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Gabriel–Ulmer duality and Lawvere theories enriched over a general base

STEPHEN LACK
School of Computing and Mathematics, University of Western Sydney, Locked Bag 1797, Penrith South DC, NSW 1797, Australia
(*e-mail:* s.lack@uws.edu.au)

JOHN POWER
Department of Computer Science, University of Bath, Claverton Down, Bath BA2 7AY, UK
(*e-mail:* a.j.power@bath.ac.uk)

Abstract

Motivated by the search for a body of mathematical theory to support the semantics of computational effects, we first recall the relationship between Lawvere theories and monads on *Set*. We generalise that relationship from *Set* to an arbitrary locally presentable category such as *Poset* and ωCpo or functor categories such as $[Inj, Set]$ and $[Inj, \omega Cpo]$. That involves allowing the arities of Lawvere theories to be extended to being size-restricted objects of the locally presentable category. We develop a body of theory at this level of generality, in particular explaining how the relationship between generalised Lawvere theories and monads extends Gabriel–Ulmer duality.

1 Introduction

Over the 20 years since Eugenio Moggi wrote the seminal papers (Moggi 1989, 1991), the notion of monad has become a valuable tool in the study of functional-programming languages, both for call-by-value languages like *ML* and for call-by-name or call-by-need languages like *Haskell* (Benton *et al.* 2002), specifically in regard to the modelling of computational effects. Over the past 10 years, substantial progress has been made, especially in regard to the theoretical study of combining effects, by observing that almost all monads of computational interest on *Set* arise naturally from countable Lawvere theories (Plotkin & Power 2002; Hyland *et al.* 2006, 2007; Hyland & Power 2006), with the combinations of effects determined principally by the sum or tensor, sometimes the distributive tensor, of the corresponding Lawvere theories. We recall the definition and leading examples in Section 2.

Lawvere theories are a category-theoretic formulation of universal algebra for which the notion of operation is primitive (Lawvere 1963). The definition of Lawvere theory axiomatises the notion of the clone of an equational theory. Unlike equational theories, Lawvere theories are presentation independent; i.e. the category of models determines a Lawvere theory uniquely up to coherent isomorphism. Every Lawvere theory generates a monad on *Set*, generating precisely the finitary monads on

Set. The relationship between Lawvere theories, equational theories and finitary monads on *Set* is one of the deepest relationships in category theory (Hyland & Power 2007).

But *Set* is not the base category of primary interest for computation: the category ωCpo is more interesting, as it incorporates an account of recursion. The definition of monad extends routinely from base category *Set* to an arbitrary base category and hence to ωCpo, but the definition of Lawvere theory does not. To some extent, that can be resolved by appeal to the notion of an enriched Lawvere theory (Power 2000), as was used in Hyland *et al.* (2006, 2007) and Hyland & Power (2006). But enrichment is less appropriate when one wants to replace *Set* by categories such as the functor categories $[C, Set]$ or $[C, \omega Cpo]$, for instance in order to model local effects, notably local state, as investigated in O'Hearn & Tennet (1997), Plotkin & Power (2002) and Power (2006). So we seek a generalisation of the definition of Lawvere theory that applies to categories such as those cited above, with the relationship with monads respected by the generalisation.

A start on that question was made in the mathematical paper by Nishizawa & Power (2009), and in this paper, we develop it further and explain it in a computational setting. For ease of exposition, we shall ignore enrichment beyond saying that everything we write enriches without fuss, using the techniques of Kelly (1982a) and Nishizawa & Power (2009), explained in the setting of computational effects in Hyland *et al.* (2006). We provide references throughout the paper to background material expressed in the enriched setting.

The mathematical foundation of the paper is Gabriel–Ulmer duality. The axiom we require of a base category is that it is locally finitely presentable or, slightly more generally, locally countably presentable. Gabriel–Ulmer duality asserts that every locally finitely presentable category A is equivalent to the category $FL(A_f^{op}, Set)$ of finite-limit-preserving functors from A_f^{op} to *Set*, where A_f is determined by what are called the finitely presentable objects of A: these generalise the notion of finite set. The situation for countability is similar. We outline the main ideas of Gabriel–Ulmer duality in Section 3.

Based upon Gabriel–Ulmer duality, in Section 4, we recall the definition of Lawvere A-theory for a locally finitely presentable category A from Nishizawa & Power (2009) and give a new definition of model, which we prove equivalent to that of Nishizawa & Power (2009): the latter is more directly applicable to examples, but the former allows for a more elegant explanation of the relationship with monads as supported by Gabriel–Ulmer duality. We explain the relationship between our definitions and the building blocks of Gabriel–Ulmer duality in Section 5. The central result of the paper, in Section 6, is Corollary 23, which expresses the equivalence between Lawvere A-theories and finitary monads on A as a lifting of Gabriel–Ulmer duality over A.

We extend Gabriel–Ulmer duality to examine change of base in Section 7: one seeks not only a characterisation of the monads with countable rank on a category such as ωCpo but also a relationship between such monads on ωCpo, equivalently countable Lawvere ωCpo-theories, and monads with countable rank on *Set*, equivalently ordinary countable Lawvere theories.

2 Ordinary Lawvere theories

In this section, we recall the notion of Lawvere theory, first defined in Lawvere's thesis (1963), its relationship with monads on Set and its relevance to functional programming with computational effects. The examples are taken from Hyland *et al.* (2006), which in turn was motivated by the desire to refine and develop Moggi's modelling of computational effects by monads in Moggi (1989, 1991).

Definition 1
A *Lawvere theory* consists of a small category L with finite products together with a strict finite-product-preserving identity-on-objects functor $I : Nat^{op} \longrightarrow L$, where Nat is the category of all natural numbers and maps between them (Barr & Wells 1985, 1990). A *model* of a Lawvere theory L in a category C with finite products is a finite-product-preserving functor from L to C.

Implicit in Definition 1 is the fact that the objects of L are exactly the natural numbers. A map of Lawvere theories from $I : Nat^{op} \longrightarrow L$ to $I : Nat^{op} \longrightarrow L'$ is a functor from L to L' that respects I. Any such functor is necessarily strictly finite product preserving and identity-on-objects. With the usual composition of functors, this yields a category **Law**.

Note the distinction in the definition between strict preservation, as used in defining a Lawvere theory, and preservation, as used in defining a model. The latter means that finite products need only be preserved up to coherent isomorphism rather than equality. The distinction is essential, as on one hand, the objects of L are exactly the natural numbers, but on the other, if we demand strict preservation in the definition of model, we would eliminate almost all examples of interest (Power 1995).

The definition of Lawvere theory provides a category-theoretic formulation of universal algebra, with the notion of operation taken as primitive: a map in L from n to m is to be understood as being given by m operations of arity n. Unlike the notion of equational theory, the concept of Lawvere theory is presentation independent; i.e. if a pair of Lawvere theories have equivalent categories of models, the two theories are isomorphic.

The definition of model extends to the definition of the category $Mod(L, C)$ of models of L in C: maps of models are defined to be natural transformations. Note that naturality forces maps of models to respect the finite product structure in the definition of model.

For any Lawvere theory L and any locally finitely presentable category C (characterised in Section 3), the functor $ev_1 : Mod(L, C) \longrightarrow C$ has a left adjoint, given by the free model construction, inducing a monad T_L on C. Thus in particular, every Lawvere theory L determines a monad T_L on Set.

There is a converse construction.

Proposition 2
Given any monad T on Set, if one factorises the canonical composite

$$Nat \longrightarrow Set \longrightarrow Kl(T)$$

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where $Kl(T)$ denotes the Kleisli category of T, as an identity-on-objects functor followed by a fully faithful functor, one obtains (the opposite of) a Lawvere theory

$$I : Nat \longrightarrow L_T^{op}$$

Proof
By construction, the functor I is identity-on-objects. Moreover, the canonical functor

$$Set \longrightarrow Kl(T)$$

strictly preserves colimits and in particular all coproducts. Restricting to Nat and factorising as above ensures that I strictly preserves all finite coproducts. Applying $(-)^{op}$, we are done. □

If one started with a Lawvere theory L, constructed T_L and then constructed L_{T_L}, one would recover L. But the converse is not true: starting with a monad T, one does not in general recover T for size reasons: one recovers T if and only if T is finitary, i.e. if and only if T preserves filtered colimits: these are a special form of colimit, for which a precise understanding is not essential here. Putting this together, with care for coherence detail, yields the following (Power 2000).

Theorem 3
The constructions of a monad T_L on Set from a Lawvere theory L together with that of a Lawvere theory L_T from a monad T on Set extend canonically to an equivalence of categories:

$$\mathbf{Law} \simeq \mathbf{Mnd_f}$$

where $\mathbf{Mnd}_f$ is the category of finitary monads on Set. Moreover, for any Lawvere theory L, the category $Mod(L, Set)$ is coherently equivalent to $T_L\text{-}Alg$.

The usual way in which one obtains Lawvere theories is by means of sketches or, equivalently, equational theories, with the Lawvere theory given freely on the sketch: Barr & Wells (1990) treat sketches in loving detail and give a range of examples of both sketches and Lawvere theories. The Lawvere theory is an axiomatisation of the notion of the clone of an equational theory and equivalently of a sketch.

Example 4
The Lawvere theory L_E for exceptions is the free Lawvere theory generated by an E-indexed family of nullary operations with no equations. The monad on Set induced by L_E is $T_E = - + E$. More generally, if C is any category with finite products and all sums, then $Mod(L_E, C)$ is equivalent to the category of algebras for the monad $- + \underline{E}$, where $\underline{E}$ is the E-fold coproduct of copies of 1, i.e. $\coprod_E 1$.

Interactive input/output works similar to exceptions (Hyland *et al.* 2006); so we omit details. For the next example, we use the evident generalisation of the notion of Lawvere theory to countable Lawvere theory as used in Hyland *et al.* (2006): it simply allows us to use countable arities.

Example 5
Let Loc be a finite set of locations, and let V be a countable set of values. The countable Lawvere theory L_{GS} for side effects, sometimes called global state, where $S = V^{Loc}$, is the free countable Lawvere theory generated by the operations $lookup : V \longrightarrow Loc$ and $update : 1 \longrightarrow Loc \times V$ subject to the seven natural equations listed in Plotkin & Power (2002), four of them specifying interaction equations for *lookup* and *update* and three of them specifying commutation equations. Our presentation of the operations here is in terms of generic effects, corresponding to the evident functions of the form $Loc \longrightarrow (S \times V)^S$ and $Loc \times V \longrightarrow S^S$ respectively (Hyland *et al.* 2006): to give a generic effect is equivalent, via the Yoneda embedding, to giving an operation (Plotkin & Power 2003). It is shown in Plotkin & Power (2002) that L_{GS} induces the side-effects monad. More generally, if C is any category with countable products and copowers, then, slightly generalising the result in Plotkin & Power (2002), $Mod(L_{GS}, C)$ is equivalent to the category of algebras for the monad $(S \times -)^S$, where we write $(S \times -)$ for the S-fold coproduct $\coprod_S -$ and $(-)^S$ for the S-fold product $\prod_S -$.

Example 6
The Lawvere theory L_N for (binary) non-determinism is the Lawvere theory freely generated by a binary operation $\vee : 2 \longrightarrow 1$ subject to equations for associativity, commutativity and idempotence, i.e. the Lawvere theory for a semilattice. The induced monad on Set is the finite non-empty subset monad, $\mathscr{F}^+$.

Example 7
The Lawvere theory L_P for probabilistic non-determinism is that freely generated by $[0,1]$-many binary operations $+_r : 2 \longrightarrow 1$ subject to natural equations generalising associativity, commutativity and idempotence (Heckmann 1994). The induced monad on Set is the distributions with finite support monad $\mathscr{D}_f$.

For a non-example, consider the monad $(-)_\perp$ on $Poset$ or ωCpo for the addition of a least element. The monad $(-)_\perp$ does not arise from an ordinary Lawvere theory, as one cannot express as an equation the assertion that for all x, one has $\perp \leqslant x$. So one needs to go beyond ordinary Lawvere theories in order to include such monads on categories such as $Poset$ or ωCpo. In Hyland *et al.* (2006), enriched Lawvere theories were used, with enrichment in $Poset$ or ωCpo. In this paper, we propose a different generalisation that works similarly well for $(-)_\perp$ and better than enriched Lawvere theories for base categories such as functor categories $[C, Set]$ or $[C, \omega Cpo]$ as used to model local state (O'Hearn & Tennent 1997; Plotkin & Power 2002; Power 2006).

3 Gabriel–Ulmer duality

The definition of a Lawvere theory, Definition 1, involves the category Nat of natural numbers, with natural numbers forming the possible arities of an operation. So, if we are to axiomatise the definition, we need to be able to speak meaningfully of the finite objects of a category A, as the finite objects of A are the possible arities for an

A-based Lawvere theory. That problem was definitively resolved several decades ago by the notion of a locally finitely presentable category and the theory of Gabriel–Ulmer duality (Kelly 1982b; Adámek & Rosický 1994): the appropriate objects are called the *finitely presentable* objects of A.

The definition of a locally finitely presentable category is quite complex (Adámek & Rosický 1994). But the central result of Gabriel–Ulmer duality characterises the notion in simple terms as follows (Kelly 1982b; Adámek & Rosický 1994): Let **FL** denote the 2-category of all small categories with finite limits, finite-limit-preserving functors and all natural transformations. Given a category C with finite limits, let $FL(C, Set)$ denote the full subcategory of the functor category $[C, Set]$ determined by those functors that preserve finite limits. And let $\mathbf{LocPres}_f$ denote the 2-category of locally finitely presentable categories, filtered colimit preserving functors that have left adjoints and natural transformations.

Theorem 8 (*Gabriel–Ulmer duality*)
The construction that sends a small category C with finite limits to the category $FL(C, Set)$ extends canonically to a bi-equivalence of 2-categories:

$$\mathbf{FL} \sim \mathbf{LocPres}_f^{op}$$

So the study of locally finitely presentable categories is equivalent to the study of categories of the form $FL(C, Set)$, where C is a small category with finite limits.

Examples of locally finitely presentable categories in the computer science literature include Set, Set^k, $Poset$, Cat and all functor categories $[C, A]$ for which C is a small category and A is a locally finitely presentable category (Barr & Wells 1990; Robinson 2002). Gabriel–Ulmer duality extends routinely from finiteness to countability, allowing examples to include ωCpo and functor categories of the form $[C, \omega Cpo]$ and hence the categories of primary interest for recursion and local effects. Papers such as Hyland *et al.* (2006) were written primarily in terms of countability, whereas the relevant mathematical literature is usually phrased in terms of finiteness.

The converse construction for Theorem 8 sends a locally finitely presentable category A to a skeleton of the opposite of the full subcategory of finitely presentable objects of A. We shall duly write A_f for a skeleton of the full subcategory of A given by the finitely presentable objects of A, and let $\iota : A_f \longrightarrow A$ denote the inclusion functor. For example, the finitely presentable objects of Set are the finite sets, and so Set_f is Nat. The finitely presentable objects of $Poset$ are the finite posets, and so $Poset_f$ contains one isomorphic copy of each finite poset, with maps given by all maps of posets. Extending to the countable setting, the countably presentable objects of ωCpo include all countable ω-cpo's and also uncountable ω-cpo's that have a countable presentation. For more details on the computing literature, see Robinson (2002). In practice, one almost only ever needs to know some of the countably presentable objects of a locally countably presentable category, e.g. knowing that the countable ω-cpo's are among the countably presentable ones. A central fact about A_f is as follows.

Proposition 9
For any locally finitely presentable category A, the category A_f has all finite colimits, and they are preserved by the inclusion $\iota : A_f \longrightarrow A$.

We denote the composite functor

$$A \xrightarrow{\;Y\;} [A^{op}, Set] \xrightarrow{[\iota^{op}, Set]} [A_f^{op}, Set]$$

by $\tilde{\iota}$, where Y is the Yoneda embedding. For example, Set_f is *Nat*, and the functor $\tilde{\iota}$ sends a set X to the functor $Set(\iota -, X)$, i.e. to $X^{(-)}$.

Since ι preserves all finite colimits, and representable functors preserve limits, $\tilde{\iota}$ factors through $FL(A_f^{op}, Set)$. So we sometimes consider $\tilde{\iota}$ as a functor from A to $FL(A_f^{op}, Set)$. Unwinding the converse construction for Theorem 8, one has the following.

Theorem 10
For any locally finitely presentable category A, the functor $\tilde{\iota}$ induces an equivalence of categories:

$$A \simeq FL(A_f^{op}, Set)$$

We shall return to Theorem 10 when we discuss models.

4 Lawvere A-theories

In generalising Definition 1, a tentative definition of an A-based Lawvere theory might be a small category L with finite products together with a strict finite-product-preserving identity-on-objects functor $I : A_f^{op} \longrightarrow L$. But such a definition would not be delicate enough to allow us to generalise the relationship between Lawvere theories and monads as the following example illustrates.

Example 11
Let $A = Poset$. The category $Poset_f$ is (equivalent to) the full subcategory of *Poset* determined by the finite posets. Given a monad T on *Poset*, the canonical composite

$$Poset_f \longrightarrow Poset \longrightarrow Kl(T)$$

preserves all finite colimits, and so the restriction

$$I : Poset_f \longrightarrow L_T^{op}$$

strictly preserves all finite colimits. But that is a strictly stronger condition than that of strict preservation of finite coproducts. For example consider the following push-out in $Poset_f$:

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{2} \\ \downarrow & & \downarrow \\ \mathbf{2} & \longrightarrow & \mathbf{3} \end{array}$$

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where **2** denotes the poset with two elements, with one less than the other, i.e. Sierpinski space; **3** is similar but with three elements and with the evident maps between the various posets. Preservation of this push-out is not implied by preservation of finite coproducts, but if we are to axiomatise the relationship between Lawvere theories and monads so that it extends to $Poset$, this push-out must be preserved in the definition of a $Poset$-based Lawvere theory, as every $Poset$-based Lawvere theory must arise from a monad on $Poset$.

Guided by this example and relying upon Proposition 9, we make the following definition. The definition of Lawvere A-theory we give here is identical to that given in Nishizawa & Power (2009), but the definition of model we give here does not, *a priori*, agree with that given in Nishizawa & Power (2009): later, we shall prove that the two definitions of model agree up to coherent isomorphism.

Definition 12
Given a locally finitely presentable category A, a *Lawvere A-theory* is a small category L together with a strict finite-limit-preserving identity-on-objects functor $I : A_f^{op} \to L$. A *model* of a Lawvere A-theory $I : A_f^{op} \longrightarrow L$ is a functor $M : L \longrightarrow Set$ for which the composite MI preserves finite limits.

The restriction of models to be Set-valued functors in Definition 12 while models were taken in any category C with finite products in Definition 1 is essentially a convenience for exposition. We discuss the general situation at the end of the paper.

A map of Lawvere A-theories from L to L' is an identity-on-objects functor from L to L' that commutes with the functors from A_f^{op}. Together with the usual composition of functors, Lawvere A-theories and their maps yield a category we denote by $\mathbf{Law}_A$.

The definition of model routinely extends to the definition of the category $Mod(L)$ of models of L, and Theorem 10 induces a canonical functor

$$U_L : Mod(L) \longrightarrow A$$

Compare Definition 1 with Definition 12: the definition of ordinary Lawvere theory required that L have finite products and that the functor from Nat^{op} to L strictly preserve finite products, whereas here, we have asked for strict preservation of finite limits but have made no further assumption of existence of any kind of limits in L. So the following result is not entirely routine.

Proposition 13 (*Nishizawa & Power 2009*)
An ordinary Lawvere theory is a Lawvere Set-theory and conversely.

Proof
Let L be an ordinary Lawvere theory. It corresponds to a finitary monad T on Set, and L is isomorphic to the restriction of $Kl(T)^{op}$ to the natural numbers, with the functor $I : Nat^{op} \to L$ given by the restriction of the canonical functor $Set \longrightarrow Kl(T)$. So $I : Nat^{op} \to L$ strictly preserves all finite limits of Nat, as the corresponding finite colimits are preserved both by the inclusion into Set and by the canonical functor into $Kl(T)$. So every ordinary Lawvere theory is a Lawvere

Set-theory. The converse is easier: L has precisely the objects of A_f^{op}, with I strictly preserving all finite limits; so L has all finite products, and they are preserved by I, although L need not have pullbacks for example. □

Proposition 14
The definitions of a model of an ordinary Lawvere theory and of a Lawvere Set-theory agree.

Proof
Set_f^{op} is both the free category with finite products on 1 and the free category with finite limits on 1 (Kelly 1982b; Adámek & Rosický 1994). So all finite-product-preserving functors out of Set_f^{op} preserve all finite limits. The result now follows routinely from Definition 12. □

A definition of model of a Lawvere A-theory appeared in Nishizawa & Power (2009), but it was quite complex, not flowing directly from the definition of Lawvere A-theory. We now show that our definition agrees with it up to coherent isomorphism.

Proposition 15
Given a Lawvere A-theory $I : A_f^{op} \longrightarrow L$, the category $Mod(L)$ is given, up to coherent equivalence, by the pullback in the category Cat of locally small categories:

$$\begin{array}{ccc} P & \xrightarrow{P_L} & [L, Set] \\ {\scriptstyle U_L}\downarrow & \lrcorner & \downarrow{\scriptstyle [I, Set]} \\ A & \xrightarrow[\tilde{i}]{} & [A_f^{op}, Set] \end{array}$$

Proof
By Theorem 10, since A is locally finitely presentable, it is equivalent to $FL(A_f^{op}, Set)$ coherently with respect to the inclusion i. Moreover, the functor $[I, Set]$ admits the lifting of isomorphisms; i.e. for any functor $M : L \longrightarrow Set$ together with a natural isomorphism of the form $MI \cong M' : A_f^{op} \longrightarrow Set$, the domain of M' and the natural isomorphism extend from A_f^{op} to L. So, to give an object of the pullback P is equivalent to giving a functor $M : L \longrightarrow Set$ such that the composite of M with $I : A^{op} \longrightarrow L$ lies in $A \simeq FL(A_f^{op}, Set)$, i.e. such that MI preserves finite limits. But that is equivalent to giving a model of L. And that extends routinely to maps of models. □

By Proposition 15, up to coherent isomorphism, a model of L consists of an object X of A together with data and axioms arising from those maps in L that are not already in A_f^{op}. In practice, one usually uses this characterisation of the definition of model, but for abstract theory, the definition as stated here, i.e. Definition 12, is typically more helpful.

Example 16
Let $A = Poset$. So A_f is (up to equivalence) the full subcategory of $Poset$ determined by the finite posets. Let **0** denote the empty poset, **1** denote the one element poset and **2** denote Sierpinski space.

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Consider the Lawvere *Poset*-theory $L_\perp$ freely generated by two maps

$$\mathbf{0} \longrightarrow \mathbf{1} \qquad\qquad \mathbf{1} \longrightarrow \mathbf{2}$$

subject to commutativity of the following two diagrams:

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{2} \\ \downarrow & & \downarrow{\scriptstyle 0} \\ \mathbf{0} & \longrightarrow & \mathbf{1} \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{2} \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle 1} \\ & & \mathbf{1} \end{array}$$

where the horizontal maps are the generating maps of $L_\perp$; the left-hand vertical map is the map in $Poset_f^{op}$ determined by the unique map in $Poset_f$ from $\mathbf{0}$ to $\mathbf{1}$; and the other two vertical maps are determined by the two maps in $Poset_f$ from $\mathbf{1}$ to $\mathbf{2}$ that choose the first and second element of $\mathbf{2}$, labelled 0 and 1 respectively.

By Proposition 15, a model of any Lawvere *Poset*-theory L consists of a poset P and a functor $M : L \longrightarrow Set$ such that the composite functor $MI : Poset_f^{op} \longrightarrow Set$ is $Poset(\iota-, P)$. So M must send $\mathbf{0}$ to the one element set 1; it must send $\mathbf{1}$ to the set of elements of P; and it must send $\mathbf{2}$ to the carrier of the poset $P^{\leqslant}$ of pairs (x, y) of elements of P for which $x \leqslant y$, ordered pointwise.

So, in particular, a model of $L_\perp$ consists of a poset P with maps of posets

$$1 \longrightarrow P \qquad\qquad P \longrightarrow P^{\leqslant}$$

subject to commutativity of the following two diagrams:

$$\begin{array}{ccc} P & \longrightarrow & P^{\leqslant} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ 1 & \longrightarrow & P \end{array} \qquad\qquad \begin{array}{ccc} P & \longrightarrow & P^{\leqslant} \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \pi_1} \\ & & P \end{array}$$

where the horizontal maps are determined by the generating maps of $L_\perp$, and the other maps are determined by the structure of the category *Poset*.

The commutativity of the two diagrams implies that the map

$$P \longrightarrow P^{\leqslant}$$

is fully determined by the other data: it must send x to the pair $(\perp, x)$, where $\perp$ is the image of the map $1 \longrightarrow P$. So, for every element x of P, the two commutativities imply that $\perp \leqslant x$.

Thus a model of $L_\perp$ consists of a poset P with a least element $\perp$. It will follow that $L_\perp$ generates the monad $T_\perp$ on *Poset* for partiality.

Example 16 is, by construction, an example of a Lawvere *Poset*-theory. The various examples of ordinary Lawvere theories of Section 2 systematically extend to become Lawvere *Poset*-theories too. One can see that this is true by considering the monads on *Poset* generated by the various examples; then directly by observing or using the equivalence between finitary Lawvere *Poset*-theories and finitary monads

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on $Poset$ we shall soon describe that every finite set is a finite poset, and so one can regard the generating operations and equations of each example of an ordinary Lawvere theory as generating operations and equations of a Lawvere $Poset$-theory.

Example 17

Let $A = [Inj, Set]$. Then A is a locally finitely presentable category (see Kelly 1982a), used as the base category for modelling local state by O'Hearn & Tennent (1997) and then by Plotkin & Power (Plotkin & Power 2002; Power 2006). The study of state inherently involves countability, as one's set V of values is typically countable. So in modelling local state, we need to generalise from local finite presentability to local countable presentability. But A, being locally finitely presentable, is necessarily locally countably presentable, and our general analysis extends routinely to countability.

The category A_c, which we define to be a skeleton of the full subcategory of A given by the countably presentable objects of A, is given, for the case of $A = [Inj, Set]$, by the closure under countable colimits of the full subcategory of $[Inj, Set]$ given by the representable functors. That may be calculated to be the full subcategory of $[Inj, Set]$ given by $[Inj, Set_c]$, i.e. those functors from Inj to Set whose values are countable sets.

In particular, for any countable set V, the constant functor at V, which, by mild overloading of notation, we denote by V, is countably presentable and so lies in A_c. The functor $L = Inj(1, -) : Inj \longrightarrow Set$, being representable, also lies in A_c, as does the product $L \times V$. It follows from the general theory of Kelly (1982a) that the definition in Plotkin & Power (2002) of a category denoted in that paper as $LS([I, Set])$ systematically yields a presentation of a Lawvere $[Inj, Set]$-theory L_{LS} such that the category $Mod(L_{LS})$ is the category of algebras for the monad T_{LS} for local state on $[Inj, Set]$.

It is not clear yet how best one can describe a denotational semantics for local state: a monad for local state has existed for some time; one of the main motivations of Plotkin & Power (2002) was the observation that if one starts with operations and equations, local state can be seen semantically to extend global state; in Power (2006), that was taken further by the introduction of a notion of indexed Lawvere theory; and the work in this paper suggests a still further perspective, dispensing with the double enrichment of Plotkin & Power (2002) and without the explicit indexing of Power (2006).

We have not yet developed an account of the various ways to combine Lawvere A-theories, but the sum of theories certainly exists, as $\mathbf{Law}_A$ is cocomplete, allowing the theory of Hyland *et al.* (2006) to extend routinely (see Lüth and Ghani, 2002 for an explanation of the value of the sum in functional programming). Extending the tensor, analysed in Hyland *et al.* (2006), and the distributive tensor, analysed in Hyland & Power (2006), will be more complex.

5 Preservation of finite limits

There is a delicate relationship between Definition 12 and the notions of existence and preservation of finite limits. Suppose C is a small category with finite limits, D is

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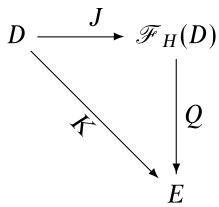
a small category, and $H : C \longrightarrow D$ preserves finite limits but with D not necessarily having all finite limits. One can speak of the free completion $\mathscr{F}_H(D)$ of D under finite limits that respects the finite limits of C (Kelly 1982b). By definition, the category $\mathscr{F}_H(D)$ has finite limits, and there is a canonical functor $J : D \longrightarrow \mathscr{F}_H(D)$ for which the composite $JH : C \longrightarrow \mathscr{F}_H(D)$ preserves finite limits, and it is the universal construct; i.e for any small category E with finite limits and any functor $K : D \longrightarrow E$ for which the composite KH preserves finite limits, there is a finite-limit-preserving functor $Q : \mathscr{F}_H(D) \longrightarrow E$ making the triangle



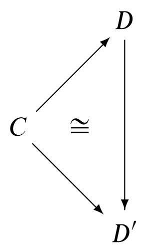
commute, unique up to coherent isomorphism.

It follows from Definition 12 that for any Lawvere A-theory $I : A_f^{op} \longrightarrow L$, one can characterise $Mod(L)$ by observing that composition with $J : L \longrightarrow \mathscr{F}_I(L)$ yields an equivalence of categories between $Mod(L)$ and $FL(\mathscr{F}_I(L), Set)$).

But **FL** is one side of Gabriel–Ulmer duality, and with a little effort, we can extend the above observation into a relationship between the category $\mathbf{Law}_A$ of all Lawvere A-theories and the 2-category **FL** and then use Gabriel–Ulmer duality to explain the relationship between models of Lawvere A-theories and finitary monads on A.

The details require 2-categorical care. Objects of a category are often isomorphic to each other without being equal to each other, and so functors between categories are often naturally isomorphic to each other without being equal to each other. So when one considers a 2-category such as **FL**, one usually needs systematically to relax equalities to become isomorphisms, isomorphisms to become equivalences, functors to become pseudo-functors and so on; see Power (1995) for further discussion of this in a computational setting.

In particular, a pseudo-slice 2-category is the natural generalisation of the notion of slice category. Specifically, given a small category C with finite limits, the pseudo-slice 2-category $C/\!\!/\mathbf{FL}$ has objects given by finite-limit-preserving functors with domain C, with arrows given by triangles of the form



consisting of a finite-limit-preserving functor from D to D' and a natural isomorphism between the two functors from C to D': in particular, note that the diagram need not commute, and the isomorphism inside it is part of the data. The 2-cells of the pseudo-slice 2-category $C /\!\!/ \mathbf{FL}$ are given by natural transformations that respect the isomorphisms in the respective triangles. Using that definition and systematically relaxing the notion of functor between categories to pseudo-functor between 2-categories, similar to natural transformations, we have the following.

Theorem 18
The category $\mathbf{Law}_A$ is a pseudo-coreflective subcategory of the pseudo-slice 2-category $A_f^{op} /\!\!/ \mathbf{FL}$

$$\mathbf{Law}_A \longrightarrow A_f^{op} /\!\!/ \mathbf{FL}$$

where a Lawvere A-theory $I : A_f^{op} \longrightarrow L$ is sent to the free completion $\mathscr{F}_I(L)$ of L under finite limits that respects the finite limits of A_f^{op}, and the pseudo-coreflection sends a finite-limit-preserving functor $A_f^{op} \longrightarrow C$ to its (identity-on-objects, fully faithful) factorisation.

In general, it is not easy to give a concrete characterisation of the free completion $\mathscr{F}_I(L)$ of a Lawvere A-theory under finite limits. But we can give a general description of $\mathscr{F}_I(L)$, albeit not a concrete one. It follows from Proposition 15 that for any Lawvere A-theory L, the Yoneda embedding restricts to a fully faithful functor

$$Y : L^{op} \longrightarrow Mod(L)$$

It then follows from Gabriel–Ulmer duality that $\mathscr{F}_I(L)^{op}$ is the full subcategory of $Mod(L)$ given by closing L^{op} under finite colimits in $Mod(L)$. Colimits in $Mod(L)$ are generally awkward to calculate, although filtered colimits are easy (see Theorem 20). But we do not need a concrete description of $\mathscr{F}_I(L)$ anyway; we only ever use its defining universal property.

As an indication of how the paper is to develop from here, compare Theorem 18 with the following.

Theorem 19
The category $\mathbf{Mnd}_f(A)$ of finitary monads on a locally finitely presentable category A is a pseudo-coreflective subcategory of $(\mathbf{LocPres}_f /\!\!/ A)^{op}$

$$\mathbf{Mnd}_f(A) \longrightarrow (\mathbf{LocPres}_f /\!\!/ A)^{op}$$

where a finitary monad T is sent to the forgetful functor $T\text{-}Alg \longrightarrow A$, and a filtered colimit preserving functor $G : B \longrightarrow A$ with left adjoint F is sent to the monad GF.

6 Theories and monads

We now work towards relating Lawvere A-theories with monads on A, extending the main result of Nishizawa & Power (2009) by explaining it in the light of Theorems 18 and 19.

Theorem 20
For any Lawvere A-theory $I : A_f^{op} \longrightarrow L$, the category $Mod(L)$ is locally finitely presentable, and the functor $U_L : Mod(L) \longrightarrow A$ is a map of locally finitely presentable categories.

Proof
Let $\mathscr{F}_I(L)$ denote the free completion of L under finite limits that respects the finite limits of A_f^{op} (cf. Theorem 18). It follows immediately from the universal property of $\mathscr{F}_I(L)$ that $Mod(L)$ is equivalent to $FL(\mathscr{F}_I(L), Set)$, which, by Theorem 8, is locally finitely presentable. Moreover, U_L is determined by composition with the canonical composite functor

$$A_f^{op} \longrightarrow L \longrightarrow \mathscr{F}_I(L)$$

which preserves finite limits by construction. So, by a further application of Theorem 8, U_L is a map of locally finitely presentable categories. □

Theorem 20 is fundamental, yielding a string of corollaries. The proof implies a little more than the theorem as stated. Specifically, it yields the following.

Corollary 21
For any Lawvere A-theory $I : A_f^{op} \longrightarrow L$, the two vertical functors U_L and $[I, Set]$ in the diagram of Proposition 15 have left adjoints, yielding a square that is commutative up to natural isomorphism as follows:

$$\begin{array}{ccc} Mod(L) & \xrightarrow{P_L} & [L, Set] \\ {\scriptstyle F_L}\uparrow & \cong & \uparrow{\scriptstyle Lan_I} \\ A & \xrightarrow[\tilde{\imath}]{} & [A_f^{op}, Set] \end{array}$$

where Lan_I denotes the left Kan extension along the functor I (Kelly 1982a).

Corollary 22
For any Lawvere A-theory $I : A_f^{op} \longrightarrow L$, the functor $U_L : Mod(L) \longrightarrow A$ is finitarily monadic.

Proof
By Theorem 20, the functor U_L is finitary and has a left adjoint. Let f, g be a U_L-split coequaliser pair in $Mod(L)$. Since $[L, Set]$ is cocomplete, $P_L f$ and $P_L g$ have a coequaliser, and the coequaliser can be chosen so that it is strictly preserved by $[I, Set]$. Since a split coequaliser of $U_L f$ and $U_L g$ is also preserved by $\tilde{\imath}$, f and g have a coequaliser in $Mod(L)$, and U_L strictly preserves it. So by Beck's monadicity theorem (Barr & Wells 1985), U_L is monadic. □

Let T_L be the finitary monad on A induced by L. The construction of T_L from L extends routinely to a functor

$$T_{-} : \mathbf{Law}_A \longrightarrow \mathbf{Mnd}_f(A)$$

We can combine that functor with the pseudo-functors in Theorems 8, 18 and 19 as follows.

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Corollary 23
The diagram

$$\begin{array}{ccc} \mathbf{Law}_A & \xrightarrow{T_-} & \mathbf{Mnd}_f(A) \\ {\scriptstyle \mathscr{F}_I(-)}\big\downarrow & & \big\downarrow{\scriptstyle (-)\text{-}Alg} \\ A_f^{op}/\!\!/\mathbf{FL} & \longrightarrow & (\mathbf{LocPres}_f/\!\!/A)^{op} \end{array}$$

commutes.

Gabriel–Ulmer duality, Theorem 8, asserts that the bottom line of the diagram is a bi-equivalence of 2-categories. The central theorem of Nishizawa & Power (2009) asserts that the top line is an equivalence of categories. The main line of results goes as follows.

Corollary 24
For any Lawvere A-theory $I : A_f^{op} \longrightarrow L$, one recovers $I^{op} : A_f \longrightarrow L^{op}$ from F_L as the (identity-on-objects, fully faithful) factorisation of $F_L \circ \iota$.

$$\begin{array}{ccc} L^{op} & \xrightarrow{I'} & Mod(L) \\ {\scriptstyle I^{op}}\big\uparrow & & \big\uparrow{\scriptstyle F_L} \\ A_f & \xrightarrow[\iota]{} & A \end{array}$$

Proof
For any finite-limit-preserving functor $H : C \longrightarrow D$, Gabriel–Ulmer duality, Theorem 8, asserts that H is the restriction of $F : FL(C, Set) \longrightarrow FL(D, Set)$, where F is the left adjoint to the functor

$$FL(H, Set) : FL(D, Set) \longrightarrow FL(C, Set)$$

given by composition with H. Considering the special case in which $C = A_f^{op}$, $D = \mathscr{F}_I(L)$ and H is the canonical composite, it follows from Corollary 21 that the diagram commutes if I' is taken to be the Yoneda embedding regarded as having codomain in $Mod(L)$. Fully faithfulness of I' follows from fully faithfulness of the Yoneda embedding. $\square$

So for an arbitrary finitary monad T on A, define (K_T, I_T, ι_T) by taking the (identity-on-objects, fully faithful) factorisation of $F^T \circ \iota$:

$$\begin{array}{ccc} K_T & \xrightarrow{\iota_T} & Kl(T) \\ {\scriptstyle I_T}\big\uparrow & & \big\uparrow{\scriptstyle F^T} \\ A_f & \xrightarrow[\iota]{} & A \end{array}$$

Since ι and F^T preserve finite colimits and ι_T reflects finite colimits, I_T is an identity-on-objects strict finite-colimit-preserving functor. So we define L_T to be K_T^{op}.

Theorem 25 (Nishizawa & Power 2009)
For a finitary monad T on A, let $F^T \dashv G^T$ be the canonical adjunction between the Eilenberg–Moore category T-Alg and A, and let Q^T send a T-algebra α to $T\text{-}Alg(\iota_T -, \alpha)$. Then, if we allow Q^T to be replaced by a canonically isomorphic functor, the following square yields a pullback:

$$\begin{array}{ccc} T\text{-}Alg & \xrightarrow{Q^T} & [L_T, Set] \\ {\scriptstyle G^T}\downarrow & \lrcorner & \downarrow{\scriptstyle [I_T^{op}, Set]} \\ A & \xrightarrow[\tilde{\iota}]{} & [A_f^{op}, Set] \end{array}$$

Corollary 26
The construction of T_L from an arbitrary Lawvere A-theory L and that of L from an arbitrary finitary monad T on A extend canonically to an equivalence of categories $\mathbf{Law}_A \simeq \mathbf{Mnd}_f(A)$. Moreover, the categories $Mod(L)$ and T_L-Alg are coherently equivalent.

Proof
By Theorem 25, $T \cong T_{L_T}$ for an arbitrary finitary monad T on A. Conversely, given an arbitrary Lawvere A-theory L, the Lawvere A-theory L_{T_L} is defined to be the (identity-on-objects, fully faithful) factorisation of $F^{T_L} \circ \iota \colon A_f \to T_L\text{-}Alg$. By Corollary 24 and since $Mod(L) \simeq T_L\text{-}Alg$, this factorisation agrees with L, and so L_{T_L} is isomorphic to L. The two constructions routinely extend to an equivalence of categories. □

7 Change of base

In this section, further developing our axiomatisation and extension of the relationship between ordinary Lawvere theories and monads on Set, we consider the effect of change of base. Specifically, given locally finitely presentable categories A and B and a map of locally finitely presentable categories from A to B, i.e. a filtered colimit-preserving functor $U : A \longrightarrow B$ that has a left adjoint F, we study the relationship between Lawvere A-theories, equivalently finitary monads on A, and Lawvere B-theories, equivalently finitary monads on B, as induced by U.

Every map $F \dashv U : A \longrightarrow B$ of locally finitely presentable categories routinely induces a 2-functor

$$(\mathbf{LocPres}_f /\!/ A)^{op} \xrightarrow[(\mathbf{LocPres}_f /\!/ U)^{op}]{} (\mathbf{LocPres}_f /\!/ B)^{op}$$

This 2-functor has a left biadjoint, which we denote by U^*, given by pseudo-pullback, which, in the situation of primary interest to us, is equivalent to an ordinary pullback (Joyal & Street 1993).

Trivially, every finitary monad T_A on A induces a finitary monad UT_AF on B. So there must be a corresponding construct for Lawvere theories cohering with the inclusion pseudo-functors of Theorems 18 and 19. The coherence is subtle. In terms of monads, it is as follows.

Theorem 27
Every map $F \dashv U : A \longrightarrow B$ of locally finitely presentable categories canonically induces an adjunction

$$F_{mnd} \dashv U_{mnd} : \mathbf{Mnd}_f(A) \longrightarrow \mathbf{Mnd}_f(B)$$

for which the diagram of left adjoints

$$\begin{array}{ccc} \mathbf{Mnd}_f(B) & \xrightarrow{F_{mnd}} & \mathbf{Mnd}_f(A) \\ \downarrow & & \downarrow \\ (\mathbf{LocPres}_f /\!\!/ B)^{op} & \xrightarrow[U^*]{} & (\mathbf{LocPres}_f /\!\!/ A)^{op} \end{array}$$

commutes.

Proof
U_{mnd} sends a finitary monad T_A on A to UT_AF with the evident monad structure on it. And F_{mnd} sends a finitary monad T_B on B to the monad induced by considering the pullback

$$\begin{array}{ccc} P & \longrightarrow & T\text{-}Alg \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[U]{} & B \end{array}$$

in Cat and observing that P is finitarily monadic over A and then taking the induced monad. Commutativity of the diagram follows by construction of F_{mnd} (cf. Joyal & Street 1993). □

Commutativity of the diagram in Theorem 27, subject to a mild 2-categorical subtlety (Joyal & Street 1993), determines a canonical 2-natural transformation

$$\begin{array}{ccc} \mathbf{Mnd}_f(A) & \xrightarrow{U_{mnd}} & \mathbf{Mnd}_f(B) \\ \downarrow & \Downarrow & \downarrow \\ (\mathbf{LocPres}_f /\!\!/ A)^{op} & \xrightarrow[(\mathbf{LocPres}_f /\!\!/ U)^{op}]{} & (\mathbf{LocPres}_f /\!\!/ B)^{op} \end{array}$$

The component at T_A of that 2-natural transformation is the canonical comparison functor

$$T_A\text{-}Alg \longrightarrow UT_AF\text{-}Alg$$

The above relationships can duly be expressed equally in terms of Lawvere A-theories and Lawvere B-theories as follows.

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Theorem 28
Every map $F \dashv U : A \longrightarrow B$ of locally finitely presentable categories canonically induces an adjunction

$$F_{Law} \dashv U_{Law} : \mathbf{Law}_A \longrightarrow \mathbf{Law}_B$$

for which the diagram of left adjoints

$$\begin{array}{ccc}
\mathbf{Law}_B & \xrightarrow{F_{Law}} & \mathbf{Law}_A \\
\downarrow & & \downarrow \\
(\mathbf{LocPres}_f /\!/ B)^{op} & \xrightarrow[U^*]{} & (\mathbf{LocPres}_f /\!/ A)^{op}
\end{array}$$

commutes.

Proof
U_{Law} sends a Lawvere A-theory $I : A_f^{op} \longrightarrow L_A$ to the (identity-on-objects, fully faithful) factorisation of the composite

$$B_f^{op} \longrightarrow A_f^{op} \longrightarrow L_A$$

And F_{Law} sends a Lawvere B-theory $I : B_f^{op} \longrightarrow L_B$ to the Lawvere A-theory generated by taking the push-out

$$\begin{array}{ccc}
B_f^{op} & \xrightarrow{F} & A_f^{op} \\
{\scriptstyle I}\downarrow & & \downarrow{\scriptstyle J} \\
L_B & \longrightarrow & P
\end{array}$$

and then taking $\mathscr{F}_J(P)$. Commutativity of the diagram in the statement of the theorem follows by construction of F_{Law}. □

Commutativity of the diagram in Theorem 28 determines a canonical 2-natural transformation

$$\begin{array}{ccc}
\mathbf{Law}_A & \xrightarrow{U_{Law}} & \mathbf{Law}_B \\
\downarrow & \Downarrow & \downarrow \\
(\mathbf{LocPres}_f /\!/ A)^{op} & \xrightarrow[(\mathbf{LocPres}_f /\!/ U)^{op}]{} & (\mathbf{LocPres}_f /\!/ B)^{op}
\end{array}$$

The component at L_A of that natural transformation is determined by composition.

Example 29
The forgetful functor $U : Poset \longrightarrow Set$ is a map of locally finitely presentable categories. Its left adjoint F takes a set X to itself, regarded as a discrete poset. So the functor U_{mnd} sends a monad T on $Poset$ to the monad on Set that sends a set

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X to the underlying set of TX. For instance, it sends the monad $(S \times -)^S$ on $Poset$ for global state to the monad on Set for global state. The behaviour of F_{mnd} seems less natural in regard to computational effects, as it is determined by its behaviour on T-Alg rather than on $Kl(T)$: the monad F_{mnd} necessarily exists, but we do not have any comprehensible concrete description of it in general; given a monad T on Set, the monad $F_{mnd}(T)$ on $Poset$ sends a poset P to the free poset Q equipped with T-structure on the underlying set UQ of Q.

Example 30

A class of examples of change of base arises when one considers local state. In this paper, following Plotkin & Power (2002) and Power (2006), we have focused on $[Inj, Set]$ as an appropriate base category in which to study local state. But it is not the only base presheaf category to have been used. For instance, the categories $[Nat, Set]$ and $[Iso, Set]$, where Iso is the category of natural numbers and permutations, or more complex variants, primarily in the work of O'Hearn & Tennent (1997), have appeared. Change of base applies to these.

Any functor $H : C \longrightarrow D$ between small categories C and D generates a map

$$[H, Set] : [D, Set] \longrightarrow [C, Set]$$

of locally finitely presentable categories, with the left adjoint to $[H, Set]$ given by left Kan extension. So, applying Theorems 27 and 28, H induces adjunctions

$$F_{mnd} \dashv [H, Set]_{mnd} : \mathbf{Mnd}_f([D, Set]) \longrightarrow \mathbf{Mnd}_f([C, Set])$$

and

$$F_{Law} \dashv [H, Set]_{Law} : \mathbf{Law}_{[D,Set]} \longrightarrow \mathbf{Law}_{[C,Set]}$$

In particular, for example, H might be the inclusion of Inj into Nat, thus yielding an adjunction between $\mathbf{Mnd}_f([D, Set])$ and $\mathbf{Mnd}_f([C, Set])$ and equivalently between $\mathbf{Law}_{[D,Set]}$ and $\mathbf{Law}_{[C,Set]}$.

There is a second, more delicate approach to change of base as follows: We have analysed the category $Mod(L)$ for a Lawvere A-theory L and shown that it supports a forgetful functor $U_L : Mod(L) \longrightarrow A$. We have further shown that U_L is finitarily monadic, and the construction of U_L characterises the finitary monads on A. But in Section 2, we considered models of an ordinary Lawvere theory in any base category with finite products, not only in Set. So one wonders whether we can consider models of a Lawvere A-theory in categories other than A.

In fact, we can do that, but the situation is subtle. Let A be a locally finitely presentable category, and let $I : A_f^{op} \longrightarrow L$ be a Lawvere A-theory. Consider any category of the form $FL(A_f^{op}, B)$, where B has finite limits. We can define the category

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$Mod(L, FL(A_f^{op}, B))$ of models of L in $FL(A_f^{op}, B)$ to be the pullback

$$\begin{array}{ccc} Mod(L, FL(A_f^{op}, B)) & \xrightarrow{P_L} & [L, B] \\ {\scriptstyle U_L}\downarrow & \lrcorner & \downarrow{\scriptstyle [I,B]} \\ FL(A_f^{op}, B) & \xrightarrow[inc]{} & [A_f^{op}, B] \end{array}$$

generalising the characterisation in Proposition 15 of the category of models of L in A.

A priori, this may look special, not recovering the idea of a model of an ordinary Lawvere theory in an arbitrary category with finite products as in Definition 1. But that is illusory: if $A = Set$, the category A_f^{op} is Nat_f^{op}, which is the free category with finite limits on 1. So, for any category B with finite limits, $FL(A_f^{op}, B)$ is equivalent to B. And so, in the case of $A = Set$, the generality we assert here means we can take models of a Lawvere A-theory in any category B with finite limits.

Thus the generality we propose here covers all examples of interest to us. With care, we can go even further: both in Definition 1 and here, we do not actually need all finite products or all finite limits in B respectively; we just need some specific ones. So, with care, it is routine give a further generalisation beyond the assertion that B has finite limits to include Definition 1 entirely, but the lack of examples makes it seem complex to the point of distraction to give the details here.

8 Conclusions

The notion of Lawvere theory, as introduced by Lawvere in his PhD thesis (1963), has become increasingly valuable over recent years in analysing computational effects, allowing a more refined denotational semantics than that provided by monads (Hyland *et al.* 2006, 2007). Classically, the relationship between Lawvere theories and monads has only been properly understood for base category Set and more recently for base V-category V (Power 2000). That does not fully cover the range of situations in which one seeks to model effects, as, in particular, local effects are typically modelled in presheaf categories such as $[Inj, Set]$, with enrichment in $[Inj, Set]$ looking out of place (O'Hearn & Tennent 1997; Plotkin & Power 2002; Power 2006).

So, in this paper, extending mathematical ideas from Nishizawa & Power (2009), we have addressed the situation, developing a notion of Lawvere A-theory, where one does not insist upon enrichment of the base category A in itself. Giving a mathematically unified account of the situation led us to explicate Gabriel–Ulmer duality, as it yields an account of change of base by considering pseudo-slice 2-categories.

This is one of a number of recent extensions of the notion of Lawvere theory, others being given by discrete Lawvere theories (Hyland & Power 2006) and indexed Lawvere theories (Power 2006). Each of these extensions has been devised with particular applications in mind, all of them relevant to computational effects. It is not clear yet precisely what combined extension of the notion of Lawvere theory might be optimal. So that remains an open question, partly because the various mathematical developments have given rise to new computational questions, such as the classification of effects into constructors, deconstructors and logical effects mooted in Hyland *et al.* (2006).

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