A NOTE ON AFFINE PAPPUS CONDITIONS

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Let ℓ, m, n be three mutually distinct lines in the projective plane. The (ℓ, m, n)-Pappus condition can be described as follows.

Let A, B, C, A', B', C' be any six mutually distinct points such that A, B, C lie on ℓ ; A', B', C' lie on m ; and none of these points lies on $\ell \cap m$, $m \cap n$, or $n \cap \ell$. If the points $AB' \cap BA'$ and $BC' \cap CB'$ both lie on n, then the point $AC' \cap CA'$ also lies on n . (cf. Fig. 1, omitting R)

REMARK. The dual of a Pappus configuration is called a Thomsen configuration; cf. [4, p. 134].

G. Pickert has shown in [5] that if we assume the (ℓ, m, n)-Pappus condition for a fixed pair of lines m and n and for every line ℓ which passes through neither $m \cap n$ nor one other fixed point $R \in n$, then the (ℓ, m, n)-Pappus condition holds for all choices of ℓ, m, n in the plane (Fig. 1).

If we designate n as the line at infinity, the above result contains the following affine Pappus condition as a special case:

If the (ℓ, m)-affine Pappus condition holds for a fixed line m, and for every line ℓ which is neither parallel to m nor to a given line r, $r \nparallel m$, then the (ℓ, m)-affine Pappus condition also holds for all pairs ℓ and m with $\ell \neq m$ (Fig. 2, 3).

We discuss a weaker form of the last condition in this note:

If the (ℓ, m)-affine Pappus condition holds for all pairs of lines ℓ, m such that $\ell \nparallel m$, then it also holds for all pairs ℓ, m with $\ell \parallel m$.

We shall give a proof of the last result using only incidences.

Consider a rudimentary affine plane satisfying only the axioms in [1, p. 52-53], namely:

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AXIOM 1. Two distinct points determine a line.

AXIOM 2. If P is a point and ℓ is a line, then there is a unique line through P which is parallel to ℓ.

AXIOM 3. There exist at least three mutual distinct and non-collinear points.

We restate our conditions.

CONDITION P_P (for a given point P). Let A, B, C, A', B', C', P be mutually distinct points such that P, A, B, C lie on a line ℓ, and P, A', B', C' lie on a line m; $\ell \neq m$. If $AB' \parallel BA'$ and $BC' \parallel CB'$ then $AC' \parallel CA'$.

CONDITION P_a. Let A, B, C, A', B', C' be mutually distinct points such that A, B, C, lie on a line ℓ, A', B', C' lie on a line m, $\ell \parallel m$, $\ell \neq m$. If $AB' \parallel BA'$ and $BC' \parallel CB'$, then $AC' \parallel CA'$.

Then we wish to prove the following.

THEOREM. Assume only Axioms 1, 2 and 3. Then Condition P_P for each point P implies Condition P_a.

Proof. Suppose that $CA' \nparallel AC'$ in the P_a configuration. Since AC' is not parallel to AB' or to BA', the line through C parallel to AC' will intersect BA', say at A''. As $A'' \neq A'$, A'' cannot lie on m. Since $AC' \nparallel \ell$, A'' cannot lie on ℓ. Hence C' and A'' define a line m', and $m' \nparallel \ell$. Since $AB' \parallel BA''$, the line m' will intersect AB', say at B''. Furthermore, C', B'' and A'' are mutually distinct points on m' and none of them can lie on ℓ. Hence we can apply Condition P_P to the Pappus hexagon $C'AB''CA''B$ on the intersecting lines ℓ and m'. Since $AC' \parallel CA''$ and $AB'' \parallel BA''$, we obtain that $BC' \parallel CB''$. Since $BC' \parallel CB'$ the line CB' must coincide with the line CB''. This implies that $B' = B''$ and hence that the line $C'B''$ coincides with the line m. Contradiction.

As is well-known, one can associate a commutative field with an affine geometry which satisfies initially only Axioms 1, 2, 3 and Condition P_P for each point P. The affine Desargues conditions can first be established using only incidences, as in

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[2, p.100] and [3, p.193], and the method of [1, Chapter 2] can be employed to construct the field.

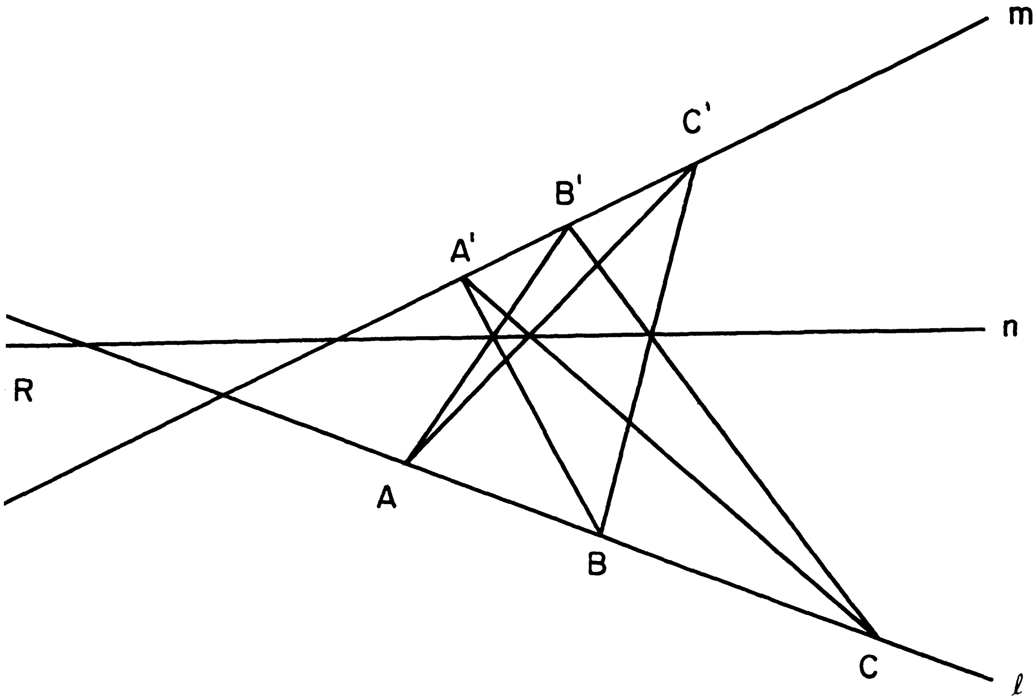

Figure 1

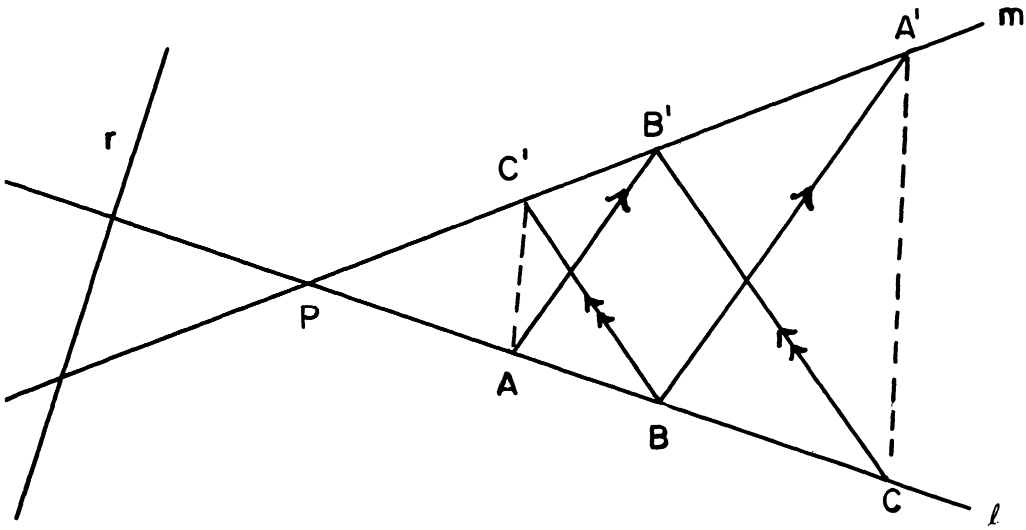

Figure 2

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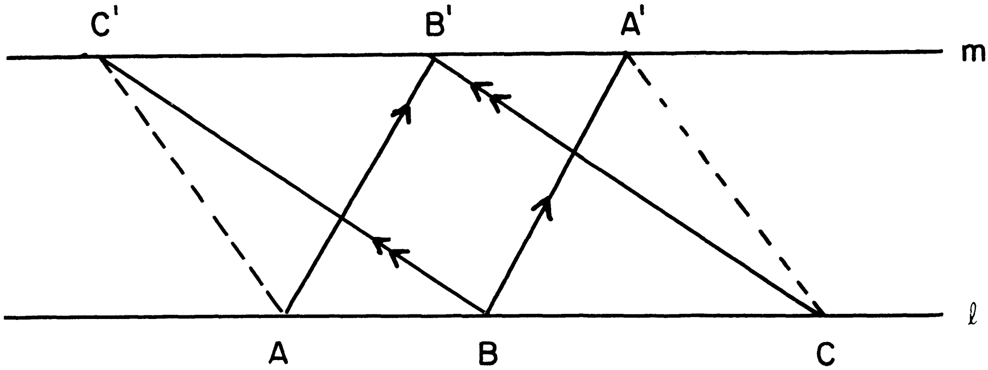

Figure 3

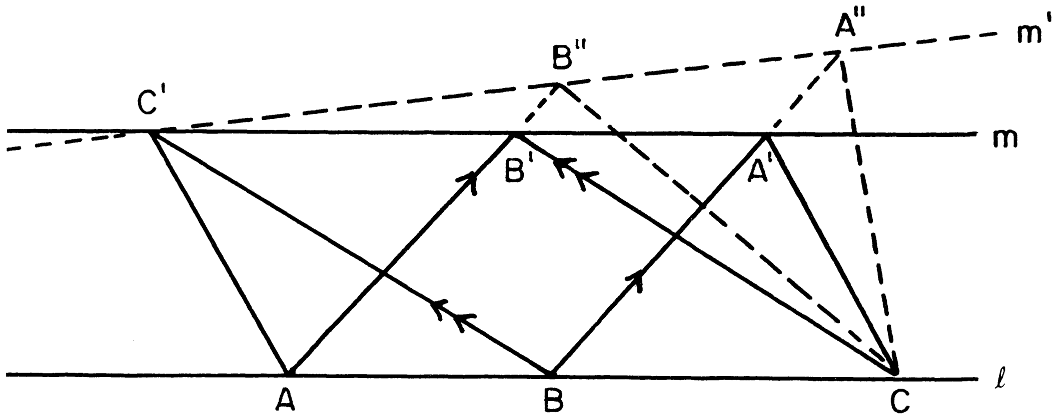

Figure 4

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REFERENCES

1. E. Artin, Geometric Algebra. Interscience (1957).

2. L.M. Blumenthal, A Modern View of Geometry. Freeman (1961).

3. H.S.M. Coxeter, Introduction to Geometry. Wiley (1961).

4. G. Pickert, Projektiven Ebenen. Springer (1955).

5. G. Pickert, Der Satz von Pappos mit Festelementen. Arch. Math., Vol. X, 1959, pp.56-61.

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