak{A}(\mathcal{A}, \mathbb{1}) := \overset{\mathrm{al}}{\otimes} c_{\mathrm{c}}(\mathcal{A}, \mathbb{K}),$$

where we write $A_1 A_2 \cdots A_n$ *instead of* $A_1 \otimes A_2 \otimes \cdots \otimes A_n$, $A_1, \dots, A_n \in \mathcal{A}$ *and the unit element is denoted by* $\mathbb{1}$.

(2) *The* universal unital $*$-algebra with generators $\mathcal{A}$ *is the* $*$*-algebra* $\mathfrak{A}(\mathcal{A} \sqcup \mathcal{A}^*, \mathbb{1})$ *equipped with the involution* $*$ *such that* $(A_1 A_2 \cdots A_n)^* = A_n^* \cdots A_2^* A_1^*$, $\mathbb{1} = \mathbb{1}^*$.

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Definition 6.9 (1) *Let $\mathfrak{R} \subset \mathfrak{A}(\mathcal{A}, \mathbb{1})$. The* unital algebra with generators $\mathcal{A}$ and relations $R = 0, \quad R \in \mathfrak{R}$, *is defined as* $\mathfrak{A}(\mathcal{A}, \mathbb{1})/\mathfrak{I}(\mathfrak{R})$.

(2) *Let $\mathfrak{R} \subset \mathfrak{A}(\mathcal{A} \cup \mathcal{A}^*, \mathbb{1})$ be $*$-invariant. The* unital $*$-algebra with generators $\mathcal{A}$ and relations $R = 0, \quad R \in \mathfrak{R}$, *is defined as* $\mathfrak{A}(\mathcal{A} \cup \mathcal{A}^*, \mathbb{1})/\mathfrak{I}(\mathfrak{R})$.

6.1.4 Super-algebras

Recall from Subsect. 1.1.15 that $(\mathcal{Y}, \epsilon)$ is a *super-space* if $\mathcal{Y}$ is a vector space and $\epsilon \in L(\mathcal{Y})$ satisfies $\epsilon^2 = \mathbb{1}$. We then have a decomposition $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{Y}_1$ into its even and odd subspace.

Definition 6.10 *$(\mathfrak{A}, \alpha)$ is called a* super-algebra *if $\mathfrak{A}$ is an algebra and α is an involutive automorphism of $\mathfrak{A}$.*

We then have a decomposition $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ into even and odd subspace. Clearly, for pure elements $A, B \in \mathfrak{A}$ of parity $|A|$, resp. $|B|$, the parity of AB is $|A| + |B|$.

Note that $\mathfrak{A}_0$ is a sub-algebra of $\mathfrak{A}$.

Definition 6.11 *We say that a super-algebra $\mathfrak{A}$ is* super-commutative *iff $AB = (-1)^{|A||B|}AB$.*

Below we give two typical examples of associative super-algebras:

Example 6.12 (1) *Let $(\mathcal{Y}, \epsilon)$ be a super-space. Then $L(\mathcal{Y})$ equipped with the involution*

$$\alpha(A) = \epsilon A \epsilon \tag{6.1}$$

is a super-algebra. It will be denoted $gl(\mathcal{Y}, \epsilon)$.

(2) $\overset{\mathrm{al}}{\Gamma}_\epsilon(\mathcal{Y})$ *equipped with $\otimes_\epsilon$ is a super-commutative super-algebra (see Subsect. 3.3.9).*

6.2 C^*- and W^*-algebras

In this section we recall basic terminology from the theory of C^*- and W^*-algebras.

6.2.1 Banach algebras

Definition 6.13 *An algebra $\mathfrak{A}$ is called a* normed algebra *if it is equipped with a norm $\|\cdot\|$ satisfying*

$$\|AB\| \leq \|A\|\|B\|, \quad A, B \in \mathfrak{A}.$$

It is called a Banach algebra *if it is complete in this norm.*

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6.2.2 C^*-algebras

Definition 6.14 *We say that* $\mathfrak{A}$ *is a* C^*-algebra *if it is a complex Banach* $*$*-algebra satisfying*

$$\|A^*\| = \|A\|, \ \|A^*A\| = \|A\|^2, \quad A \in \mathfrak{A}. \tag{6.2}$$

Definition 6.15 *Let* $\mathfrak{A}$ *be a complex normed* $*$*-algebra (not necessarily complete). We say that its norm is a* C^*-norm *if it satisfies (6.2).*

Clearly, the completion of an algebra equipped with a C^*-norm is a C^*-algebra.

If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ equipped with the Hermitian conjugation and the operator norm is a C^*-algebra.

Definition 6.16 *A norm closed* $*$*-sub-algebra of* $B(\mathcal{H})$ *is called a* concrete C^*-algebra.

Clearly, every concrete C^*-algebra is a C^*-algebra. Conversely, every C^*-algebra is $*$-isomorphic to a concrete C^*-algebra.

Any $*$-homomorphism, resp. $*$-isomorphism between two C^*-algebras is a contraction, resp. isometry.

Definition 6.17 *We define the set of* positive elements of $\mathfrak{A}$ *as the set of self-adjoint elements with spectrum in* $[0, \infty[$*, or equivalently, of elements of the form* A^*A*. The set of positive elements of* $\mathfrak{A}$ *is denoted* $\mathfrak{A}_+$*.*

Definition 6.18 *Let* $\mathfrak{A}$ *be a* C^**-algebra. A* C^*-dynamics on $\mathfrak{A}$ *is a one-parameter group* $\mathbb{R} \ni t \mapsto \tau^t \in \mathrm{Aut}(\mathfrak{A})$ *such that for each* $A \in \mathfrak{A}$ *the map* $t \mapsto \tau^t(A)$ *is continuous. Such a pair* $(\mathfrak{A}, \tau)$ *is called a* C^*-dynamical system.

6.2.3 Representations of C^*-algebras

Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{A} \subset B(\mathcal{H})$.

Definition 6.19 *The* commutant of $\mathfrak{A}$ *is defined as*

$$\mathfrak{A}' := \{B \in B(\mathcal{H}) \ : \ AB = BA, \quad A \in \mathfrak{A}\}.$$

Let $\mathfrak{A} \subset B(\mathcal{H})$ be a $*$-algebra.

Definition 6.20 $\mathfrak{A}$ *is called* irreducible *if the only closed subspaces of* $\mathcal{H}$ *invariant under* $\mathfrak{A}$ *are* $\{0\}$ *and* $\mathcal{H}$*, or equivalently if* $\mathfrak{A}' = \mathbb{C}1\!\!1$*.* $\mathfrak{A}$ *is called* non-degenerate *if* $\mathfrak{A}\mathcal{H}$ *is dense in* $\mathcal{H}$*.*

Let $\mathfrak{A}$ be a C^*-algebra.

Definition 6.21 $(\mathcal{H}, \pi)$ *is a* representation of $\mathfrak{A}$ *if* $\mathcal{H}$ *is a Hilbert space and* π *is a* $*$*-homomorphism of* $\mathfrak{A}$ *into* $B(\mathcal{H})$*.* π *is called* faithful *if* $\mathrm{Ker}\,\pi = \{0\}$*.*

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(*Faithful* in this context is the synonym of *injective*.) Since $\operatorname{Ker}\pi$ is a closed two-sided ideal of $\mathfrak{A}$, any non-trivial representation of a simple C^*-algebra is faithful. Actually, a stronger statement is true: a C^*-algebra is simple iff all its representations are faithful.

Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathfrak{A}$.

Definition 6.22 *A closed subspace $\mathcal{H}_1 \subset \mathcal{H}$ is* invariant *if $\pi(A)\mathcal{H}_1 \subset \mathcal{H}_1$ for all $A \in \mathfrak{A}$. $(\mathcal{H}_1, \pi_1)$ is a* sub-representation of $(\mathcal{H}, \pi)$ *if $\mathcal{H}_1$ is an invariant subspace of $\mathcal{H}$ and $\pi_1 = \pi\big|_{\mathcal{H}_1}$.*

Definition 6.23 *We say that $(\mathcal{H}, \pi)$ is the* direct sum of $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ *if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $(\mathcal{H}_i, \pi_i)$ are sub-representations of $(\mathcal{H}, \pi)$.*

Note that if $\mathcal{H}_1$ is invariant, then so is $\mathcal{H}_2 := \mathcal{H}_1^\perp$. $(\mathcal{H}, \pi)$ is then the direct sum of $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$, with $\pi_1 := \pi\big|_{\mathcal{H}_1}$, $\pi_2 := \pi\big|_{\mathcal{H}_2}$.

Definition 6.24 *We say that a representation $(\mathcal{H}, \pi)$ of a C^*-algebra is* irreducible *if $\pi(\mathfrak{A})$ is irreducible. Equivalently $\pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$, or π has no non-trivial sub-representations.*

Definition 6.25 *The representation $(\mathcal{H}, \pi)$ is called* non-degenerate *if $\pi(\mathfrak{A})$ is non-degenerate.*

Definition 6.26 *The representation $(\mathcal{H}, \pi)$ is called* factorial *if $\pi(\mathfrak{A}) \cap \pi(\mathfrak{A})' = \mathbb{C}\mathbb{1}$.*

Let $\mathcal{E} \subset \mathcal{H}$.

Definition 6.27 (1) *$\mathcal{E}$ is called* cyclic for π *if $\{\pi(A)\,\Phi : A \in \mathfrak{A},\ \Phi \in \mathcal{E}\}$ is dense in $\mathcal{H}$.*
(2) *$\mathcal{E}$ is called* separating for π *if*

$$\pi(A)\Phi = 0, \quad \Phi \in \mathcal{E} \quad \Rightarrow \ A = 0.$$

Clearly, if $(\mathcal{H}, \pi)$ is irreducible, all non-zero vectors in $\mathcal{H}$ are cyclic.

6.2.4 Intertwiners and unitary equivalence

Let $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$ be two representations of a C^*-algebra $\mathfrak{A}$.

Definition 6.28 *An operator $B \in B(\mathcal{H}_1, \mathcal{H}_2)$* intertwines *$\pi_1$ and π_2 if*

$$B\pi_1(A) = \pi_2(A)B, \ A \in \mathfrak{A}.$$

If π_1 and π_2 have an intertwiner in $U(\mathcal{H}_1, \mathcal{H}_2)$, they are called unitarily equivalent.

The following theorem can be called *Schur's lemma for C^*-algebras*:

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Theorem 6.29 *If $(\mathcal{H}_1, \pi_1)$, $(\mathcal{H}_2, \pi_2)$ are irreducible, then the set of intertwiners equals either $\{0\}$ or $\{\lambda U : \lambda \in \mathbb{C}\}$ for some $U \in U(\mathcal{H}_1, \mathcal{H}_2)$.*

Proof If B intertwines π_1 and π_2, B^* intertwines π_2 and π_1, hence $B^*B \in \pi_1(\mathfrak{A})'$ and $BB^* \in \pi_2(\mathfrak{A})'$. By irreducibility, $B^*B = \lambda_1 \mathbb{1}$, $BB^* = \lambda_2 \mathbb{1}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Now

$$\lambda_1^2 \mathbb{1} = BB^*BB^* = B\lambda_2 B^* = \lambda_2 \lambda_1 \mathbb{1}. \tag{6.3}$$

If $\lambda_1 = 0$, then $B = 0$, and hence $\lambda_2 = 0$. Hence (6.3) implies that $\lambda_1 = \lambda_2$, which means that $B = \lambda U$ for some $U \in U(\mathcal{H}_1, \mathcal{H}_2)$. If B_1 and B_2 are two intertwiners, then a similar argument shows that $B_1 B_2^*$ is proportional to identity. This means that B_1 is proportional to B_2. □

6.2.5 States

Let $\mathfrak{A}$ be a C^*-algebra.

Definition 6.30 *A linear functional on $\mathfrak{A}$ is called* positive *if it maps positive elements to positive numbers.*

A positive linear functional is automatically continuous.

Definition 6.31 *A positive linear functional is called a* state *if its norm is 1. In the case of a unital C^*-algebra it is equivalent to requiring that $\omega(\mathbb{1}) = 1$.*

Definition 6.32 *A state ω is called* faithful *if $\omega(A) = 0$ and $A \in \mathfrak{A}_+$ implies $A = 0$.*

Definition 6.33 *A state ω is called* tracial *if*

$$\omega(AB) = \omega(BA), \quad A, B \in \mathfrak{A}.$$

6.2.6 GNS representations

Let $(\mathcal{H}, \pi)$ be a $*$-representation of $\mathfrak{A}$, Ω a normalized vector in $\mathcal{H}$. Then

$$\omega(A) = (\Omega|\pi(A)\Omega) \tag{6.4}$$

defines a state on $\mathfrak{A}$.

Definition 6.34 *If (6.4) is true, we say that Ω is a* vector representative of ω.

Definition 6.35 *$(\mathcal{H}, \pi, \Omega)$ is called a* cyclic $*$-representation *if $(\mathcal{H}, \pi)$ is a $*$-representation and Ω is a cyclic vector.*

Theorem 6.36 (Gelfand–Najmark–Segal theorem) *Let ω be a state on $\mathfrak{A}$. Then there exists a cyclic $*$-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that Ω_ω is a vector representative of ω. Such a representation is unique up to a unitary equivalence.*

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Definition 6.37 *The cyclic $*$-representation described in Thm. 6.36 is called the* GNS representation *(for Gelfand–Najmark–Segal)* associated with ω.

6.2.7 W^*-algebras

Definition 6.38 *We say that $\mathfrak{M}$ is a W^**-algebra *if it is a C^*-algebra such that there exists a Banach space whose dual is isomorphic to $\mathfrak{M}$ as a Banach space. This Banach space is unique up to an isometry. It is called the* pre-dual of $\mathfrak{M}$ *and is denoted $\mathfrak{M}_\#$. The topology on $\mathfrak{M}$ given by the functionals from $\mathfrak{M}_\#$ (the $*$-weak topology in the terminology of Banach spaces) is called the* σ-weak topology. *Functionals in $\mathfrak{M}_\#$ are called* normal functionals.

It follows from the general theory of Banach spaces that $\mathfrak{M}_\#$ coincides with the space of all σ-weakly continuous functionals on $\mathfrak{M}$.

Definition 6.39 *The set*

$$\{B \in \mathfrak{M} \ : \ AB = BA, \ A \in \mathfrak{M}\}$$

is called the center of $\mathfrak{M}$. *A W^*-algebra with a trivial center is called a* factor.

Two-sided σ-weakly closed ideals $\mathfrak{I}$ of a W^*-algebra $\mathfrak{M}$ have a simple form: they are equal to $\mathfrak{I} = \mathfrak{M}E$, for a projection E in the center of $\mathfrak{M}$. Clearly, all two-sided σ-weakly closed ideals of a factor are trivial.

If ω is a σ-weakly continuous state, then the map π_ω given by the GNS representation is σ-weakly continuous.

Definition 6.40 *Let $\mathfrak{M}$ be a W^*-algebra. A W^**-dynamics on $\mathfrak{M}$ *is a one-parameter group $\mathbb{R} \ni t \mapsto \tau^t \in \mathrm{Aut}(\mathfrak{M})$ such that for each $A \in \mathfrak{M}$ the map $t \mapsto \tau^t(A)$ is σ-weakly continuous. Such a pair $(\mathfrak{M}, \tau)$ is called a W^**-dynamical system.

6.2.8 Von Neumann algebras

Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a W^*-algebra, since it is the dual of $B^1(\mathcal{H})$ (the space of trace-class operators on $\mathcal{H}$). Thus $B^1(\mathcal{H})$ is the pre-dual of $B(\mathcal{H})$ and the topology on $B(\mathcal{H})$ given by functionals in $B^1(\mathcal{H})$ is its *σ-weak topology.*

Definition 6.41 *Every C^*-sub-algebra of $B(\mathcal{H})$ closed w.r.t. the σ-weak topology is called a* concrete W^*-algebra. *If in addition it contains $\mathbb{1}_\mathcal{H}$, then it is called a* von Neumann algebra.

Clearly, all concrete W^*-algebras are W^*-algebras. Conversely, a W^*-algebra is isomorphic to a von Neumann algebra.

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Definition 6.42 *Let $\mathfrak{M}_i \subset B(\mathcal{H}_i)$, $i = 1, 2$. Let $\rho : \mathfrak{M}_1 \to \mathfrak{M}_2$ be an isomorphism. We say that ρ is* spatially implementable *if there exists $U \in U(\mathcal{H}_1, \mathcal{H}_2)$ such that $\rho(A) = UAU^*$, $A \in \mathfrak{M}_1$.*

If $\mathfrak{A} \subset B(\mathcal{H})$ is $*$-invariant, then $\mathfrak{A}'$ is a von Neumann algebra.

An equivalent characterization of a von Neumann algebra is given by *von Neumann's double commutant theorem*, stating that a $*$-algebra $\mathfrak{M}$ is a von Neumann algebra iff

$$\mathfrak{M} = \mathfrak{M}''.$$

The *von Neumann density theorem* says that if $\mathfrak{A} \subset B(\mathcal{H})$ is a non-degenerate $*$-algebra, then $\mathfrak{A}$ is dense in $\mathfrak{A}''$ in the weak, strong, strong*, σ-weak, σ-strong and σ-strong* topologies.

The *Kaplansky density theorem* says that if $\mathfrak{A} \subset B(\mathcal{H})$ is a $*$-algebra, then the unit ball of $\mathfrak{A}$ is σ-weakly dense in the unit ball of $\mathfrak{A}''$.

Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra, and A a closed densely defined operator on $\mathcal{H}$. Let $A = U|A|$, where U is a partial isometry, be its polar decomposition.

Definition 6.43 *A is called* affiliated to $\mathfrak{M}$ *if the operators U and $\mathbb{1}_\Delta\big(|A|\big)$ belong to $\mathfrak{M}$ for all Borel sets $\Delta \subset \mathbb{R}$.*

Clearly, a von Neumann algebra $\mathfrak{M} \subset B(\mathcal{H})$ is a factor iff $\mathfrak{M} \cap \mathfrak{M}' = \mathbb{C}\mathbb{1}_{\mathcal{H}}$, or equivalently, $(\mathfrak{M} \cup \mathfrak{M}')'' = B(\mathcal{H})$. Below we give a more elaborate criterion for being a factor.

Proposition 6.44 *Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra. Suppose that*

(1) *$\Omega \in \mathcal{H}$ is a cyclic vector for $(\mathfrak{M} \cup \mathfrak{M}')''$;*
(2) *There exists a set $\mathfrak{L} \subset (\mathfrak{M} \cup \mathfrak{M}')''$ such that $\{\Psi \in \mathcal{H} \ : \ A\Psi = 0, \ A \in \mathfrak{L}\} = \mathbb{C}\Omega$.*

Then $\mathfrak{M}$ is a factor.

Proof Suppose that $\mathfrak{M}$ is not a factor and Ω is cyclic for $(\mathfrak{M} \cup \mathfrak{M}')''$. Then there exists an orthogonal projection $P \in \mathfrak{M} \cap \mathfrak{M}'$ different from 0 and $\mathbb{1}$. If $P\Omega = 0$, then $(\mathbb{1} - P)(\mathfrak{M} \cup \mathfrak{M}')''\Omega = (\mathfrak{M} \cup \mathfrak{M}')''(\mathbb{1} - P)\Omega = (\mathfrak{M} \cup \mathfrak{M}')''\Omega$. Hence Ω is not cyclic for $(\mathfrak{M} \cup \mathfrak{M}')''$. Therefore, $P\Omega \neq 0$. Likewise, we show that $(\mathbb{1} - P)\Omega \neq 0$.

Now let $\mathfrak{L}$ be as in (2). Then since $P \in \mathfrak{M} \cap \mathfrak{M}'$ one has

$$A\left(c_1 P + c_2(\mathbb{1} - P)\right)\Omega = 0, \quad A \in \mathfrak{L}, \quad c_1, c_2 \in \mathbb{C}.$$

But for $c_1 \neq c_2$, the vector $\left(c_1 P + c_2(\mathbb{1} - P)\right)\Omega$ is not proportional to Ω. □

6.2.9 UHF algebras

In this subsection we describe an example of a C^*-algebra which plays an important role in mathematical physics, and in particular in the theory of CAR.

For any $n = 1, 2, \ldots$, we introduce the identifications

$$B(\otimes^n \mathbb{C}^2) \ni A \mapsto A \otimes \mathbb{1}_{\mathbb{C}^2} \in B(\otimes^{n+1} \mathbb{C}^2).$$

Definition 6.45 *Define*

$$\mathrm{UHF}_0(2^\infty) := \bigcup_{k=1}^{\infty} B(\otimes^n \mathbb{C}^2), \quad \mathrm{UHF}(2^\infty) := \mathrm{UHF}_0(2^\infty)^{\mathrm{cpl}}.$$

$UHF(2^\infty)$ is called the uniformly hyper-finite C^*-algebra of type 2^∞.

6.2.10 Hyper-finite type II_1 factor

We continue to consider the C^*-algebra $\mathrm{UHF}(2^\infty)$ introduced in the last subsection. On $B(\otimes^n \mathbb{C}^2)$ we have a tracial state

$$\mathrm{tr} A := 2^{-n} \mathrm{Tr}\, A.$$

This state extends to a state on the whole $\mathrm{UHF}(2^\infty)$. Let $(\pi_{\mathrm{tr}}, \mathcal{H}_{\mathrm{tr}}, \Omega_{\mathrm{tr}})$ be the GNS representation given by the state tr on $\mathrm{UHF}(2^\infty)$.

Definition 6.46 *The W^*-algebra*

$$\mathrm{HF} := \pi_{\mathrm{tr}} \left(\mathrm{UHF}(2^\infty)\right)''. \tag{6.5}$$

is called the hyper-finite type II_1 factor.

Clearly,

$$\mathrm{tr}(A) := (\Omega_{\mathrm{tr}} | A \Omega_{\mathrm{tr}})$$

defines a tracial state on HF.

6.2.11 Conditional expectations

Let $\mathfrak{N}$ be a unital C^*-sub-algebra of a C^*-algebra $\mathfrak{M}$. We assume that the unit of $\mathfrak{M}$ is contained in $\mathfrak{N}$.

Definition 6.47 *We say that $E : \mathfrak{M} \to \mathfrak{N}$ is* $\mathfrak{N}$-linear *if $A \in \mathfrak{M}$, $B \in \mathfrak{N}$ implies $E(AB) = E(A)B$, $E(BA) = BE(A)$.*

We say that E is a conditional expectation *if*

(1) $A \geq 0$ *implies* $E(A) \geq 0$,
(2) E *is* $\mathfrak{N}$*-linear,*
(3) $E(\mathbb{1}) = \mathbb{1}$.

Proposition 6.48 *Let ω be a normal tracial faithful state on a W^*-algebra $\mathfrak{M}$. Then there exists a unique conditional expectation from $\mathfrak{M}$ with range equal to $\mathfrak{N}$ such that $\omega(A) = \omega(E(A))$.*

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6.3 Tensor products of algebras

Let $\mathfrak{A}$,$\mathfrak{B}$ be algebras. Then $\mathfrak{A} \overset{\mathrm{al}}{\otimes} \mathfrak{B}$ is naturally an algebra. If in addition $\mathfrak{A}$,$\mathfrak{B}$ are $*$-algebras, then so is $\mathfrak{A} \overset{\mathrm{al}}{\otimes} \mathfrak{B}$.

One can define natural tensor products also in the category of C^*- and W^*-algebras. The definitions of these constructions are given in this section.

6.3.1 Tensor product of C^*-algebras

Let $\mathfrak{A}$,$\mathfrak{B}$ be C^*-algebras. We choose an arbitrary injective $*$-representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ and $(\mathcal{K}, \rho)$ of $\mathfrak{B}$. Then $\mathfrak{A} \overset{\mathrm{al}}{\otimes} \mathfrak{B}$ has an obvious $*$-representation in $B(\mathcal{H} \otimes \mathcal{K})$. It equips $\mathfrak{A} \overset{\mathrm{al}}{\otimes} \mathfrak{B}$ with a C^* norm. It can be shown that this norm does not depend on the representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$.

Definition 6.49 *The C^*-algebra*

$$\mathfrak{A} \otimes \mathfrak{B} := (\mathfrak{A} \overset{\mathrm{al}}{\otimes} \mathfrak{B})^{\mathrm{cpl}},$$

is called the minimal C^*-tensor product of $\mathfrak{A}$ and $\mathfrak{B}$.

6.3.2 Tensor product of W^*-algebras

Let $\mathfrak{M}$,$\mathfrak{N}$ be W^*-algebras. We choose an arbitrary injective σ-continuous $*$-representation $(\mathcal{H}, \pi)$ of $\mathfrak{M}$ and $(\mathcal{K}, \rho)$ of $\mathfrak{N}$. Then $\mathfrak{M} \overset{\mathrm{al}}{\otimes} \mathfrak{N}$ has an obvious $*$-representation in $B(\mathcal{H} \otimes \mathcal{K})$. Let $\mathcal{X}$ denote the Banach space of linear functionals on $\mathfrak{M} \overset{\mathrm{al}}{\otimes} \mathfrak{N}$ given by density matrices in $B^1(\mathcal{H} \otimes \mathcal{K})$. One can show that $\mathcal{X}$ does not depend on the choice of representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$.

Definition 6.50 *We set*

$$\mathfrak{M} \otimes \mathfrak{N} := \mathcal{X}^{\#},$$

and call it the W^*-tensor product of $\mathfrak{M}$ and $\mathfrak{N}$.

Clearly, $\mathfrak{M} \overset{\mathrm{al}}{\otimes} \mathfrak{N}$ is σ-weakly dense in $\mathfrak{M} \otimes \mathfrak{N}$. We extend the multiplication from $\mathfrak{M} \overset{\mathrm{al}}{\otimes} \mathfrak{N}$ to $\mathfrak{M} \otimes \mathfrak{N}$ by the σ-weak continuity. One can check that $\mathfrak{M} \otimes \mathfrak{N}$ is a W^*-algebra.

Remark 6.51 *According to our convention, the meaning of $\otimes$ between two algebras depends on the context. It depends on whether we treat the algebras as C^*- or W^*-algebras.*

6.4 Modular theory

In this section we give a concise resumé of the *modular theory*. The modular theory is one of the most interesting parts of the theory of operator algebras. It

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sheds light on the structure of general W^*-algebras. It plays an important role in applications of operator algebras to quantum statistical physics. Key concepts of the modular theory include the *modular automorphism* and *conjugation* due to Tomita–Takesaki, *KMS states* and *standard forms* introduced by Araki, Connes and Haagerup.

6.4.1 Standard representations

Let $\mathcal{H}$ be a Hilbert space.

Definition 6.52 *A* self-dual cone $\mathcal{H}^+$ *is a subset of* $\mathcal{H}$ *with the property*

$$\mathcal{H}^+ = \{\Phi \in \mathcal{H} \ : \ (\Phi|\Psi) \geq 0, \ \Psi \in \mathcal{H}^+\}.$$

Let $\mathfrak{M}$ be a W^*-algebra.

Definition 6.53 *A quadruple* $(\mathcal{H}, \pi, J, \mathcal{H}^+)$ *is a* standard representation of a W^*-algebra $\mathfrak{M}$ *if* $\pi : \mathfrak{M} \to B(\mathcal{H})$ *is a faithful* σ*-weakly continuous representation,* J *is a conjugation on* $\mathcal{H}$ *and* $\mathcal{H}^+$ *is a self-dual cone in* $\mathcal{H}$ *with the following properties:*

(1) $J\pi(\mathfrak{M})J = \pi(\mathfrak{M})'$,
(2) $J\pi(A)J = \pi(A)^*$ *for* A *in the center of* $\mathfrak{M}$,
(3) $J\Phi = \Phi$ *for* $\Phi \in \mathcal{H}^+$,
(4) $\pi(A)J\pi(A)\mathcal{H}^+ \subset \mathcal{H}^+$ *for* $A \in \mathfrak{M}$.

Every W^*-algebra admits a unique (up to unitary equivalence) standard representation.

The standard representation has several important properties.

Theorem 6.54 (1) *For every* σ*-weakly continuous state* ω *on* $\mathfrak{M}$ *there exists a unique vector* $\Omega \in \mathcal{H}^+$ *such that* $\omega(A) = (\Omega|A\Omega)$.

(2) *For every* $*$*-automorphism* τ *of* $\mathfrak{M}$ *there exists a unique* $U \in U(\mathcal{H})$ *such that*

$$\pi(\tau(A)) = U\pi(A)U^*, \quad U\mathcal{H}^+ \subset \mathcal{H}^+.$$

(3) *If* $\mathbb{R} \ni t \mapsto \tau^t$ *is a* W^**-dynamics on* $\mathfrak{M}$*, there exists a unique self-adjoint operator* L *on* $\mathcal{H}$ *such that*

$$\pi(\tau^t(A)) = \mathrm{e}^{\mathrm{i}tL}\pi(A)\mathrm{e}^{-\mathrm{i}tL}, \quad \mathrm{e}^{\mathrm{i}tL}\mathcal{H}^+ \subset \mathcal{H}^+. \tag{6.6}$$

Definition 6.55 *The operator* L *that appears in (6.6) is called the* standard Liouvillean of the W^* dynamics $t \mapsto \tau^t$.

Definition 6.56 *Given a standard representation* $(\mathcal{H}, \pi, J, \mathcal{H}^+)$*, we also have the* right representation $\pi_{\mathrm{r}} : \overline{\mathfrak{M}} \to B(\mathcal{H})$ *given by* $\pi_{\mathrm{r}}(\overline{A}) := J\pi(A)J$*. Note that* $\pi_{\mathrm{r}}(\overline{\mathfrak{M}}) = \pi(\mathfrak{M})'$*. We will often write* π_{l} *for* π *and call it the* left representation.

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6.4.2 Tomita–Takesaki theory

Let $\mathfrak{M}$ be a W^*-algebra, $(\mathcal{H}, \pi)$ a faithful σ-weakly continuous representation of $\mathfrak{M}$ and Ω a cyclic and separating vector for $\pi(\mathfrak{M})$.

Definition 6.57 *Define the operator S_0 with domain $\pi(\mathfrak{M})\Omega$ by*

$$S_0\pi(A)\Omega := \pi(A^*)\Omega,\ \ A \in \mathfrak{M}.$$

One can show that S_0 is closable.

Definition 6.58 *S is defined as the closure of S_0.*

For further reference let us note the following proposition, which follows by the von Neumann density theorem:

Proposition 6.59 *If $\mathfrak{A} \subset \mathfrak{M}$ is a $*$-algebra weakly dense in $\mathfrak{M}$, then $\{A\Omega\ :\ A \in \mathfrak{A}\}$ is an essential domain for S.*

Definition 6.60 *The* modular operator Δ *and* modular conjugation J *are defined by the polar decomposition:*

$$S =: J\Delta^{\frac{1}{2}}.$$

Definition 6.61 *The* natural positive cone *is defined by*

$$\mathcal{H}^+ := \left\{\pi(A)J\pi(A)\Omega\ :\ A \in \mathfrak{M}\right\}^{\mathrm{cl}}.$$

Theorem 6.62 *$(\mathcal{H}, \pi, J, \mathcal{H}^+)$ is a standard representation of $\mathfrak{M}$. Given $(\mathcal{H}, \pi)$, it is the unique standard representation such that $\Omega \in \mathcal{H}^+$.*

6.4.3 KMS states

Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system. Consider $\beta > 0$ (having the interpretation of the inverse temperature). Let ω be a normal state on $\mathfrak{M}$.

Definition 6.63 *ω is called a (τ, β)-*KMS state *if for all $A, B \in \mathfrak{M}$ there exists a function $F_{A,B}(z)$ holomorphic in the strip $I_\beta = \{z \in \mathbb{C}\ :\ 0 < \operatorname{Im} z < \beta\}$, bounded and continuous on its closure, such that the KMS boundary condition holds:*

$$F_{A,B}(t) = \omega\big(A\tau^t(B)\big), \quad F_{A,B}(t + \mathrm{i}\beta) = \omega\big(\tau^t(B)A\big), \quad t \in \mathbb{R}. \tag{6.7}$$

Below we quote a number of properties of KMS states.

Proposition 6.64 (1) *One has $|F_{A,B}(z)| \le \|A\|\|B\|$, uniformly on I_β^{cl}.*
(2) *A KMS state is τ^t-invariant.*
(3) *Let $\mathfrak{A}$ be a $*$-algebra weakly dense in $\mathfrak{M}$ and τ-invariant. If (6.7) holds for all $A, B \in \mathfrak{A}$, then it holds for all $A, B \in \mathfrak{M}$.*

Proposition 6.65 *A KMS state on a factor is faithful.*

Definition 6.66 *If $\mathfrak{M} \subset B(\mathcal{H})$ and $\Phi \in \mathcal{H}$, we say that Φ is a (τ, β)-*KMS vector *if $(\Phi| \cdot \Phi)$ is a (τ, β)-KMS state.*

6.4.4 Type I factors: irreducible representation

Definition 6.67 *Algebras isomorphic to $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, are called* type I factors.

Such algebras are the most elementary W^*-algebras. In this and the next subsection we describe various concepts of the theory of W^*-algebras as applied to type I factors.

The space of σ-weakly continuous functionals on $B(\mathcal{H})$ (the pre-dual of $B(\mathcal{H})$) can be identified with $B^1(\mathcal{H})$ (trace-class operators) by the formula

$$\psi(A) = \operatorname{Tr} \gamma A, \qquad \gamma \in B^1(\mathcal{H}), \quad A \in B(\mathcal{H}). \tag{6.8}$$

In particular, σ-weakly continuous states are determined by density matrices. A state given by a density matrix γ is faithful iff $\operatorname{Ker} \gamma = \{0\}$.

Proposition 6.68 (1) *Every $*$-automorphism of $B(\mathcal{H})$ is of the form*

$$\tau(A) = UAU^* \tag{6.9}$$

for some $U \in U(\mathcal{H})$. If $U_1, U_2 \in U(\mathcal{H})$ satisfy (6.9), then there exists $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that $U_1 = \mu U_2$.

(2) *Every W^*-dynamics $\mathbb{R} \ni t \mapsto \tau_t$ on $B(\mathcal{H})$ is of the form*

$$\tau_t(A) = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} \tag{6.10}$$

for some self-adjoint H. If H_1 is another self-adjoint operator satisfying (6.10), then there exists $c \in \mathbb{R}$ such that $H_1 = H + c$.

Definition 6.69 *In the context of (6.9) we say that U* implements *τ. In the context of (6.10) we say that H* is a Hamiltonian *of $\{\tau_t\}_{t\in\mathbb{R}}$.*

A state given by (6.8) is invariant w.r.t. the W^*-dynamics (6.10) iff H commutes with γ.

There exists a (β, τ)-KMS state iff $\operatorname{Tr} \mathrm{e}^{-\beta H} < \infty$, and then it has the density matrix $\mathrm{e}^{-\beta H}/\operatorname{Tr} \mathrm{e}^{-\beta H}$.

6.4.5 Type I factors: representation on Hilbert–Schmidt operators

Clearly, the representation of $B(\mathcal{H})$ on $\mathcal{H}$ is not in the standard form. To construct a standard form of $B(\mathcal{H})$, consider the Hilbert space of Hilbert–Schmidt operators on $\mathcal{H}$, denoted $B^2(\mathcal{H})$.

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Definition 6.70 *We introduce two injective representations:*

$$\begin{aligned} &B(\mathcal{H}) \ni A \mapsto \pi_{\mathrm{l}}(A) \in B\big(B^2(\mathcal{H})\big), \quad \pi_{\mathrm{l}}(A)B := AB, \quad B \in B^2(\mathcal{H}); \\ &\overline{B(\mathcal{H})} \ni \overline{A} \mapsto \pi_{\mathrm{r}}(\overline{A}) \in B\big(B^2(\mathcal{H})\big), \quad \pi_{\mathrm{r}}(\overline{A})B := BA^*, \quad B \in B^2(\mathcal{H}). \end{aligned} \tag{6.11}$$

We set $J_{\mathcal{H}}B := B^*$, $B \in B^2(\mathcal{H})$.

With the above notation, $J_{\mathcal{H}}\pi_{\mathrm{l}}(A)J_{\mathcal{H}} = \pi_{\mathrm{r}}(\overline{A})$ and

$$\big(B^2(\mathcal{H}), \pi_l, J_{\mathcal{H}}, B^2_+(\mathcal{H})\big)$$

is a standard representation of $B(\mathcal{H})$.

If a state on $B(\mathcal{H})$ is given by a density matrix $\gamma \in B^1_+(\mathcal{H})$, then its standard vector representative is $\gamma^{\frac{1}{2}} \in B^2_+(\mathcal{H})$. If $\tau \in \mathrm{Aut}\big(B(\mathcal{H})\big)$ is implemented by $W \in U(\mathcal{H})$, then its standard implementation is $\pi_{\mathrm{l}}(W)\pi_{\mathrm{r}}(\overline{W})$. If the W^*-dynamics $t \mapsto \tau^t$ has a Hamiltonian H, then its standard Liouvillean is $\pi_{\mathrm{l}}(H) - \pi_{\mathrm{r}}(\overline{H})$.

6.5 Non-commutative probability spaces

Throughout the section, $\mathfrak{R}$ is a W^*-algebra and ω a normal faithful tracial state on $\mathfrak{R}$.

The two most important examples of such a pair $(\mathfrak{R}, \omega)$ are as follows:

Example 6.71 (1) *Let $(Q, \mathfrak{S}, \mu)$ be a set with a σ-algebra and a probability measure. Then taking $\mathfrak{R} = L^\infty(Q, \mu)$ and*

$$\omega(F) = \int_Q F \mathrm{d}\mu, \quad F \in L^\infty(Q, \mu),$$

we obtain an example of a W^*-algebra with a normal tracial state.

(2) *The algebra* HF *with the state* tr, *described in Subsect. 6.2.10, is another example.*

Recall that the triple $(Q, \mathfrak{S}, \mu)$ of Example 6.71 (1) is called a *probability space*. Therefore, some authors call a couple consisting of a W^*-algebra and a normal tracial faithful state a *non-commutative probability space*. In any case, this section is in many ways analogous to parts of Sect. 5.1, where (commutative) probability spaces were considered.

6.5.1 Measurable operators

Let us start with an abstract construction of measurable operators.

Definition 6.72 *The* measure topology *on the W^*-algebra $\mathfrak{R}$ is given by the family $V(\epsilon, \delta)$ of neighborhoods of* 0 *defined for $\epsilon, \delta > 0$ as*

$$\begin{aligned} V(\epsilon, \delta) := \{A \in \mathfrak{R} \ : \ &\|AP\| < \epsilon, \ \omega(\mathbb{1} - P) < \delta, \\ &\textit{for some orthogonal projection } P \in \mathfrak{R}\}. \end{aligned}$$

$\mathcal{M}(\mathfrak{R})$ denotes the completion of $\mathfrak{R}$ for the measure topology. Elements of $\mathcal{M}(\mathfrak{R})$ are called (abstract) measurable operators*.*

Let us now assume that $\mathfrak{R}$ is isometrically embedded in $B(\mathcal{H})$.

Definition 6.73 *A closed densely defined operator on $\mathcal{H}$ is called a* (concrete) measurable operator *iff it is affiliated to $\mathfrak{R}$ and*

$$\lim_{R\to+\infty} \omega\big(\mathbb{1}_{[R,+\infty[}(|A|)\big) = 0.$$

It can be shown that one can identify $\mathcal{M}(\mathfrak{R})$ with the set of concrete measurable operators on $\mathcal{H}$. Thus $\mathcal{M}(\mathfrak{R})$ becomes a subset of $Cl(\mathcal{H})$.

Proposition 6.74 *Let $A, B \in \mathcal{M}(\mathfrak{R})$. Then $A + B$ and AB are closable. $(A + B)^{\mathrm{cl}}$ and $(AB)^{\mathrm{cl}}$ belong again to $\mathcal{M}(\mathfrak{R})$ and do not depend on the representation of $\mathfrak{R}$.*

Using the above proposition, we endow $\mathcal{M}(\mathfrak{R})$ with the structure of a $*$-algebra. One extends ω to the subset $\mathcal{M}_+(\mathfrak{R})$ of positive operators in $\mathcal{M}(\mathfrak{R})$ by setting

$$\omega(A) := \lim_{\epsilon\to 0^+} \omega\big(A(\mathbb{1} + \epsilon A)^{-1}\big) \in [0, +\infty].$$

6.5.2 Non-commutative L^p spaces

Definition 6.75 *For $1 \le p < \infty$ one sets*

$$L^p(\mathfrak{R}, \omega) := \left\{A \in \mathcal{M}(\mathfrak{R}) \ :\ \omega\big(|A|^p\big) < \infty\right\},$$

equipped with the norm $\|A\|_p := \omega\big(|A|^p\big)^{1/p}$.

For $p = \infty$ one sets $L^\infty(\mathfrak{R}, \omega) := \mathfrak{R}$, and $\|A\|_\infty := \|A\|$.

We will often drop ω from $L^p(\mathfrak{R}, \omega)$, where it does not cause confusion. The spaces $L^p(\mathfrak{R})$ are Banach spaces with $\mathfrak{R}$ as a dense subspace.

Note that if $A \in L^1(\mathfrak{R})$, then $\mathfrak{M} \ni B \mapsto \omega(AB) \in \mathbb{C}$ is a normal functional of norm $\|A\|_1 = \omega\big(|A|\big)$. This defines an isometric identification between $L^1(\mathfrak{R})$ and $\mathfrak{R}_\#$, the space of normal functionals on $\mathfrak{R}$.

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation for the state ω. Then $L^2(\mathfrak{R})$ can be unitarily identified with the space $\mathcal{H}_\omega$, as an extension of the map

$$\mathfrak{R} \ni A \mapsto A\Omega_\omega \in \mathcal{H}_\omega. \tag{6.12}$$

We have $L^q(\mathfrak{R}) \subset L^p(\mathfrak{R})$ if $q \ge p$.

Proposition 6.76 (1) *For $A \in L^p(\mathfrak{R})$, $1 \le p \le \infty$, one has $\|A\|_p = \|A^*\|_p$. In particular, $A \mapsto A^*$ is anti-unitary on $L^2(\mathfrak{R})$.*

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(2) *The* non-commutative Hölder's inequality *holds: for all* $1 \le r, p, q \le \infty$ *with* $p^{-1} + q^{-1} = r^{-1}$, *if* $A \in L^p(\mathfrak{R})$, $B \in L^q(\mathfrak{R})$, *then* $AB \in L^r(\mathfrak{R})$ *and*

$$\|AB\|_r \le \|A\|_p \|B\|_q. \tag{6.13}$$

(3) $\|A\|_p = \sup\{\omega(AB) \ : \ B \in \mathfrak{R}, \ \|B\|_q \le 1\}, \ p^{-1} + q^{-1} = 1, \ p > 1.$

Definition 6.77 *An element* A *of* $L^p(\mathfrak{R})$ *is* positive *if it is positive as an unbounded operator on* $\mathcal{H}$. *We denote by* $L^p_+(\mathfrak{R})$ *the set of positive elements of* $L^p(\mathfrak{R})$.

For all $1 \le p \le \infty$, $\mathfrak{R}_+$ is dense in $L^p_+(\mathfrak{R})$ and the sets $L^p_+(\mathfrak{R})$ are closed in $L^p(\mathfrak{R})$.

Lemma 6.78 (1) $A \in \mathfrak{R}_+$ *iff* $\omega(AB) \ge 0, \ B \in \mathfrak{R}_+$.
(2) $A \in L^p_+(\mathfrak{R})$ *iff* $\omega(AB) \ge 0, \ B \in L^q_+(\mathfrak{R})$.

6.5.3 Operators between non-commutative L^p spaces

Let $(\mathfrak{R}_i, \omega_i)$, $i = 1, 2$, be two W^*-algebras with normal tracial faithful states.

Definition 6.79 $T \in B\big(L^2(\mathfrak{R}_1), L^2(\mathfrak{R}_2)\big)$ *is called*

(1) positivity preserving *if* $A \ge 0 \ \Rightarrow TA \ge 0$,
(2) hyper-contractive *if* T *is a contraction and there exists* $p > 2$ *such that* T *is bounded from* $L^2(\mathfrak{R}_1)$ *to* $L^p(\mathfrak{R}_2)$.

Using Lemma 6.78 we see as in the commutative case that T is positivity preserving iff T^* is.

Let $(\mathfrak{R}, \omega)$ be a W^*-algebra with a normal tracial faithful state.

Definition 6.80 $T \in B\big(L^2(\mathfrak{R})\big)$ *is called* doubly Markovian *if it is positivity preserving and* $T\mathbb{1} = T^*\mathbb{1} = \mathbb{1}$.

Theorem 6.81 *A doubly Markovian map* T *extends to a contraction on* $L^p(\mathfrak{R})$ *for all* $1 \le p \le \infty$.

Proof Using that $\pm T \le \|T\|_\infty \mathbb{1}$ and the fact that T is positivity preserving, we obtain that T is a contraction on $L^\infty(\mathfrak{R})$. Applying Prop. 6.76 (3) and the above result to T^*, we see that T is a contraction on $L^1(\mathfrak{R}, \omega)$. By the non-commutative version of Stein's interpolation theorem (see Prop. 3 of Gross (1972)), this extends to all $1 < p < \infty$. □

6.5.4 Conditional expectations on non-commutative spaces

Let $\mathfrak{R}_1$ be a W^*-sub-algebra of $\mathfrak{R}$. Let ω_1 be the restriction of ω to $\mathfrak{R}_1$. Clearly, $L^p(\mathfrak{R}_1, \omega_1)$ injects isometrically into $L^p(\mathfrak{R}, \omega)$.

Definition 6.82 *Denote by $E_{\mathfrak{R}_1}$ the orthogonal projection from $L^2(\mathfrak{R},\omega)$ onto $L^2(\mathfrak{R}_1,\omega_1)$.*

Proposition 6.83 (1) *$E_{\mathfrak{R}_1}$ uniquely extends to a contraction from $L^p(\mathfrak{R})$ into $L^p(\mathfrak{R}_1)$ for all $1 \le p \le \infty$.*

(2) *$E_{\mathfrak{R}_1}$ is doubly Markovian.*

(3) *Let $A \in L^p(\mathfrak{R})$, $B \in L^q(\mathfrak{R}_1)$, $p^{-1}+q^{-1}=1$. Then*

$$E_{\mathfrak{R}_1}(AB) = E_{\mathfrak{R}_1}(A)B, \qquad E_{\mathfrak{R}_1}(BA) = BE_{\mathfrak{R}_1}(A).$$

(4) *$E_{\mathfrak{R}_1}$ considered as an operator on $L^\infty(\mathfrak{R}) = \mathfrak{R}$ is the unique conditional expectation onto $\mathfrak{R}_1$ described in Prop. 6.48, that is, satisfying*

$$\omega(A) = \omega(E(A)), \quad A \in \mathfrak{R}.$$

6.6 Notes

A comprehensive reference to operator algebras is the three-volume monograph of Takesaki. In particular, Takesaki (1979) contains basics and Takesaki (2003) contains the modular theory. Another useful reference, aimed at applications in mathematical physics, is the two-volume monograph of Bratteli–Robinson (1987, 1996). In particular, proofs of the properties of KMS states of Subsect. 6.4.2 can be found in Bratteli–Robinson (1996).

Non-commutative probability spaces are analyzed in Takesaki (2003), following Segal (1953a,b), Kunze (1958) and Wilde (1974).